Universal bounds for spreading on networks

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ABSTRACT

Spreading (diffusion) of innovations is a stochastic process on social networks. When the key driving mechanism is the peer effect (word of mouth), the rate at which the aggregate adoption level increases with time depends strongly on the network structure. In many applications, however, the network structure is unknown. To estimate the aggregate adoption level as a function of time for such innovations, we show that the minimal and maximal adoption levels are attained on a homogeneous two-node network and on a homogeneous infinite complete network, respectively. Solving the Bass model on these two networks yields explicit lower and upper bounds for the expected adoption level on any network. These bounds are tight, and they also hold for the individual adoption probabilities of nodes. The gap between the lower and upper bounds increases monotonically with the ratio of the rates of internal and external influences.

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Quantitative predictions for the adoption level of a new product from the time it is first introduced into the market are of fundamental importance to firms and investors. When most of the spreading of the new product occurs through word-of-mouth, the structure of the social network plays a key role in determining the adoption level in the market. In many cases, however, the social network structure is unknown. In such cases, one wishes to have at least some information on the minimal and maximal adoption levels. In this paper, we find explicit expressions for these lower and upper bounds and analyze the size of the gap between the two.

I. INTRODUCTION

Diffusion (spreading) of innovations in networks is an active research area in mathematics, economics, management science, social sciences, and more.^{1–3} In marketing, diffusion of new products is a classical problem.⁴

The first mathematical model of diffusion of new products was introduced by Bass.⁵ In this model, individuals adopt a new product because of *external influences* by mass media and commercials and because of *internal influences* (peer effect, word-of-mouth) by individuals who have already adopted the product. Let f denote the *adoption level* (fraction of adopters) in the population at time t. Then, according to the Bass model,

$$f'(t) = (1 - f)(p + qf), \quad t > 0, \quad f(0) = 0.$$
 (1)

Thus, the 1 - f potential adopters adopt due to external influences at the constant rate of p and due to internal influences at the rate of qf, which is proportional to the fraction of adopters. Equation (1) can be solved explicitly, yielding the S-shaped Bass formula,⁵

$$f_{\text{Bass}}(t) = \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{2}e^{-(p+q)t}}.$$
(2)

The Bass model (1) inspired a huge body of theoretical and empirical research; in 2004, it was selected as one of the 10 mostcited papers in the 50-year history of Management Science.⁶ Initially, this research was carried out using compartmental Bass models, such as Eq. (1), in which the population is divided into several compartments (e.g., nonadopters and adopters), and the transition rates between compartments are given by deterministic ordinary differential equations. Compartmental Bass models, therefore, implicitly assume that the underlying social network is a homogeneous complete graph, i.e., that all individuals within the population are equally likely to influence each other.

In order not to make these assumptions, in more recent studies, diffusion of new products has been studied using Bass models on networks.⁷⁻¹² These *agent-based models* for the stochastic adoption decision of each individual allow for implementing a heterogeneous network structure, whereby individuals can only be influenced by adopters who are also their peers.

Explicit expressions for the expected adoption level f(t) in the Bass model were only obtained for a few networks. Niu¹³

computed explicitly the expected adoption level $f_{\text{complete}}(t; M)$ on complete homogeneous networks with M nodes and showed that $\lim_{M\to\infty} f_{\text{complete}}(t; M) = f_{\text{Bass}}(t)$, see Theorem 2 below. Fibich and Gibori⁸ computed explicitly the expected adoption level $f_{\text{circle}}(t; M)$ on homogeneous circles with M nodes. They showed that the adoption level on the infinite circle, denoted by $f_{\text{1D}}(t)$

 $:= \lim_{M \to \infty} f_{\text{circle}}(t; M), \text{ is given by } f_{\text{1D}}(t) = 1 - e^{-(p+q)t+q\frac{1-e^{-pt}}{p}}.$

For most networks, explicit expressions for f(t) are not available. Moreover, in many applications, the network structure is not known. Hence, it is important, for both theoretical and practical considerations, to obtain explicit lower and upper bounds for the expected adoption level f(t).

In Ref. 8, it was conjectured that since circular and complete networks are the "least-connected" and the "most-connected" networks, the adoption level on any infinite network should be bounded from below by that on the infinite circle and from above by that on the infinite complete network, i.e., that $f_{1D}(t) \le f(t) \le f_{\text{Bass}}(t)$. So far, this conjecture has remained open.

In this study, we settle this conjecture. We prove that $f(t) \leq f_{Bass}(t)$ for any finite or infinite network. Thus, as was conjectured in Ref. 8, f_{Bass} is a universal upper bound for the adoption level. Moreover, this upper bound is tight and is strict for non-complete networks. The tight universal upper bound for the individual adoption probabilities of nodes (i.e., for the probability of any node to adopt the product before time t) is also given by f_{Bass} .

The universal lower bound for f(t) on general finite or infinite networks, however, is not f_{1D} . Rather, we prove that $f(t) \ge f_{M=2}^{\text{hom}}(t)$ for any network, where $f_{M=2}^{\text{hom}} := 1 - e^{-pt} \frac{qe^{-pt} - pe^{-qt}}{q-p}$ is the expected adoption level on a homogeneous two-node network. This universal lower bound is also tight, and it also holds for the individual adoption probabilities of nodes. Thus, the conjecture from Ref. 8 that f_{1D} is a universal lower bound for all infinite networks is wrong (note, however, that for any $D \ge 1$, f_{1D} is the tight lower bound for the adoption level $f_D(t)$ on infinite D-dimensional Cartesian network where each node is connected to its 2D nearest neighbors with edges of weight $\frac{q}{2D}$; see Ref. 14 for more details).

Let us motivate the "success" of the conjecture from Ref. 8 regarding the upper bound, and its "failure" regarding the lower bound. As noted, the compartmental Bass model (1) corresponds to a complete network, which is indeed the "most-connected" network, in the sense that each node can be directly influenced by all other nodes. A one-sided circle, where each node can only influenced by the node to its left, however, is not the "least-connected" network. This is because each node is also indirectly influenced by all other nodes. Rather, the "least-connected" network is a collection of disjoint pairs of nodes, where each node can be directly influenced by the other node in the pair but cannot be indirectly influenced by any other node.

