Beam self-focusing in the presence of a small normal time dispersion

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We present a system of *modulation equations* that approximate the focusing of the nonlinear Schrödinger equation in the presence of a small normal time dispersion (TDNLS). Since the modulation equations are much easier for analysis and for numerical simulations, they can be used to get a general picture of the TDNLS focusing. Analytical and numerical agreement between the modulation equations and the TDNLS is established.

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I. INTRODUCTION

The nonlinear Schrödinger equation with time dispersion (TDNLS)

$$i\psi_z+\Delta_\perp\psi-\epsilon\psi_{tt}+|\psi|^2\psi=0, \quad \psi(z=0,\mathbf{r},t)=\psi_0(\mathbf{r},t) \tag{1}$$

arises in the study of the propagation of ultrashort laser pulses in media with a Kerr nonlinearity. Here ψ is the rescaled amplitude of the electric field, z is the axial coordinate in the direction of the beam, $\Delta_\perp=\partial_{rr}+\partial_r/r$ is the Laplacian in the transverse $\mathbf{r}=(x,y)$ plane, ϵ is the time dispersion coefficient, and t is the time in a coordinate system moving with the group velocity. Initial conditions are given at $z=0$ for all x, y, and t. Thus z plays the role of time and t the role of a third spatial variable in this problem.

The time dispersion parameter ϵ depends on the optical properties of the medium. In the case of ultrashort laser-tissue interactions, where pulses in the visible regime propagate through aqueous media, its value is given by [1]

$$\epsilon=\left(\frac{a}{cT}\right)^2,$$

where a is the beam width, c is the speed of light, and T is the pulse duration. Hence ϵ is positive (*normal* time dispersion) and is proportional to the ratio of the radial pulse width to its axial length, indicating that time dispersion is still small for "cigarlike" pulses (i.e., long and narrow) but is dominant for "disklike" pulses.

When time dispersion is negligible each t cross section of the pulse [i.e., the plane $t=$const in (x,y,t) space] evolves independently according to the Schrödinger equation with a cubic nonlinearity (CNLS):

$$i\psi_z+\Delta_\perp\psi+|\psi|^2\psi=0. \tag{2}$$

In that case, CNLS theory predicts that sufficiently intense beams undergo self-focusing and blow up in a finite propagation distance.

The experimental evidence that ultrashort laser interactions depend on the pulse duration [2–4] is related to the increasing importance of time dispersion. Numerical simulations have shown that even a small amount of normal time dispersion in the TDNLS can have a substantial effect on the focusing and lead to the temporal splitting of the pulse into two peaks [5–8]. The peak splitting phenomenon has attracted attention because it delays the focusing and may provide a mechanism for its arrest. Although the onset of pulse splitting was explained based on a local analysis of self-similar solutions very near the point of peak intensity [6,8], this analysis is not valid past the onset of pulse splitting. In addition, numerical simulation of the TDNLS cannot continue very far after the peak splitting, at present. Thus the general question of whether or not normal time dispersion arrests collapse is still open.

In this paper we analyze the TDNLS focusing when time dispersion is small using an alternative system of modulation equations [9]. This provides a theoretical understanding of the focusing behavior well past the onset of pulse splitting. We start by reviewing the CNLS focusing (Sec. II). In Sec. III we derive the modulation equations by treating the TDNLS as a small perturbation of the CNLS. We establish the validity of the modulation equations by demonstrating a correspondence between their analytical properties and those of TDNLS (Sec. III C) and by extensive numerical computations (Sec. V). We analyze special solutions of the modulation equations in Sec. III D.

Vysloukh and Matveeva [10] have analyzed the effects of time dispersion on the propagation of planar waveguides and have shown that normal time dispersion suppresses the modulation instability of CNLS with a single transverse dimension and slows the self-focusing rate considerably. As a result, the pulse splitting into one-dimensional (1D) solitons is delayed. One cannot, however, extend this result to higher dimensions: Self-focusing is always balanced by radial dispersion in one dimension but not in higher dimensions, where it can result in wave collapse. Indeed, the sensitivity of self-focusing to small perturbations in two dimensions has to do with it being the critical dimension for blowup.

In several papers it is argued that the paraxial approximation used to derive nonlinear Schrödinger equations may be inappropriate when intense self-focusing occurs [11,12]. An important issue in understanding experimental results is whether time dispersion is the reason collapse is not observed or whether it is the breakdown of the paraxial approximation of the wave equation. The answer to this question depends on the initial pulse shape, since nonparaxiality arrests focusing when the beam width becomes comparable to a few wavelengths [13]. The beam width at which the effects of time dispersion become important depends also on the initial conditions. Other interesting phenomena have been observed in simulations of perturbed CNLS equations. While some of these perturbations have a direct physical origin (e.g., anomalous dispersion [14], others (e.g., saturable nonlinearity [15]) have a general, mathematical form.

II. REVIEW OF THE CNLS FOCUSING

Since we will derive the modulation equations by considering TDNLS as a perturbation of CNLS, we start by giving a brief review of the CNLS focusing. For more comprehensive presentations see [13,16,17,15]. We consider only radially symmetric solutions in this paper.

Two important invariants of the CNLS (2) are mass

$$N(\psi)=\int_0^\infty |\psi|^2\, r dr=\text{const}$$

and energy

$$H(\psi)=\int_0^\infty |\psi_r|^2\, r dr-\frac{1}{2}\int_0^\infty |\psi|^4\, r dr=\text{const}.$$

The variance identity

$$\frac{\partial^2 V}{\partial z^2}=8H,\quad V=\int_0^\infty r^2|\psi|^2\, r dr$$

can be used to show the existence of solutions that blow up, because when $H(\psi_0)<0$ there is a finite z for which $V=0$, if the solution existed up to that point.