More generally, this study presents a collection of analytic results that share a common principle: *weak influences by numerous adopters lead to a faster spreading than strong influences by a few adopters.* The results that the lower and upper bounds are attained on two-node and infinite complete networks, respectively, can be viewed as extreme manifestations of this principle.

To quantify the influence of the social-network structure on the adoption level of new products, we study the size of the gap between the lower and upper bounds. The gap size is a monotonically increasing function of the ratio $\frac{q}{p}$ of the rates of internal and external influences. For products that spread predominantly through word of mouth, we obtain an explicit approximation for the gap size. This explicit approximation shows that the network structure indeed has a large influence on the adoption level of such products.

The practical implications of this study are as follows:

- (1) Availability of explicit lower and upper bounds for the expected adoption level as a function of time.
- (2) The insight that when $p \ll q$, the network structure has a large effect on the speed at which a new product will spread.
- (3) The principle that weak influences by numerous adopters lead to a faster spreading than strong influences by a few adopters suggests that promotional strategies that lead to a small increase in peer effects by numerous individuals will be more effective than those that lead to a large increase in peer effects by few individuals.

The paper is organized as follows. Section II presents the Bass model on a general network. Section III presents the main results of this paper on the universal lower and upper bounds. Section IV considers the size of the gap between the lower and upper bounds. Section V lists some open research problems. The detailed proofs are given in Sec. VI.

II. BASS MODEL ON NETWORKS

We begin by introducing the Bass model on a general heterogeneous network. This model is stochastic, unlike the compartmental Bass model (1) that is a deterministic ODE.

A new product is introduced at time t = 0 to a network with M individuals, denoted by $\mathcal{M} := \{1, ..., M\}$, where M can be finite or infinite. We denote by $X_j(t)$ the *state* of individual j at time t so that

$$X_j(t) = \begin{cases} 1, & \text{if } j \text{ is an adopter of the product at time } t, \\ 0, & \text{otherwise.} \end{cases}$$

Since the product is new, all individuals are initially nonadopters, i.e.,

$$X_j(0) = 0, \quad j \in \mathcal{M}. \tag{3a}$$

The underlying social network is represented by a weighted directed graph such that if there is an edge from k to j, the rate of internal influence of adopter k on nonadopter j to adopt is $q_{k,j} > 0$, and $q_{k,j} = 0$ if there is no edge from k to j. The edges and influence rates are *not* assumed to be symmetric, i.e., $q_{k,j}$ may be different from $q_{j,k}$. Since nonadopters do not self-influence to adopt,

$$q_{j,j} \equiv 0, \quad j \in \mathcal{M}$$

In contrast to similar models in epidemiology on networks,¹⁵ such as the susceptible infected (SI) model, *j* also experiences external influences to adopt by mass media and commercials at a constant rate of $p_j > 0$. Internal and external influences are assumed to be additive. Thus, the adoption time T_j of nonadopter *j* is piecewise

exponentially distributed at the rate of

$$\lambda_j(t) := p_j + \sum_{k \in \mathcal{M}} q_{k,j} X_k(t), \quad j \in \mathcal{M}, \quad t > 0,$$
(3b)

which increases whenever k adopts and $q_{k,j} > 0$. Finally, it is assumed that once an individual adopts the product, she or he remains an adopter for all time. Therefore, the stochastic adoption of $j \in \mathcal{M}$ in the time interval $(t, t + \Delta t)$ as $\Delta t \rightarrow 0$ is given by

$$\mathbb{P}(X_j(t + \Delta t) = 1 | \mathbf{X}(t))$$

$$= \begin{cases} 1, & \text{if } X_j(t) = 1, \\ \left(p_j + \sum_{k \in \mathcal{M}} q_{k,j} X_k(t)\right) \Delta t, & \text{if } X_j(t) = 0, \end{cases} (3c)$$

where $\mathbf{X}(t) := (X_1(t), \dots, X_M(t))$ is the state of the network at time *t*. Note that the time variable is continuous.

The maximal rate of internal influences that can be exerted on node j (which is when all its neighbors/peers are adopters) is

$$q_j := \sum_{k \in \mathcal{M}} q_{k,j}.$$
 (4a)

For simplicity, we assume that each node can be influenced by at least one node, i.e.,

$$q_j > 0, \quad j \in \mathcal{M}.$$
 (4b)

We do not assume, however, that the network only consists of a single connected component. The underlying network of the Bass model (3) is denoted by

$$\mathcal{N} = \mathcal{N}\left(\mathcal{M}, \{p_k\}_{k \in \mathcal{M}}, \{q_{k,j}\}_{k,j \in \mathcal{M}}\right).$$
(5)

The adoption level at time t is $\frac{1}{M} \sum_{j \in \mathcal{M}} X_j(t)$. Our goal is to obtain lower and upper bounds for the *expected adoption level* (fraction of adopters),

$$f(t; \mathcal{N}) := \frac{1}{M} \mathbb{E}\left[\sum_{j \in \mathcal{M}} X_j(t)\right].$$

To do that, we will compute lower and upper bounds for the adoption probabilities of nodes

$$f_j(t; \mathcal{N}) := \mathbb{P}(X_j(t) = 1) = \mathbb{E}[X_j(t)], \quad j \in \mathcal{M}$$

and then use

$$f = \begin{cases} \frac{1}{M} \sum_{j=1}^{M} f_j, & M < \infty, \\ \lim_{M \to \infty} \frac{1}{M} \sum_{j=1}^{M} f_j, & M = \infty. \end{cases}$$
(6)

The dependence of the adoption level and of the adoption probabilities of nodes on the external and internal influence rates is monotonic:

Theorem 1 (Ref. 16): Consider the Bass model (3) on network \mathcal{N} , see (5). Let t > 0. Then $f(t; \mathcal{N})$ is monotonically increasing, and $\{f_m(t; \mathcal{N})\}$ is monotonically non-decreasing, with respect to each p_j and each $q_{k,j}$.