The CNLS has waveguide solutions

$$\psi=e^{iz}R(r),$$

where $R(r)$ satisfies

$$\Delta_\perp R-R+R^3=0,\quad R'(0)=0,\quad R(\infty)=0. \tag{3}$$

The unique positive solution of this equation is monotonically decreasing and is called the *Townes soliton.* A necessary condition for blowup is that the initial mass be above a critical value

$$N\geq N_c,$$

which is equal to the mass of the Townes soliton

$$N_c=\int_0^\infty R^2\, r dr\approx 1.86.$$

The CNLS has a similarity transformation where a linear function $L(z)$ is used to rescale the solution as well as the independent variables:

$$\psi\to\frac{1}{L}\psi(\zeta,\xi)\exp\left(\frac{iLL_z\xi^2}{4}\right),\quad \xi=\frac{r}{L},$$

$$\zeta=\int_0^z L^{-2}(s)\, ds,\quad L=Z_c-z,\quad z<Z_c.$$

Based on this transformation and on the waveguide solution, one can construct an exact CNLS solution that blows up in a finite distance Z_c:

$$\psi=\frac{1}{Z_c-z}R\left(\frac{r}{Z_c-z}\right)\exp\left(-i\frac{r^2/4+1}{Z_c-z}\right).$$

This solution is unstable and has not been observed in numerical simulations because it has exactly the critical mass for blowup. However, it motivates looking for blowup solutions that have a quasi-self-similar structure in the vicinity of the singularity

$$\psi=\frac{1}{L}V(\zeta,\xi)\exp\left(i\zeta+\frac{iLL_z\xi^2}{4}\right),\quad \xi=\frac{r}{L},$$

$$\zeta=\int_0^z L^{-2}(s)\, ds.$$

Here V approaches the Townes soliton R and L is not necessarily linear in z. The equation for V is

$$iV_\zeta+\Delta_\perp V-V+|V|^2V+\tfrac{1}{4}\xi^2\beta V=0,$$

where

$$\beta=-L^3L_{zz} \tag{4}$$

is an important parameter in this problem.

As the pulse focuses, $\beta\searrow 0$ and V can be expanded in an asymptotic series of the form

$$V\sim V_0+V_1+\cdots, \tag{5}$$

where the leading-order term V_0 depends on ζ only through β and satisfies [16,17]

$$\Delta V_0-V_0+V_0^3+\tfrac{1}{4}\beta\xi^2V_0-i\nu(\beta)V_0=0,$$

$$\nu(\beta)\sim e^{-\pi/\sqrt{\beta}}. \tag{6}$$

Although $\nu(\beta)$ is exponentially small compared with the other terms, it has to be included in (6) to ensure the right behavior for $\xi\gg 1$, since in its absence the solution would have large oscillations $V_0\sim\xi^{-d/2}e^{i\sqrt{\beta}\xi^2}$.

When $\beta\ll 1$, it is related to the excess mass above critical of the focusing part of the solution [15]

$$\beta \sim \frac{N_{\rm sol}-N_{\rm cr}}{M}, \quad N_{\rm sol}=\int_0^\infty |V_0|^2 r dr,$$

$$M=\frac{1}{4}\int_0^\infty R^2 r^3 \, dr \approx 0.55. \qquad (7)$$

The rate of radial mass loss becomes very slow compared with the focusing rate:

$$\frac{\partial}{\partial z} N_{\rm sol} \cong -M\nu(\beta), \qquad (8)$$

which can be also written as

$$\beta_\zeta \sim -\nu(\beta). \qquad (9)$$

In [16,17,6], β is written in the form $\beta=a^2+a_\zeta$ so that $a=-L_\zeta/L$. If we then regard a_ζ as small compared to a^2, (9) becomes

$$a_\zeta \sim \frac{1}{2a} e^{-\pi/a}, \qquad (10)$$

which appeared in [16,17] without the exact exponential factor π. This equation can be solved asymptotically, leading to the log-log *law* [18,16,17]

$$L \sim \left(\frac{2\pi(Z_c-z)}{\ln\ln[1/(Z_c-z)]} \right)^{1/2}.$$

One can, however, consider (10) or better yet (9) directly and note that changes in β are exponentially slow compared with changes in L. This then leads to the *adiabatic law* [15] for the variation of the beam amplitude L^{-1}

$$L=\sqrt{2\beta^{1/2}(Z_c-z)}.$$

Asymptotic analysis and numerical simulations show that the adiabatic law, with β evaluated from (7) rather than from (9), is valid even in the early stages of self-focusing, while the log-log asymptotes are reached only at huge focusing factors [13].

III. MODULATION THEORY FOR THE TDNLS

Self-focusing in the CNLS depends on the initial mass and is also sensitive to perturbations of the power of the nonlinearity: If the power is less than 2 (subcritical case) the solution exists for all z, while if it is greater than 2 (supercritical case) the solution blows up in general. As a result, even a small time dispersion term can have an important effect on the TDNLS focusing. Focusing is much easier to understand in the case of anomalous time dispersion ($\epsilon<0$) because this case is supercritical for self-focusing as the dimension of the transverse variables is 3. It is the difference in signs between the Laplacian and the normal time dispersion that complicates the TDNLS analysis.

A. Derivation of the modulation equations

At the initial stages of the TDNLS focusing the effect of small time dispersion is negligible and each t cross section follows the adiabatic law

$$L=\sqrt{2\beta^{1/2}[Z_c(t)-z]}, \qquad (11)$$

where $Z_c(t)$ is the location of blowup. As a result, the relative size of the time dispersion term

$$\frac{\psi_{tt}}{\Delta_\perp \psi} \sim \epsilon L_t^2$$

increases, indicating that the TDNLS solution is eventually not a small perturbation of (11).

In addition, as the pulse is focusing according to (11), V approaches R and the nonlinearity and the Laplacian almost cancel each other. Hence time dispersive effects become important when $\epsilon\psi_{tt}$ becomes comparable to $(\Delta_\perp \psi + |\psi|^2\psi)$. Since in this regime $\epsilon\psi_{tt}$ is still a small term, for each t the solution is a small perturbation the CNLS and it is natural to look for it in the form

$$\psi(z,r,t)=\frac{1}{L(z,t)} V(\zeta,\xi,t) \exp\left(i\zeta + i\frac{L_z}{L}\frac{r^2}{4}\right),$$

$$\xi=\frac{r}{L(z,t)}, \quad \zeta=\int_0^z \frac{1}{L^2(s,t)}\, ds, \qquad (12)$$

where now the radial scale $L(z,t)$ depends also on t. We call this the generalized Talanov transformation. Using this in (1) we get the following equation for V:

$$iV_\zeta + \Delta_\perp V - V + |V|^2 V + \beta \frac{\xi^2}{4} V$$
$$-\epsilon L^3 \left[\frac{V}{L} \exp\left(i\zeta + i\frac{L_z}{L}\frac{r^2}{4}\right) \right]_{tt} \exp\left(-i\zeta - i\frac{L_z}{L}\frac{r^2}{4}\right) = 0. \qquad (13)$$

As in the stationary case, β is defined by (4) and we assume that the cross-sectional mass is slightly above critical

$$0<\beta \ll 1. \qquad (14)$$

The modulation equation can be derived from the mass balance between nearby t cross sections of the solution. From (1) we get the conservation relation

$$\frac{\partial}{\partial z}\int_0^\infty |\psi|^2 r \, dr = 2\epsilon \, \mathrm{Im} \int_0^\infty \psi^* \psi_{tt} r \, dr. \qquad (15)$$

For ψ of the form (12) the rate of the cross-sectional mass change is computed as in (7) and (8)

$$\frac{\partial}{\partial z}\int_0^\infty |\psi|^2 r \, dr \cong M[\beta_z + \nu(\beta)]. \qquad (16)$$

To estimate the right-hand side of (15) we use (12) and (14) to approximate the time dispersive term

$$\mathrm{Im}\left\{\left[\frac{V}{L}\exp\left(i\zeta+i\frac{L_z}{L}\frac{r^2}{4}\right)\right]_{tt}\exp\left(-i\zeta-i\frac{L_z}{L}\frac{r^2}{4}\right)\right\}$$

$$\sim\frac{R}{L}\zeta_{tt}+2\zeta_t\left(\frac{1}{L}\right)_t(R\xi)_\xi. \quad (17)$$

Using (16), (17), and the identity

$$\int_0^\infty (R\xi)_\xi R\ \xi d\xi=0 \quad (18)$$

the mass balance (15) reduces to

$$\beta_z+\nu(\beta)=\frac{2\epsilon N_c}{M}\zeta_{tt}. \quad (19)$$

The exponentially small mass radiation effect had to be retained in the analysis of CNLS focusing because it is the only mass-reducing mechanism. However, radial mass losses are now negligible compared with the temporal mass flux so the term $\nu(\beta)$ can be omitted in (19). Equations (4) and (19) and the ζ,L relation in (12) form a closed system, the *modulation equations*

$$\beta_z=\frac{2\epsilon N_c}{M}\zeta_{tt}, \quad (20)$$

$$L_{zz}=-\beta L^{-3}, \quad (21)$$

$$\zeta_z=L^{-2}. \quad (22)$$

The variables in the modulation system for the TDNLS focusing are the pulse width L, the excess mass above critical β, and the local axial phase ζ. When $\epsilon=0$ we recover the adiabatic law (11) and ζ has its maximum at the peak mass cross section. Hence normal time dispersion results in mass loss to the neighboring cross sections leading to the pulse splitting, while anomalous time dispersion ($\epsilon<0$) tends to enhance the focusing.

To leading order in β Eq. (20) can be written as a conservation law

$$\frac{\partial N_s}{\partial z}=\frac{\partial}{\partial t}[uN],$$

where

$$u=2\epsilon\zeta_t$$

is the velocity in the t direction. The modulation equation can also be derived from energy balance arguments and from the solvability condition for linearized CNLS operator about the Townes soliton (Appendix).

B. Linear stability analysis

To check the linear stability of uniform solutions of the modulation system we express it as a single equation in L:

$$-(L^3L_{zz})_{zz}=\frac{2N_c}{M}\left(\frac{1}{L^2}\right)_{tt}.$$

We look for solutions of the form $L=L_0+\delta L$, where $L_0\equiv$const and the perturbation δL is small compared to L_0. The linearized equation for δL is

$$(\delta L)_{zzzz}=\frac{4N_c}{M}\frac{1}{L_0^6}(\delta L)_{tt}.$$

Substituting $\delta L=\exp(ikt-i\omega z)$ we get the *dispersion relation*

$$\omega^4=-\frac{2C}{L_0^6}k^2,$$

which shows that the system is linearly unstable for all k. However, it should be remembered the $L_0\equiv$const corresponds to the waveguide solution

$$\psi=\frac{1}{L_0}e^{i\zeta}R\left(\frac{r}{L_0}\right),$$

which is unstable in the radial direction as well.

C. Lagrangian formulation

We can use the Lagrangian of the TDNLS

$$I=\int\mathscr{L}\,d\mathbf{x}\,dz,\quad \mathscr{L}=\mathrm{Im}(\psi\psi_z^*)-|\psi_r|^2+\epsilon|\psi_t|^2+\tfrac{1}{2}|\psi|^4,$$

$$\mathbf{x}=(x,y,t)$$

to derive a Lagrangian for the modulation equations, by following the same approach used in deriving the modulation equations, namely, using (12) and (14) and averaging in the radial direction (see the upper part of Fig. 1)

$$\bar{\mathscr{L}}=\int\mathscr{L}\,r\,dr=\frac{\epsilon N_c}{M}(\zeta_t)^2+\frac{(\zeta_{zz})^2}{4(\zeta_z)^3}.$$

The modulation Lagrangian $\bar{\mathscr{L}}$ can be written as a constrained one using only first derivatives:

$$\bar{\mathscr{L}}=M(L_z)^2+\epsilon N_c(\zeta_t)^2+M\beta\left(\frac{1}{L^2}-\zeta_z\right),$$

with $M\beta$ being the Lagrange multiplier.

Conservation laws

Based on Noether's theorem [19], we can use the transformation groups that leave the action integral

$$I=\int\left[\frac{\epsilon N_c}{M}(\zeta_t)^2+\frac{(\zeta_{zz})^2}{4(\zeta_z)^3}\right]dz\,dt \quad (23)$$

invariant to derive conservation laws for the modulation system. Invariance to phase (i.e., the identity), time, and space translations leads to mass, energy, and momentum conservation

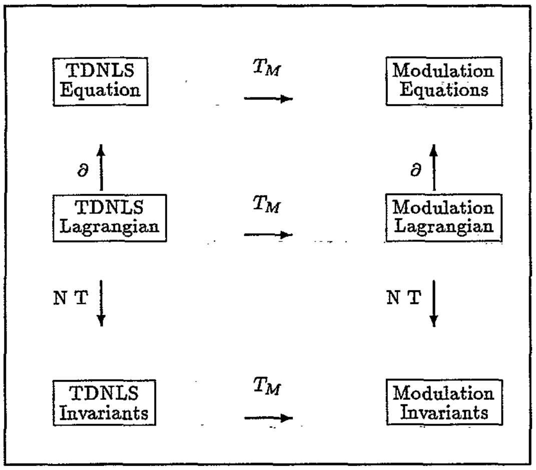

FIG. 1. Summary of the relations between the TDNLS (1) and the modulation equations (20)–(22). T_M is the transformation used to derive the modulation equations from the TDNLS, namely, changing from ψ to the modulation variables using (12) and (14). Since the modulation Lagrangian is derived by applying T_M to the TDNLS Lagrangian, the operators T_M and the variational derivative (∂) commute. Likewise, using the Noether theorem (NT) to derive a conservation law for the TDNLS based on a symmetry group of the TDNLS Lagrangian and applying T_M to this law results in the same conservation law for the modulation system as if we used the symmetry group for the modulation Lagrangian.

$$\int \beta\, dt=\text{const},$$

$$\int \left[-\tfrac{1}{2}(L^2)_{zz}+\frac{\epsilon N_c}{M}(\zeta_t)^2\right] dt=\text{const},$$

$$\int [\beta\zeta_t-2L_zL_t]\, dt=\text{const}.$$

These conservation laws could also be derived from the corresponding conservation laws for the TDNLS (Fig. 1).