A. Homogeneous complete networks

Let $f_{\text{complete}}(t; p, q, M)$ denote the expected adoption level in the Bass model (3) on the homogeneous complete network $\mathcal{N}_{\text{complete}}(p, q, M)$, defined as

$$p_j \equiv p, \quad q_{k,j} = \begin{cases} \frac{q}{M-1}, & k \neq j, \\ 0, & k = j, \end{cases} \quad j, k \in \mathcal{M}.$$

$$(7)$$

As *M* increases, each node is influenced by more nodes, but the weight of each node decreases, so that the maximal rate of internal influences $q_j \equiv q$ remains unchanged, see Eq. (4a). Nevertheless, the expected adoption level increases with *M*:

Lemma 1 (Ref. 17): Let t, p, q > 0. Then, $f_{\text{complete}}(t; p, q, M)$ is monotonically increasing in M.

As $M \to \infty$, the Bass model (3) on complete networks approaches the original compartmental Bass model:

Theorem 2 (Ref. 13): $\lim_{M\to\infty} f_{\text{complete}}(t; p, q, M) = f_{\text{Bass}}(t; p, q)$, where f_{Bass} is given by (2).

From Lemma 1 and Theorem 2, we have **Corollary 1:** Let t, p, q > 0. Then

$$f_{\text{complete}}(t; p, q, M) < f_{\text{Bass}}(t; p, q), \quad M = 1, 2, \dots$$

III. MAIN RESULTS

In this section, we present the main results of this paper. The proofs are given in Sec. VI. For clarity, we formulate the results for networks that are homogeneous in $\{p_i\}$ and $\{q_i\}$, i.e.,

$$p_j \equiv p, \quad q_j \equiv q, \quad j \in \mathcal{M}.$$
 (8)

Networks that do not satisfy (8) are discussed in Sec. III D.

The condition (8) can be satisfied by any graph structure that satisfies (4b) and not just by the complete network (7). For example, for any given network $\mathcal{N}(\mathcal{M}, \{p_j\}, \{q_{k,j}\})$, define network $\widetilde{\mathcal{N}}(\mathcal{M}, \{\widetilde{p}_j\}, \{\widetilde{q}_{k,j}\})$ such that $\widetilde{p}_j := p$ and $\widetilde{q}_{k,j} := q_{k,j} \frac{q}{q_j}$. Then, $\widetilde{\mathcal{N}}$ satisfies (8), and it has the same nodes/edges structure as \mathcal{N} .

A. Non-tight universal bounds

The following universal lower and upper bounds are immediate:

Lemma 2: Consider the Bass model (3) on a network N that is homogeneous in $\{p_i\}$ and $\{q_i\}$, see (8). Then,

$$1 - e^{-pt} \le f_m(t) \le 1 - e^{-(p+q)t}, \quad t \ge 0, \quad m \in \mathcal{M},$$
 (9a)

and so

$$1 - e^{-pt} \le f(t) \le 1 - e^{-(p+q)t}, \quad t \ge 0.$$
 (9b)

Proof. Since $X_k(t) \in \{0, 1\}$ for any $k \in \mathcal{M}$, the adoption rate of node *m* is bounded by, see (3b) and (4a),

$$p = p_m \le \lambda_m(t) \le p_m + \sum_{k \in \mathcal{M}} q_{k,m} = p + q, \quad m \in \mathcal{M}, \quad t \ge 0.$$

Hence, (9a) follows, and so (9b) follows by (6).

Thus, the lower and upper bounds (9a) for $f_m(t)$ correspond to the extreme cases when none of the other individuals adopted by

time *t* and when all the other individuals adopted at t = 0+, respectively. Therefore, these bounds are not expected to be tight, as indeed we will show below.

B. Tight upper bound

If one adds edges to a network, this increases the adoption level f (Theorem 1). The following two observations suggest a stronger result, namely, that even if as we add edges, we lower the weights of the edges while keeping $q_j \equiv q$ unchanged, the adoption level increases:

- The adoption level *f*_{complete}(*t*; *p*, *q*, *M*) in homogeneous complete networks is monotonically increasing in *M* (Lemma 1).
- (2) The adoption level $f_D(t; p, q)$ in infinite *D*-dimensional Cartesian networks, where each node is connected to its 2*D* nearest neighbors, and the weights of these edges is $\frac{q}{2D}$, is monotonically increasing in *D* (this was shown numerically and asymptotically in Ref. 8).

Thus, for networks that satisfy (8), *numerous weak edges lead to a faster diffusion than a few strong ones*. Therefore, we can expect that among all networks with *M* nodes that satisfy (8), the fastest diffusion would be on the complete network $\mathcal{N}_{complete}(p, q, M)$, see (7), as formulated in Conjecture 1 below. If that is indeed the case, then by Corollary 1, the adoption levels on all networks should be bounded from above by f_{Bass} . Indeed, we can rigorously prove the following:

Theorem 3: Consider the Bass model (3) on a network N that is homogeneous in $\{p_i\}$ and $\{q_i\}$, see (8). Then

$$f_m(t;\mathcal{N}) \le f_{\text{Bass}}(t;p,q), \quad t \ge 0, \quad m \in \mathcal{M}, \tag{10}$$

where f_{Bass} is given by (2), and so

$$f(t;\mathcal{N}) \le f_{\text{Bass}}(t;p,q), \quad t \ge 0.$$
(11)

In Lemma 2, we derived the upper bound $f_j(t), f(t) \le 1 - e^{-(p+q)t}$. The upper bound of Theorem 3 is better (i.e., lower), since by (2),

$$f_{\text{Bass}}(t;p,q) = \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}} < 1 - e^{-(p+q)t}$$

We can further show that f_{Bass} is the *tight* universal upper bound: Lemma 3: The universal upper bound in Theorem 3 is tight, in

the sense that

$$\sup_{\mathcal{N} \mid (\&) \text{ holds}\}} f(t; \mathcal{N}) = \sup_{\{\mathcal{N} \mid (\&) \text{ holds}\}, m \in \mathcal{M}} f_m(t; \mathcal{N}) = f_{\text{Bass}}(t; p, q).$$

While the upper bound f_{Bass} is attained for an infinite homogeneous complete network (Theorem 2), it is strict for nodes that have a finite indegree, hence for networks with a positive fraction of nodes with finite indegree:

Theorem 4: Assume the conditions of Theorem 3.

(1) If node m has a finite indegree, then

$$f_m(t;\mathcal{N}) < f_{\text{Bass}}(t;p,q), \quad t > 0.$$
(12)

(2) If there is a positive fraction of nodes in the network with a finite indegree, then

$$f(t;\mathcal{N}) < f_{\text{Bass}}(t;p,q), \quad t > 0.$$
(13)

Therefore, the upper bound f_{Bass} is strict for any network that is not infinite and complete (up to a vanishing fraction of nodes). In particular, assume that the network type is one of the following:

- A finite network.
- An infinite (homogeneous or heterogeneous) D-dimensional Cartesian network.
- An infinite scale-free network.¹⁸
- An infinite small-worlds network.¹⁹
- The infinite sparse random networks $\lim_{M\to\infty} G\left(M, \frac{\lambda}{M}\right)^{20}$

Since all these finite and infinite networks have a positive fraction of finite-indegree nodes, Theorem 4 implies that $f < f_{\text{Bass}}$ for all these network types.

C. Tight lower bound

Let $\mathcal{N}_{M=2}^{\mathrm{hom}}(p,q)$ denote the homogeneous network with two nodes, where

$$\mathcal{M} = \{1, 2\}, \quad p_1 = p_2 = p, \quad q_{1,2} = q_{2,1} = q, \quad q_{1,1} = q_{2,2} = 0.$$
(14)

The expected adoption level on $\mathcal{N}_{M=2}^{\text{hom}}$ can be explicitly calculated (see, e.g., Ref. 16), giving

$$f_{M=2}^{\text{hom}}(t;p,q) = 1 - e^{-pt} \frac{q e^{-pt} - p e^{-qt}}{q-p}, \quad p \neq q.$$
 (15)

Note that there is only one homogeneous network with two nodes. Thus, $f_{M=2}^{\text{hom}}(t; p, q) = f_{\text{complete}}^{\text{hom}}(t; p, q, M = 2) = f_{\text{circle}}^{\text{hom}}(t; p, q, M = 2).$

As noted informally in Sec. III B, for networks that satisfy (8), few strong edges lead to a slower spreading than numerous weak ones. Hence, it is intuitive to expect that for given p and q, the adoption level is lowest when the influence q on any node in the network is exerted by a single node. This requirement is satisfied when the network is a one-sided circle or a collection of disjoint one-sided circles. Among all circles, the lowest adoption is on a two-node circle.¹⁷ This is because on a two-node circle each node can only be influenced by one node, whereas on longer circles, each node can also be indirectly influenced by additional nodes. Indeed, we now prove that $f_{m=1}^{\text{hom}}$ is a universal lower bound for $\{f_m\}$, hence, for f,

Theorem 5: Assume the conditions of Theorem 3. Then

$$f_m(t;\mathcal{N}) \ge f_{M=2}^{\text{hom}}(t;p,q), \quad t \ge 0, \quad m \in \mathcal{M},$$
(16a)

and so

$$f(t; \mathcal{N}) \ge f_{M=2}^{\text{hom}}(t; p, q), \quad t \ge 0.$$
 (16b)

In Lemma 2, we derived the lower bound $f_j(t), f(t) \ge 1 - e^{-pt}$. The lower bound in Theorem 5 is better (i.e., larger), since by Theorem 1,

$$f_{M=2}^{\text{hom}}(t; p, q) > f_{M=2}^{\text{hom}}(t; p, q = 0) = 1 - e^{-pt}.$$

Moreover, $f_{M=2}^{\text{hom}}$ is the *tight* universal lower bound: **Lemma 4:** Let $M \in \{2, 4, ...\}$. Then

$$\inf_{\{\mathcal{N}|(8) \text{ holds}\}} f(t;\mathcal{N}) = \inf_{\{\mathcal{N}|(8) \text{ holds}\}, m \in \mathcal{M}} f_m(t;\mathcal{N}) = f_{M=2}^{\text{hom}}(t;p,q).$$

The lower bound $f_j \ge f_{M=2}^{\text{hom}}$ is attained if and only if *j* belongs to an isolated pair of nodes.

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Theorem 6: Assume the conditions of Theorem 3. Let $j \in M$. If *j* can only be influenced by a single node, denoted by *k*, and if *k* can only be influenced by *j*, then

$$f_j(t; \mathcal{N}) = f_{\text{hom}}^{M=2}(t; p, q), \quad t \ge 0.$$
 (17)

Otherwise,

$$f_j(t; \mathcal{N}) > f_{\text{hom}}^{M=2}(t; p, q), \quad t > 0.$$
 (18)

Therefore, $f > f_{M=2}^{\text{fom}}$ for any finite network with an odd number of nodes, which is homogeneous in $\{p_j\}$ and $\{q_j\}$.

D. Bounds for networks inhomogeneous in $\{p_j\}$ or $\{q_j\}$

We can extend the upper-bound results to networks that are not homogeneous in $\{p_i\}$ and in $\{q_i\}$ as follows:

Corollary 2: Theorem 3, Lemma 3, and Theorem 4 remain valid if we replace condition (8) with

$$p = \max_{j \in \mathcal{M}} p_j, \quad q = \max_{j \in \mathcal{M}} q_j. \tag{19}$$

Proof. This follows from Theorem 1. \Box

Similarly, we can extend the lower-bound results to networks that are not homogeneous in $\{p_i\}$ and $\{q_i\}$.