A symmetry group for (23) that does not exist in the TDNLS is the dilation transformation $\zeta(z,t)\to\lambda\,\zeta(\lambda z,t/\lambda)$. The resulting conservation law is

$$\int_{t_1}^{t_2}\left[(L^2)_z+\beta\zeta+z\left(-\tfrac{1}{2}(L^2)_{zz}+\frac{\epsilon N_c}{M}(\zeta_t)^2\right)\right.$$
$$\left.-t(\beta\zeta_t-2L_zL_t)\right]dt=\text{const}+t\left[\left((L_z)^2-\frac{\epsilon N_c}{M}(\zeta_t)^2\right)\right]_{t_1}^{t_2}.$$

D. Special solutions of the modulation equations

Let us look for solutions of the modulation equations (20)–(22) under the assumptions that there is a singularity curve $Z_c(t)$ of the solution in the (z,t) plane and that in the neighborhood of this curve the solution depends only on distance from the curve. In this case solutions of the modulation equations have the form

$$L(z,t)=L(Z_c(t)-z),\quad \beta(z,t)=\beta(Z_c(t)-z),$$
$$\zeta(z,t)=\zeta(Z_c(t)-z)\ . \tag{24}$$

Therefore

$$\zeta_{tt}=-\ddot{Z}_c\zeta_z+\dot{Z}_c{}^2\zeta_{zz}, \tag{25}$$

where the overdot stands for differentiation with respect to time. Equation (20) reduces to

$$\beta_z=\gamma(-\ddot{Z}_c\zeta_z+\dot{Z}_c{}^2\zeta_{zz}),\quad \gamma=\frac{2N_c}{M}\epsilon.$$

Integrating this last equation gives

$$\beta=\beta_0+\gamma(-\ddot{Z}_c\zeta+\dot{Z}_c{}^2\zeta_z), \tag{26}$$

where

$$\beta_0(t)=\beta(0,t)-\gamma\dot{Z}_c{}^2\frac{1}{[L_0(t)]^2}$$

and $L_0(t)=L(0,t)$ is the width of the beam at $z=0$. It is convenient to make a change of variable to the reciprocal of the radial width of the beam

$$A=\frac{1}{L}. \tag{27}$$

Then $\beta=A_{\zeta\zeta}/A$ and (26) can be rewritten in the form

$$A_{\zeta\zeta}=(\beta_0-\gamma\ddot{Z}_c\zeta)A+\gamma\dot{Z}_c{}^2A^3. \tag{28}$$

Introduce also a new independent variable by

$$s=\frac{\beta_0}{(\gamma\ddot{Z}_c)^{2/3}}-(\gamma\ddot{Z}_c)^{1/3}\zeta.$$

Then Eq. (28) becomes

$$A_{ss}=sA+2\eta^2A^3, \tag{29}$$

where

$$\eta^2=\frac{\gamma^{1/3}\dot{Z}_c{}^2}{2\ddot{Z}_c{}^{2/3}}. \tag{30}$$

The initial condition at $z=0$ or $\zeta=0$ is now at

$$s_0=\frac{\beta_0}{(\gamma\ddot{Z}_c)^{2/3}}$$

and we note that for sufficiently small time dispersion

$$s_0\sim\beta_0\epsilon^{-2/3}\gg1.$$

The focusing of each t cross section is described by Eq. (29), which defines the second Painlevé transcendent function [20]. When analyzing blowup in (29) we have to distinguish between three cases, depending on the relative size of the terms in (29).

At the time t_0 where $Z_c(t)$ attains its minimum we have

$$\dot{Z}_c(t_0)=0, \quad \ddot{Z}_c(t_0)>0. \tag{31}$$

At this t cross section (29) reduces to the Airy equation $A_{ss}=sA$ and its solution is

$$A=\frac{A_0}{\mathrm{Ai}(s_0)}\mathrm{Ai}(s), \quad A_0=\frac{1}{L_0}. \tag{32}$$

The asymptotic form of the Airy function is [21]

$$\mathrm{Ai}(s)\sim\frac{\sqrt{\pi}}{2}s^{-1/4}\exp(-\tfrac{2}{3}s^{3/2}), \quad s\gg 1. \tag{33}$$

Using this in (32), we find that for $0\leq\zeta\ll\beta_0/\epsilon$

$$A\sim A_0 e^{\sqrt{\beta_0}\zeta}. \tag{34}$$

This shows that during the initial stage of self-focusing the solution agrees with that of the one in the dispersion-free case, in which $\epsilon=0$ in (28):

$$A_{\zeta\zeta}=\beta_0 A.$$

To express this dispersion-free solution in terms of the original variables we use (12), (27), and (34) to get

$$(L^2)_z=(L^2)_\zeta\zeta_z=-2\beta_0,$$

from which the adiabatic law of critical collapse [13,15] follows:

$$L=\sqrt{2\beta_0^{1/2}(Z_c-z)}. \tag{35}$$

The effects of time dispersion become important for $\zeta\gg\beta_0/\epsilon$. In particular, since the Airy function attains its maximum around $s_{\max}\cong-1.02$ and decreases for $s<s_{\max}$, [21] the collapse at t_0 gets arrested at

$$z_{\max}=\int_{s_0}^{s_{\max}} z_\zeta\zeta_s\,ds=\frac{1}{(\gamma\ddot{Z}_c)^{1/3}}\int_{s_{\max}}^{s_0}\frac{1}{A^2}\,ds.$$

Using (33) we find that for $s_0\gg 1$

$$\int_{s_{\max}}^{s_0}\frac{1}{A^2}\,ds=L_0^2\int_{s_{\max}}^{s_0}\frac{A^2i(s_0)}{A^2i(s)}\,ds\sim\frac{L_0^2}{2s_0^{1/2}}$$

and hence the collapse is arrested at

$$z_{\max}\cong\frac{L_0^2}{2\beta_0^{1/2}},$$

which is the location Z_c of blowup of the dispersion-free solution, as we can see by setting $z=0$ in (35) [13]. The maximum amplification factor at t_0 can also be estimated:

$$\frac{A(z_{\max})}{A(0)}=\frac{\mathrm{Ai}(s_{\max})}{\mathrm{Ai}(s_0)}\cong\frac{0.54}{\mathrm{Ai}(s_0)}$$

$$\sim\frac{1.08}{\sqrt{\pi}}\frac{\beta_0^{1/4}M^{1/6}}{[2N_c\ddot{Z}_c(t_0)\epsilon]^{1/6}}e^{\beta_0^{3/2}M/2N_c\ddot{Z}_c(t_0)\epsilon}.$$

In addition, from (26) the excess power at t_0 is

$$\beta=\beta_0-\gamma\ddot{Z}_c\zeta \tag{36}$$

and it goes below critical at $s=0$, prior to the arrest of the collapse.