Corollary 3: Theorem 5, Lemma 4, and Theorem 6, remain valid if we replace condition (8) with

$$p = \min_{j \in \mathcal{M}} p_j, \quad q = \min_{j \in \mathcal{M}} q_j.$$
(20)

Both extensions, however, are quite crude. Indeed, in Ref. 21, it was proved that on vertex-transitive graphs (A graph is called "vertex transitive" if for any two nodes $i, j \in M$, there is a permutation of the indices $\{1, \ldots, M\}$ of the nodes, which maps $i \rightarrow j$ and leaves the graph invariant.), the difference between the expected adoption level on a network that is heterogeneous in $\{p_j\}$ and $\{q_j\}$ and on the corresponding homogeneous network with $\bar{p} := \frac{1}{M} \sum_{j=1}^{M} p_j$ and $\bar{q} := \frac{1}{M} \sum_{j=1}^{M} q_j$ is $O(\epsilon^2)$ small, where ϵ is the level of heterogeneity in $\{p_i\}$ and $\{q_i\}$. Moreover, numerical simulations in Ref. 22 showed that heterogeneity in $\{p_i\}$ and $\{q_i\}$ has a minor effect on the expected adoption level in the Bass model on complete networks. In,8 simulations of the Bass model on 1D and 2D Cartesian networks showed that when the values of $\{p_i\}$ and $\{q_i\}$ are uniformly distributed within $\pm 20\%$ of the corresponding values of *p* and *q* of the homogeneous individuals, the heterogeneous and homogeneous expected adoption levels are nearly indistinguishable. Even when the heterogeneity level was increased to $\pm 50\%$, the heterogeneous and homogeneous expected adoption level were still very close. Similar results were obtained in Ref. 23 in simulations of the Bass-SIR model on 1D and 2D Cartesian networks, on small-worlds networks, and on scale-free networks. Therefore, all the analytic and numerical evidence suggests that $f(t; \{p_j\}, \{q_j\}) \approx f(t; \bar{p}, \bar{q})$. Hence, for all practical purposes, one can bound $f(t; \{p_i\}, \{q_i\})$ from below by $f_{M=2}^{\text{hom}}(t; \bar{p}, \bar{q})$ and from above by $f_{\text{Bass}}(t; \bar{p}, \bar{q})$.

IV. GAP BETWEEN LOWER AND UPPER BOUNDS

Consider any network N that is homogeneous in $\{p_j\}$ and $\{q_j\}$, see (8). By Theorems 3 and 5, the expected adoption level and the

adoption probability of nodes are bounded by

$$f_{M=2}^{\text{hom}}(t;p,q) \leq f(t;\mathcal{N}), f_m(t;\mathcal{N}) \leq f_{\text{Bass}}(t;p,q), \quad t \geq 0.$$

Therefore, it is natural to consider the *size of the gap* between the explicit lower and upper bounds $f_{M=2}^{hom}$ and f_{Bass} , which expresses the dependence of the diffusion on the network structure.

The explicit bounds can be written in a dimensionless form as

$$f_{M=2}^{\text{hom}}(t; p, q) = f_{M=2}^{\text{hom}}(\widetilde{t}; \widetilde{q}), \quad f_{\text{Bass}}(t; p, q) = f_{\text{Bass}}(\widetilde{t}; \widetilde{q}),$$

where $\tilde{t} = qt$ and $\tilde{q} = \frac{q}{p}$. The nondimensional parameter \tilde{q} expresses the ratio of internal and external influences. Since network effects are only due to internal influences, they increase with $\frac{q}{p}$. Thus, when q = 0, there are no network effects, and so the two bounds are identical, i.e.,

$$f_{M=2}^{\text{hom}}(t; p, q = 0) = f_{\text{Bass}}(t; p, q = 0) = 1 - e^{-pt}.$$

When $\frac{q}{p} \ll 1$, the network has a minor effect on the diffusion, and so $f_{M=2}^{\text{hom}} \approx f_{\text{Bass}}$, see Fig. 1(a). For products that spread predominantly through word-of-mouth, however, the regime of relevance is $\frac{q}{p} \gg 1$, typically $10 \le \frac{q}{p} \le 100.5$ As can be expected, the difference between $f_{M=2}^{\text{hom}}$ and f_{Bass} is significant for $\frac{q}{p} = 10$ [Fig. 1(b)] and even larger for $\frac{q}{p} = 100$ [Fig. 1(c)]. Note that for any network \mathcal{N} , $f(t; \mathcal{N})$ lies in the shaded region between $f_{M=2}^{\text{hom}}(t)$ and $f_{\text{Bass}}(t)$.

It is instructive to compare the adoption levels on different networks using the "half-life" $T^{1/2}$ for half of the population to adopt. In particular, we can use $T^{1/2}$ to compare the bounds f_{Bass} and $f_{M=2}^{\text{hom}}$. The ratio $\frac{T_{\text{Bass}}^{1/2}}{T_{M=2}^{1/2}}$ can be estimated asymptotically, yielding

$$\frac{T_{\text{Bass}}^{1/2}}{T_{M=2}^{1/2,\text{hom}}} \sim \frac{2}{\log 2} \frac{p}{q} \log \frac{q}{p}, \quad \frac{q}{p} \gg 1.$$
(21)

Figure 1(d) confirms that $\frac{T_{Rass}^{1/2}}{T_{M=2}^{1/2,\text{hom}}}$ decreases with $\frac{q}{p}$ and approaches the asymptotic limit (21) as $\frac{q}{p} \rightarrow \infty$. This limit goes to zero as $\frac{q}{p} \rightarrow \infty$, showing that the network structure has a large effect on diffusion when $\frac{q}{p} \gg 1$, i.e., for products that diffuse primarily by internal influences.

V. OPEN PROBLEMS

This manuscript settles the conjecture from Ref. 8 but leads to some new questions, which are currently open. Indeed, the upper and lower bounds in Theorems 3 and 5 are tight for networks with any number of nodes. Can these bounds be improved if we restrict ourselves to networks with a fixed number of nodes?

Thus, let

$$\mathcal{G}(p, q, M) := \{\mathcal{N} \mid \mathcal{N} has Mnodes, (8) holds\}$$

be the set of all networks with M nodes that are homogeneous in $\{p_j\}$ and $\{q_j\}$. In the beginning of Sec. III B, we argued that the fastest diffusion in $\mathcal{G}(p, q, M)$ should occur on the homogeneous complete network (7). Therefore, we formulate

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FIG. 1. The expected adoption level f(t) of any network \mathcal{N} lies in the shaded region between the lower bound $f_{M=2}^{hom}$ (orange dashed) and upper bound f_{Bass} (blue solid). (A) $\frac{q}{p} = 0.1$. The two bounds are nearly indistinguishable. (B) $\frac{q}{p} = 10$. (C) $\frac{q}{p} = 100$. (D) $\frac{T_{Bass}^{1/2}}{T_{M=2}^{1/2,hom}}$ as a function of $\frac{q}{p}$ (blue solid) and its asymptotic approximation (21) (black dashes).

Conjecture 1:

$$\sup_{\mathcal{N}\in\mathcal{G}(p,q,M)}f(t;\mathcal{N})=f_{\text{complete}}(t;p,q,M).$$

We note, however, that the rate of convergence of f_{complete} to f_{Bass} as $M \to \infty$ is $O\left(\frac{1}{M}\right)$, see Ref. 24. Therefore, the difference between these two upper bounds becomes negligible for large (e.g., $M = 10^6$) networks.

Consider now the lower bound. Let *M* be even, and let network $\tilde{\mathcal{N}}$ be composed of $\frac{M}{2}$ pairs of nodes, each of which is of type $\mathcal{N}_{M=2}^{\text{hom}}$, see Eq. (14). Then $f(t; \tilde{\mathcal{N}}) = f_{M=2}^{\text{hom}}(p, q)$. Therefore,

$$\inf_{\mathcal{N}\in\mathcal{G}(p,q,M)}f(t;\mathcal{N})=f_{M=2}^{\mathrm{hom}}(p,q),\quad M \text{ even}.$$

Thus, the lower bound $f_{M=2}^{\text{hom}}$ cannot be improved (i.e., increased) for networks with a fixed even number of nodes. The tight lower bound for *M* odd, however, is an open problem.

Another open question is the tight lower bound of f among *connected* networks with M nodes (even or odd) that are homogeneous in $\{p_j\}$ and $\{q_j\}$. Here, one may need to distinguish between connected undirected networks, weakly connected directed graphs (there is an undirected path between any pair of vertices), and strongly connected directed graphs (there is a directed path between every pair of vertices).

VI. PROOF OF RESULTS

A. Master equations

Denote the nonadoption probability of node *j* by

$$[S_j](t) := 1 - f_j(t) = \mathbb{P}(X_j(t) = 0).$$
(22)

Then, $[S_j]$ satisfies the master equation,²⁵

$$\frac{d}{dt}[S_j](t) = -(p_j + q_j)[S_j] + \sum_{k \in \mathcal{M}} q_{k,j}[S_j, S_k](t), \quad [S_j](0) = 1,$$
(23)

where q_j is given by (4a), and

$$[S_i, S_k](t) := \mathbb{P}(X_i(t) = X_k(t) = 0).$$

In general, to close these equations, one adds the master equations for all pairs $\{[S_j, S_k]\}$, all triplets $\{[S_j, S_k, S_m]\}$, etc., see Ref. 25. For the purpose of obtaining the lower and upper bounds, however, we will only need the following result:

Lemma 5: Consider the Bass model (3). Then, for any $i, j \in \mathcal{M}$,

$$[S_i](t)[S_i](t) \le [S_i, S_i](t) \le e^{-2pt}, \quad 0 \le t < \infty.$$
(24)

Proof. The left inequality is proved in Ref. 14. For the right inequality, we note that the joint nonadoption probability of a pair

 $\{i, j\}$ of isolated nodes $(q_j = q_i = 0)$ is given by

$$[S_i, S_j](t; p, q_j = q_i = 0) = e^{-2pt},$$

see, e.g., Ref. 16. Hence, the right inequality follows from Theorem 1. $\hfill \square$

B. Differential and integral Bass inequalities

Let us recall the following result:

Lemma 6 (Ref. 8): Let p, q > 0 and let f(t) satisfy the differential Bass inequality,

$$\frac{df}{dt} < (1-f)(p+qf), \quad t > 0, \quad f(0) = 0.$$

Then, $f(t) < f_{\text{Bass}}(t; p, q)$ for $0 < t < \infty$.

Let $[S_{\text{Bass}}] := 1 - f_{\text{Bass}}$ denote the nonadoption level in the compartmental Bass model. Then, by Eq. (1),

$$\frac{d}{dt}[S_{\text{Bass}}](t) = -(p+q)[S_{\text{Bass}}] + q[S_{\text{Bass}}]^2, \quad [S_{\text{Bass}}](0) = 1.$$
(25)

If we replace the equality sign in Eq. (25) by an inequality, the solution of this inequality is bounded from below by $[S_{Bass}]$:

Lemma 7: Let p, q > 0, and let [S](t) satisfies the differential Bass inequality

$$\frac{d}{dt}[S](t) > -(p+q)[S] + q[S]^2, \quad t > 0, \quad [S](0) = 1.$$

Then, $[S](t) > [S_{Bass}](t)$ for $0 < t < \infty$.

Proof. This follows from Lemma 6 and $[S_{Bass}] = 1 - f_{Bass}$. Multiplying Eq. (25) by $e^{(p+q)t}$, integrating between zero and

t, and using the initial condition give the integral form of the compartmental Bass model,

$$[S_{\text{Bass}}](t) = e^{-(p+q)t} + q \int_0^t e^{-(p+q)(t-\tau)} [S_{\text{Bass}}]^2(\tau) \, d\tau.$$
(26)

If we replace the equality sign in Eq. (26) by an inequality, the solution of the resulting integral Bass inequality is bounded from below by $[S_{Bass}]$:

Lemma 8: Let p, q > 0, and let [S](t) be non-negative and continuous in $[0, \infty)$.

(1) If [S] satisfies the integral Bass inequality

$$[S](t) \ge e^{-(p+q)t} + q \int_0^t e^{-(p+q)(t-\tau)} [S]^2(\tau) \, d\tau, \quad t > 0, \quad (27)$$

then $[S](t) \ge [S_{Bass}](t; p, q)$ for $t \ge 0$. (2) If inequality (27) is strict, then $[S](t) > [S_{Bass}](t; p, q)$ for t > 0.