For t cross sections that are not in the neighborhood of t_0 the first term on the right-hand side of (29) becomes negligible. This leads to solutions of the form

$$A\sim\frac{1}{\eta}\frac{1}{s-s_c(t)},$$

and since

$$z_s=z_\zeta\zeta_s=-(s-s_c)^2\frac{\eta^2}{(\gamma\ddot{Z}_c)^{1/3}}$$

we get by integrating this equation solutions with a *one-third power law* for self-focusing collapse

$$L\sim[a(Z_c-z)]^{1/3}, \quad a=3\left(\frac{N_c}{M}\right)^{1/2}|\dot{Z}_c|\epsilon^{1/2}. \tag{37}$$

In order to estimate the size of the neighborhood of t_0 where collapse is arrested a more careful analysis is required. Let

$$B=\eta A$$

so that

$$B_{ss}=sB+2B^3. \tag{38}$$

The behavior of the solutions of (38) is characterized by the following result [22,23]. Any solution of (38) satisfying

$$\lim_{s\to\infty}B(s)=0$$

is asymptotic to $k\mathrm{Ai}(s)$ for some k. If $|k|<1$, then as $s\to-\infty$

$$B(s)=O(|s|^{-1/4}),$$

and if $|k|>1$, $B(s)$ has a pole at a finite s_c, depending on k.

To apply this result we express k in terms of the parameters of the problem and note that A should agree with (34) in the domain $\zeta\ll\beta_0/\epsilon$, when it is given by (32). Thus

$$B\sim k\mathrm{Ai}(s), \quad k=\frac{\eta A_0}{\mathrm{Ai}(s_0)}, \quad s\sim s_0$$

and the solution does not blow up only if

$$|\eta|<L_0Ai(s_0). \tag{39}$$

From (30) and (31) we see that near t_0

$$\eta\sim 2^{-1/2}\gamma^{1/6}\ddot{Z}_c^{2/3}(t-t_0)$$

or, using (39),

$$|t-t_0|\leq L_0\sqrt{\frac{\pi}{2}}[\ddot{Z}_c(t_0)]^{-1/6}\beta_0^{-1/4}e^{-M\beta_0^{3/2}/3N_c\ddot{Z}_c\epsilon},$$

which is an exponentially small in ϵ neighborhood of t_0.

The above analysis of the modulation equations suggests the following picture of self-focusing with small, normal time dispersion. Solutions of the form (24) blowup for nearly all t cross sections following the one-third power law (37). However, collapse is arrested in an exponentially small temporal neighborhood of the cross section t_0 for which the initial focusing is fastest. Power will move away from the t_0 cross section to the nearby cross sections and the initial peak at t_0 will split into two peaks that will continue to focus.

There are, however, at least two problems with this picture.

(i) The arrest of collapse at t_0 becomes inconsistent with $Z_c(t_0)$ being the earliest z for which collapse takes place.

(ii) The *one-third power law* is not really valid for the TDNLS focusing since it implies that the corresponding β is

$$\beta = -L^3 L_{zz} \sim (Z_c - z)^{-2/3} \to \infty,$$

which blows up, violating the basic assumption in the derivation of the modulation equations requiring that β be small.

Regarding the first problem, the initial stage of the self-focusing is described by (11) with $Z_c(t)$ the singularity curve in the absence of time dispersion and in the above analysis we assume that $Z_c(t)$ in (24) is this singularity curve, since time dispersion is small. With this interpretation, t_0 is the cross section of fastest initial self-focusing. Time dispersive effects, however, make the power go below critical at t_0, followed by temporal peak splitting, arrest of the collapse at t_0, and departure from the form (24) of solutions that is based on the dispersion free singularity curve $Z_c(t)$. Away from t_0 where collapse is arrested there may be a different singularity curve for solutions of the modulation equations with the one-third power law. So the first problem is due to the way $Z_c(t)$ is defined.

The second problem indicates that solutions that follow the one-third power law ultimately violate the assumptions for the validity of the modulation equations. In that case, another theory for the advanced stages of the self-focusing is needed. It is not clear that there are initial conditions for which the solution will follow the one-third power law, unless β is very small for all time cross sections of the pulse. In our numerical simulations of both TDNLS and the modulation equations we did not observe the one-third power law.

IV. THE NUMERICAL SCHEME

We have carried out extensive numerical simulations that compare solutions of the full TDNLS to those constructed with the modulation equations. In this section we outline briefly the numerical method used in the simulations. For more details, see [24].

A. TDNLS

The TDNLS (1) is solved by a split-step method, using a uniform (in t) *dynamic rescaling* in the radial direction. More specifically, under the rescaling transformation

$$\psi(z,r,t) = \frac{1}{\bar{L}(z)} u(\bar{\zeta},\xi,t), \quad \xi = \frac{r}{\bar{L}(z)},$$

$$\bar{\zeta}(z) = \int_0^z \frac{1}{\bar{L}^2(s)}\, ds,$$

$u(\bar{\zeta},\xi,t)$ satisfies

$$u_{\bar{\zeta}} = i\Delta_\perp u - i\epsilon \bar{L}^2 u_{tt} + \bar{a}(u\xi)_\xi + i|u|^2 u,$$

where

$$\bar{a}(\bar{\zeta}) = \bar{L}\frac{d\bar{L}}{dz} = \frac{1}{\bar{L}}\frac{d\bar{L}}{d\bar{\zeta}}$$

and the overbar indicates that $\bar{a}$, $\bar{L}$, and $\bar{\zeta}$ are independent of t.

The split-step method has two stages.