Proof. Let $u := [S] - [S_{Bass}]$. Subtracting Eq. (26) from Eq. (27) gives

$$u(t) \ge q \int_0^t e^{-(p+q)(t-\tau)} \left([S]^2 - [S_{\text{Bass}}]^2 \right) (\tau) \, d\tau.$$

Therefore,

$$u(t) \ge \int_0^t \phi(\tau) u(\tau) \, d\tau, \quad \phi(\tau) := q e^{-(p+q)(t-\tau)} \left([S] + [S_{\text{Bass}}] \right)(\tau).$$
(28)

Since [S] and [S_{Bass}] are continuous and non-negative, then so is ϕ . Let

$$v(t) := e^{-\int_0^t \phi} \int_0^t \phi(\tau) u(\tau) \, d\tau.$$
 (29)

Then v(0) = 0 and

$$\frac{dv}{dt} = e^{-\int_0^t \phi} \phi(t) \left(u(t) - \int_0^t \phi(\tau) u(\tau) \, d\tau \right) \ge 0,$$

where the inequality follows from Eq. (28). Therefore, for $t \ge 0$, $v(t) \ge 0$. Hence, by Eq. (29), $\int_0^t \phi(\tau) u(\tau) d\tau \ge 0$, and so by Eq. (28), $u(t) \ge 0$.

If inequality (27) is strict, we replace in the above proof all " \geq " signs by ">" signs.

C. Upper bound

We begin with an auxiliary result.

Lemma 9: Consider the Bass model (3). Let (19) hold, and let

$$[\underline{S}](t) := \inf_{i \in \mathcal{M}} \left\{ [S_j](t) \right\}.$$
(30)

Then, $[\underline{S}](t)$ *is non-negative and continuous.*

Proof. The non-negativity of [S] follows from that of $\{[S_j]\}$. Let $j \in \mathcal{M}$. Since all probabilities are bounded between 0 and 1, then using (23) and (19),

$$\left|\frac{d}{dt}[S_j]\right| \leq (p+q)[S_j] + \sum_{k \in \mathcal{M}} q_{k,j}\left[S_j, S_k\right] \leq p+q + \sum_{k \in \mathcal{M}} q_{k,j} \leq \kappa,$$

where $\kappa := p + 2q$. Hence, by the mean-value theorem, for any $t, t^* > 0$, $|[S_j](t) - [S_j](t^*)| \le \kappa |t - t^*|$, and so $-[S_j](t^*) \le -[S_j](t) + \kappa |t - t^*| \le -[\underline{S}](t) + \kappa |t - t^*|$. Taking the supremum of the left-hand side yields $-[\underline{S}](t^*) \le -[\underline{S}](t) + \kappa |t - t^*|$, and so $[\underline{S}](t) - [\underline{S}](t^*) \le \kappa |t - t^*|$. Swapping t and t^* gives the inverse estimate, and so $[\underline{S}](t)$ is continuous.

Proof of Theorem 3. Since $1 - f_m = [S_m] \ge [\underline{S}]$, see Eqs. (22) and (30), it is sufficient to show that

$$[\underline{S}](t) \ge [S_{\text{Bass}}](t, p, q). \tag{31}$$

By Eq. (23) with $q_j = q$, see (8),

$$[S_j] = e^{-(p+q)t} + \int_0^t e^{-(p+q)(t-\tau)} \sum_{k \in \mathcal{M}} q_{k,j} [S_j, S_k](\tau) \, d\tau.$$
(32)

Therefore, by the lower bound in Eqs. (24) and (30),

$$\begin{split} [S_j] &\geq e^{-(p+q)t} + \int_0^t e^{-(p+q)(t-\tau)} \sum_{k \in \mathcal{M}} q_{k,j} [S_j](\tau) [S_k](\tau) \, d\tau \\ &\geq e^{-(p+q)t} + q \int_0^t e^{-(p+q)(t-\tau)} [\underline{S}]^2(\tau) d\tau. \end{split}$$

Taking the infimum over all j gives

$$[\underline{S}] \ge e^{-(p+q)t} + q \int_0^t e^{-(p+q)(t-\tau)} [\underline{S}]^2(\tau) \, d\tau.$$

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Therefore, since $[\underline{S}]$ is non-negative and continuous (Lemma 9), we can use the integral Bass inequality (Lemma 8) to get inequality (31), from which Eq. (10) follows. Therefore, by Eqs. (6) and (11) follows.

Proof of Lemma 3. The result for f follows from Theorem 2. Since the complete network (7) is homogeneous, $f_m \equiv f$ for all $m \in \mathcal{M}$. Hence, the result holds for any f_m as well. Proof of Theorem 4. Let

$$A_d(\mathcal{N}) := \{m \in \mathcal{M} \mid \text{indegree}(m) = d\}$$

denote the set of all nodes with indegree *d* in network \mathcal{N} . Then it is sufficient to prove that for all networks that satisfy (8) and for all $d \in \mathbb{N}$,

$$[S_m](t;\mathcal{N}) > [S_{\text{Bass}}](t;p,q), \quad t > 0, \quad m \in A_d(\mathcal{N}).$$
(33)

We prove Eq. (33) by induction on *d*. When d = 0, node $m \in A_0$ is not influenced by any other node, and so

$$[S_m](t;\mathcal{N}) = e^{-pt} = [S_{\text{Bass}}](t;p,q=0) > [S_{\text{Bass}}](t;p,q), \quad (34)$$

where the inequality follows from Theorem 1.

For the induction stage, we assume that Eq. (33) holds for all networks that satisfy Eq. (8) and for all $m \in A_{d-1}$ and prove that it holds for all networks that satisfy Eq. (8) and for all $m \in A_d$, as follows. Let $m \in A_d$, where $d \ge 1$, and denote by $\{k_1, \ldots, k_d\}$ the *d* nodes that can influence *m*. The master equation for $[S_m]$ is, see Eqs. (8) and (23),

$$\frac{d}{dt}[S_m] = -(p+q)[S_m] + \sum_{i=1}^d q_{k_i,m}[S_m, S_{k_i}], \quad [S_m](0) = 1.$$
(35)

By the indifference principle, we can compute each of the *d* probabilities $\{[S_m, S_{k_i}]\}_{i=1}^d$ on a modified network $\widetilde{\mathcal{N}}_i$, in which we remove the edge $k_i \to m$. Thus, $[S_m, S_{k_i}] = [\widetilde{S_m}, \widetilde{S_{k_i}}]$, where the tilde sign refers to probabilities in $\widetilde{\mathcal{N}}_i$. In this modified network, node *m* has indegree d-1, and so by the induction assumption (In the modified network $\widetilde{\mathcal{N}}_i$, we reduced q_m by $q_{k_i,m} > 0$. Therefore, $\widetilde{q}_m < q$, and so we cannot apply the induction assumption directly for $\widetilde{\mathcal{N}}_i$. By Theorem 1, however, since the induction assumption holds when $\widetilde{q}_m = q$, see Eq. (8), it also holds when $\widetilde{q}_m < q$.),

$$\widetilde{S_m}$$
] > [S_{Bass}].

In addition, by Theorem 3,

$$[\widetilde{S_{k_i}}] \geq [S_{\text{Bass}}].$$

Combining the above and Eq. (24), we have that

$$[S_m, S_{k_i}] = [\widetilde{S_m}, \widetilde{S_{k_i}}] \ge [\widetilde{S_m}][\widetilde{S_{k_i}}] > [S_{\text{Bass}}]^2.$$

Therefore,

$$\sum_{i=1}^{d} q_{k_i,m} \left[S_m, S_{k_i} \right] > \sum_{i=1}^{d} q_{k_i,m} \left[S_{\text{Bass}} \right]^2 = q \left[S_{\text{Bass}} \right]^2.$$
(36)

By Eqs. (35) and (36),

$$\frac{d}{dt}[S_m] + (p+q)[S_m] > q[S_{\text{Bass}}]^2, \quad [S_m](0) = 1.$$

This is the differential Bass inequality (Lemma 6), written in terms of [*S*], see Eq. (7). Hence, $[S_m] > [S_{Bass}]$, as needed.

D. Lower bound

Proof of Theorem 5. To prove the lower bound (16a) for f_m , it is sufficient to show that

$$[S_m](t) \le [S_{M=2}^{\text{hom}}](t;p,q) := 1 - f_{M=2}^{\text{hom}}(t;p,q) = e^{-pt} \frac{q e^{-pt} - p e^{-qt}}{q-p},$$

where $[S_m] = 1 - f_m$. By the upper bound in Eqs. (24) and (32), we have that

$$\begin{split} [S_m] &\leq e^{-(p+q)t} + \int_0^1 e^{-(p+q)(t-\tau)} \sum_{k \in \mathcal{M}} q_{k,m} e^{-2p\tau} \, d\tau \\ &= e^{-(p+q)t} + q \int_0^t e^{-(p+q)(t-\tau)} e^{-2p\tau} \, d\tau \\ &= \left(1 - \frac{q}{q-p}\right) e^{-(p+q)t} + \frac{q}{q-p} e^{-2pt} = \left[S_{M=2}^{\text{hom}}\right](t;p,q) \end{split}$$

Therefore, we proved Eq. (16a), which implies Eq. (16b).

Proof of Lemma 4. When M = 2, this bound is attained by $\mathcal{N} = \mathcal{N}_{M=2}^{\text{hom}}(p, q)$. Moreover, this bound is also attained by any finite or infinite network, which is a collection of disjoint pairs of nodes, each of which is of type $\mathcal{N}_{M=2}^{\text{hom}}(p, q)$.

Proof of Theorem 6. The only inequality in the proof of Theorem 5 arises from using the upper bound in Eq. (24). Therefore, the lower bound (16a) for $[S_m]$ becomes an equality if and only if $[S_m, S_k] = e^{-2pt}$ for all $k \in \mathcal{M} \setminus m$ for which $q_{k,m} > 0$. A minor modification of Theorem 1 shows that

$$[S_i, S_k] = e^{-2pt}$$

 \iff *j* and *k* cannot be influenced by any other node.

Therefore, the result follows.

E. Asymptotic evaluation of $\frac{\tau_{\text{Bass}}^{1/2}}{\tau_{M=2}^{1/2,\text{hom}}}$

By Eq. (15),
$$T^{1/2} := T_{M=2}^{1/2,hom}$$
 is the solution of

$$e^{-pT^{1/2}}\frac{qe^{-pT^{1/2}}-pe^{-qT^{1/2}}}{q-p}=\frac{1}{2}.$$
(37)

Let $X := e^{-pT^{1/2}}$ and $\lambda := \frac{q}{p}$. Then, $e^{-qT^{1/2}} = X^{\frac{q}{p}} = X^{\lambda}$. Plugging this into Eq. (37) and noting that 0 < X < 1 and $\lambda > 0$ give

$$X^{2} - \frac{1}{2} = \frac{p}{q - p} \left(-X^{2} + X^{\lambda} \right) = O\left(\frac{1}{\lambda}\right), \quad \lambda \gg 1.$$

Therefore,

$$X^2 \sim rac{1}{2}, \quad \lambda \gg 1.$$

Hence, by the definition of *X*,

$$T_{M=2}^{l/2,\text{hom}} = \frac{1}{2p} \log(X^{-2}) \sim \frac{\log(2)}{2p}, \quad \lambda \gg 1.$$

Chaos **34**, 053101 (2024); doi: 10.1063/5.0191197 © Author(s) 2024 Finally, by Lemma 11 in Ref. 8,

$$T_{\mathrm{Bass}}^{1/2} = rac{\log\left(2+rac{q}{p}
ight)}{p+q} \sim rac{\log\left(rac{q}{p}
ight)}{q}, \quad \lambda \gg 1,$$

and so Eq. (21) follows.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Gadi Fibich: Conceptualization (equal); Formal analysis (equal). **Tomer Levin:** Conceptualization (equal); Formal analysis (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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