(i) Solve for each t cross section ($t=$const)

$$u_{\bar{\zeta}} = i\Delta_\perp u + a(u\xi)_\xi + i|u|^2 u$$

by combining a Crank-Nicholson implicit method on the Laplacian term and Adams-Bashford extrapolation on the others. The main modification to the method used in [25] for the CNLS focusing is in the way that a is chosen:

$$a(u) = \frac{1}{G_0}\int dt \int_0^\infty |u|^2 \mathrm{Im}(u\Delta_\perp u^*)\xi\, d\xi,$$

where

$$G_0 = G(u_0), \quad G(u) = \int dt \int_0^\infty |u_\xi|^2 \xi\, d\xi.$$

The global smoothness of u is maintained since $G(u) \equiv G_0$. However, since $\bar{L}$ is averaged over all t cross sections it cannot follow the fastest collapse once the temporal variations increase, which eventually causes the simulation to break down.

(ii) Solve for each r cross section ($r=$const)

$$u_{\bar{\zeta}} = -i\bar{L}^2 u_{tt}$$

using an explicit Crank-Nicholson method.

As a consistency check, we monitor the conservation of

$$M := \int dt \int_0^\infty |u|^2\, \xi d\xi, \quad G := \int dt \int_0^\infty |u_\xi|^2 \xi\, d\xi$$

and verify the convergence of the radial profile of $|u|$ to a Townes soliton.

B. Recovering the modulation variables

Recovering the modulation variables from the TDNLS simulation results is done using

$$\zeta = \arg u(r=0), \quad \beta = \frac{N_{\mathrm{sol}} - N_c}{M}, \quad L = \left(\frac{d\zeta}{d\bar{\zeta}}\right)^{-2} \bar{L}.$$

These relations are only approximate since they are based on the asymptotic form of the focusing CNLS solution. In addition, the relation for β is only $O(\beta)$ accurate. Additional inaccuracy is caused by the numerical differentiation and by the truncation of the integral when evaluating N_{sol}.

C. Modulation equations

We solve the modulation equations (20)–(22) using a second-order line method with an adjustable "time" step

$$dz = dz_0^{\min} + L^2.$$

D. The one-third law

Since with the current code we cannot integrate very far after the peak splitting, we cannot check directly whether the solution follows the one-third law (37). However, we can detect a power-law behavior by noting that if $L \sim (Z_c - z)^m$ then $\beta \sim (Z_c - z)^{4m-2}$ and

$$L^n \beta \sim \text{const}, \quad n = \frac{2}{m} - 4 \ .$$

The value of n is estimated numerically by fitting $n \ln L + \ln \beta \sim \text{const}$.

E. Numerical comparison of the TDNLS and the modulation equations

The numerical agreement between the TDNLS and the modulation equation for the case of periodic boundary conditions

$$\psi(\cdot, t=0) = \psi(\cdot, t=1)$$

(and similarly for β, L, and ζ) was verified by comparing the modulation variables that were recovered from the TDNLS simulation with the solution of the modulation equations. The initial condition for the modulation system was the recovered value of the modulation variables at z_0 and the modulation variables were compared with those recovered from TDNLS for various $z > z_0$.

Since modulation theory is only $O(\beta)$ accurate, comparing for the same numerical value of z translates into an $O(\beta)$ error in the actual value of z, leading to an $O(\beta/L^2)$ error in the modulation variables. To overcome this difficulty we use the time averaged "distance"

$$\tilde{\zeta} = \int \zeta dt$$

instead of z as the basis for the comparison [since $z(\tilde{\zeta})$ is a monotonically increasing function, this is the same as comparing for the same z].

Resolving the increasing t gradients is limited by the grid resolution in the t direction. While this resolution can be easily refined for the modulation equations, it requires more memory and slows the computations considerably for the TDNLS (which again demonstrate the advantage of the modulation equations).

V. NUMERICAL RESULTS

Various initial conditions were used for the comparison, showing good qualitative agreement between the TDNLS and the modulation equation. However, in order to demonstrate a good quantitative agreement, the initial condition should be chosen in such a way that the error in the recovered value of the "modulation variables" (Sec. IV B) is much smaller than their slow temporal variation. It should be emphasized that this does not mean that the theory is valid only for specially constructed initial conditions, but rather reflects the difficulty of recovering the modulation variables from the TDNLS simulation with sufficient accuracy.

We have integrated the TDNLS with the initial condition

$$\psi_0(r,t) = b(t) R(r) e^{-i\sqrt{(b^2-1)N_c/M}r^2/4},$$

$$b(t) = 1.03 + 0.01 \sin(2\pi t), \tag{40}$$

with $\epsilon = 0.01$ and where R is the Townes soliton (3). The comparison results with modulation theory are shown in Figs. 2 and 3, with the comparison starting at $z_0 = 0.667$ and $z_0 = 0.886$, respectively. The modulation equations capture the temporal distribution before and after the pulse splitting. Quantitative agreement starts to deteriorate as $|\beta|$ increases. The agreement is better in Fig. 3, since the initial condition for the modulation variables is more accurate, since it is recovered at a later stage.

Results for the initial condition

$$\psi_0(r,t) = 3e^{-r^2}[1 + 0.001 \sin(2\pi t)], \tag{41}$$

which has 20% mass above critical with $\epsilon = 1$, are presented in Fig. 4. Although the initial t modulation is very small, large temporal gradients are observed as the pulse is focusing. During the initial nonadiabatic stage $\int_0^1 \beta dt$ is decreasing because the large excess mass above critical is removed by radiation. No peak splitting is observed in $|\psi|$ for a focusing factor of over 2000, although there is one in β. The "flip" of the minimum and maximum of β and L is typical for the nonadiabatic stage of the focusing and was also observed in other simulations. Note that most of the focusing occurs over a very short distance z while this domain is automatically stretched by the dynamical rescaling variable ζ, increasing z resolution.

VI. DISCUSSION

A. Comparison with previous studies

Zharova *et al.* [8] predicted the *peak splitting* phenomenon, using arguments based on asymptotes and on numerical simulations. They went on to suggest that the new peaks would continue to split, resulting in a fractal collapse. Peak splitting was later observed in numerical simulations by Rothenberg [7] and by Chernev and Petrov [5].

Luther *et al.* [6] have considered solutions of the TDNLS of the form

$$\psi(z,r,t) = \psi(Z_c(t) - z, r),$$

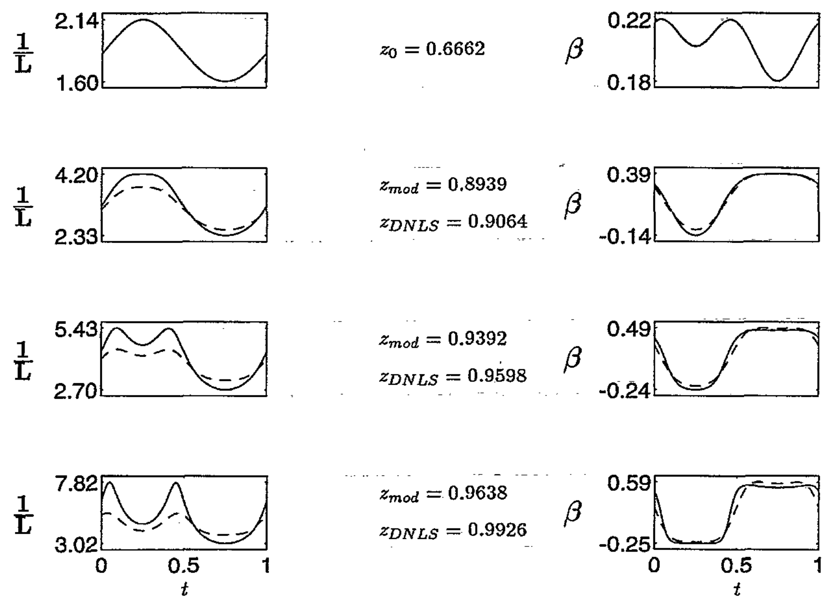

FIG. 2. Comparison of the TDNLS (dashed line) and the modulation equations (solid line). The initial condition for the TDNLS is (40) with $\epsilon=0.01$. The comparison was started at $z_0=0.6662$ when the peak splitting in β had already formed. The peak splitting in the field amplitude $(1/L)$ follows later. For each line $z_{\rm mod}$ and $z_{\rm TDNLS}$ are the recovered values of z from the modulation equations and from the TDNLS simulation, respectively.

where $Z_c(t)$ is the singularity curve of the stationary CNLS. They showed that the evolution of the t_0 cross section of the peak mass [i.e., where $Z_c(t)$ attains its minimum] is described by

$$\beta_\zeta=-\nu(\beta)-\ddot{Z}_c(t_0)\left(\frac{2N_c}{M}-2(a^2+\beta)\right). \tag{42}$$

Using phase plane analysis of (42) and

$$a_\zeta=a^2-\beta, \quad L_\zeta=aL \tag{43}$$

they showed that a becomes negative, leading to an arrest of the focusing at t_0. They also demonstrated a numerical agreement between the TDNLS solution at t_0 and (42) and (43) in the regime where the pulse peak intensity has increased by a factor of 2 up to the *peak splitting* and then decreased by a factor of 25%. In this comparison $\ddot{Z}_c(t_0)$ was calculated separately from CNLS simulations with the same initial condition.

The results of Luther *et al.*. fall within the framework of modulation theory.

(i) Equation (42) corresponds to Eq. (36), which was derived from the modulation equation (20) under the same assumption of the special 2D form (24). The terms in (42) that are missing in (36) were neglected in the derivation of (20), since they are of lower order.

(ii) The arrest of the focusing at t_0 was derived from modulation theory in Sec. III D.

(iii) Both Eqs. (42) and (43) and the modulation equations are in numerical agreement with the peak splitting of the TDNLS (Figs. 2 and 3).

The main difference between the approach of Luther *et al.* and ours is that they neglected the second term on the right-

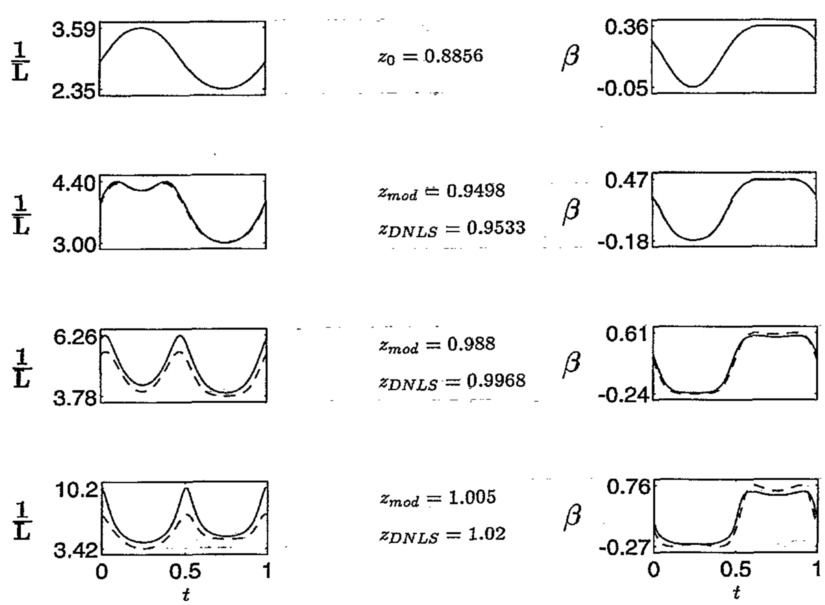

FIG. 3. Comparison of the TDNLS (dashed line) and the modulation equations (solid line) for the same conditions as in Fig 2. The comparison was started at a later "time" $z_0=0.8856$, at which the error in recovering the initial values of the "modulation variables" is smaller. As a result, the agreement is much better than in Fig. 2.

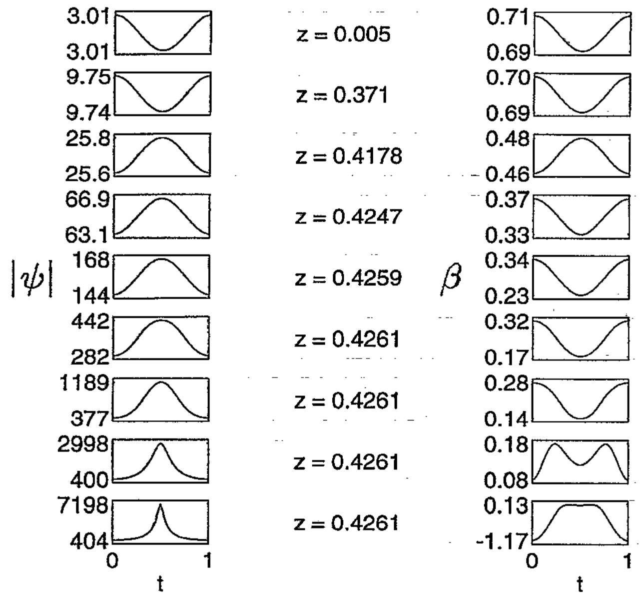

FIG. 4. Evolution in z of $|\psi(t,r=0)|$ and $\beta(t)$ for the initial condition (41). β is a measure of the excess cross-sectional mass above critical and is evaluated using (7).

hand side of (25) while we retained both terms. This allowed us to show that this term in (25) can be neglected only in a region around t_0, which is exponentially small in the time dispersion parameter ϵ. Therefore, while the modulation equations are valid for all t cross sections, Eqs. (42) and (43) are valid only in an exponentially small region around t_0. In particular, the arrest of the collapse occurs only in a very small temporal section of the pulse. Moreover, the modulation equations do not depend on the unknown value $\ddot{Z}_c(t_0)$ and continue to be valid for some distance after the onset of peak splitting when the solution departs from the 2D form (24).

B. The emerging picture of focusing in the TDNLS

The main stages of the focusing in the presence of small normal time dispersion are the following.

Nonadiabatic 2D focusing. Initially, time dispersion is negligible and each t cross section undergoes a 2D nonadiabatic collapse, during which the focusing solution at each cross section sheds by radiation most of its excess cross-sectional mass above critical while approaching a Townes profile.

Adiabatic 2D focusing. Each cross section continues to undergo a 2D self-similar collapse according to the adiabatic law [Eq. (11)], where the fastest collapse is at the cross section with the peak cross-sectional mass.

3D modulation focusing. As the higher temporal gradients become comparable to the balance of the Laplacian with the nonlinearity, temporal mass flux becomes important and the dynamics becomes three dimensional [i.e., (x,y,t)].

Modulation theory covers the adiabatic 2D focusing stage and the 3D modulation focusing stage. It is still an open question whether at a certain point $\epsilon\psi_{tt}$ becomes comparable to the other terms or β becomes large, so that the validity of modulation theory breaks down. Although the time dispersion term is increasing, this does not necessarily mean that modulation theory breaks down, since the Laplacian and the nonlinearity terms have also increased in the meantime. In our simulations with periodic boundary conditions and in [7] for short pulses, β does increase at some cross sections. It is not clear, however, whether this increase is large enough so as to invalidate the modulation equations.

C. Why peak splitting, why only one?

Peak splitting received a lot of attention in TDNLS research because it is the most conspicuous phenomenon that is observed in numerical simulations and also because it may lead to the arrest of collapse. We have seen in Sec. III D that peak splitting is related to the departure of the solution from the self-similar 2D structure of the focusing CNLS. Since peak splitting occurs in the transition between the adiabatic 2D focusing stage and the 3D modulation focusing stage, the new peaks are unlikely to split again, since by now the dynamics is fully three dimensional.

Numerical simulations of both the TDNLS and the modulation equation support this explanation for peak splitting. In particular, they predict correctly that peak splitting in β would occur before peak splitting in L (Fig. 3 and [7]) and explains why the splitting of new peaks has not been observed. It also explains why a solution with large initial mass may focus without peak splitting (Fig. 4): Since the 2D nonadiabatic focusing stage is longer, the temporal gradients will become large by the time the solution approaches a Townes profile, thus skipping the 2D adiabatic focusing stage.

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APPENDIX: OTHER DERIVATIONS OF THE MODULATION EQUATIONS

1. Energy balance

Based on (1), we can write an equation for the energy balance between the t cross sections

$$\frac{\partial}{\partial z}H+2\epsilon\mathrm{Re}\int \psi_z^*\psi_{tt}\, r\, dr=0,$$

$$H=\int |\psi_r|^2 r\, dr-\frac{1}{2}\int |\psi|^4 r\, dr. \qquad \text{(A1)}$$

Since in the modulation theory ansatz

$$H\cong M(L_z^2-\beta/L^2)=\frac{M}{2}(L^2)_{zz} \qquad \text{(A2)}$$

and

$$2\epsilon\mathrm{Re}\int \psi_z^*\psi_{tt}\, r\, dr\cong \epsilon N_c\frac{\zeta_{tt}}{L^2},$$

(A1) reduces to

$$(L^2)_{zzz} = -\frac{4\epsilon N_c}{M}\frac{\zeta_{tt}}{L^2}.$$

This is equivalent to (20), since

$$L^2(L^2)_{zzz} = -2\beta_z. \tag{A3}$$

2. Solvability condition

Motivated by the original derivation of the log-log *law* [16,17], Eq. (19) can also be derived from the equation for the second-order term V_1[(13), (5), and (6)]:

$$\Delta_\perp V_1 - V_1 + 2|V_0|^2 V_1 + V_0^2 V_1^* + \beta\frac{\xi^2}{4}V_1 - i\nu(\beta)V_1$$

$$= -i\left[\frac{\partial V_0}{\partial \beta}\beta_\zeta + \nu(\beta)V_0\right]$$

$$+ \epsilon L^3(\zeta)\left[\frac{V_0}{L}\exp\left(i\zeta + i\frac{L_z}{L}\frac{r^2}{4}\right)\right]_{tt}$$

$$\times\exp\left(-i\zeta - i\frac{L_z}{L}\frac{r^2}{4}\right). \tag{A4}$$

While the equation for the real part of V_1 is always solvable [24], the solvability condition for the imaginary part of V_1 is that R is perpendicular to the right-hand side of (A4):

$$\int_0^\infty R\left(\frac{\partial V_0}{\partial \beta}\beta_\zeta + \nu(\beta)V_0 - \epsilon\, L^3 \mathrm{Im}\left\{\left[\frac{V_0}{L}\exp\left(i\zeta + i\frac{L_z}{L}\frac{r^2}{4}\right]_{tt}\right.\right.$$

$$\left.\left.\times\exp - i\zeta - i\frac{L_z}{L}\frac{r^2}{4}\right)\right\}\right)\xi d\xi = 0. \tag{A5}$$

From (17) and (18) we get

$$L^3\int R\ \mathrm{Im}\left\{\left[\frac{V_0}{L}\exp\left(i\zeta + i\frac{L_z}{L}\frac{r^2}{4}\right)\right]_{tt}\right.$$

$$\left.\times\exp\left(-i\zeta - i\frac{L_z}{L}\frac{r^2}{4}\right)\right\}\xi\, d\xi \sim \zeta_{tt}L^2\int R^2\xi\, d\xi.$$

Using the relations

$$\beta_\zeta = L^2\beta_z, \quad \int\frac{\partial V_0}{\partial \beta}R\xi d\xi = \frac{M}{2},$$

the solvability condition (A5) reduces to (19).

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