Chaos ARTICLE pubs.aip.org/aip/cha

Funnel theorems for spreading on networks

Cite as: Chaos 35, 073109 (2025); doi: 10.1063/5.0269855
Submitted: 8 March 2025 · Accepted: 22 June 2025 ·
Published Online: 7 July 2025

Gadi Fibich,[a] Tomer Levin,[b] and Steven Schochet[c]

AFFILIATIONS

Department of Applied Mathematics, Tel Aviv University, Tel Aviv, Israel

[a]**Author to whom correspondence should be addressed:** fibich@tau.ac.il
[b]levintmr@gmail.com
[c]schochet@tau.ac.il

ABSTRACT

We derive the funnel theorems for the Bass and susceptible–infected models on networks that describe the spreading of innovations and epidemics. Let j be a node and divide the remaining nodes into $L \geq 2$ disjoint sets $\{A_l\}_{l=1}^{L}$. The funnel theorems provide lower and upper bounds for the difference between the susceptibility probability of j and the product of its susceptibility probability on the L modified networks in which j can only be influenced by incoming edges from A_l. In particular, one can let L be equal to the indegree of j, so that in the modified networks, j is only influenced by one incoming edge. We illustrate how the funnel theorems can be used to obtain exact explicit expressions for the adoption/infection probabilities of nodes and for the expected adoption/infection level in various types of networks, both with and without cycles.

Spreading of epidemics or innovations is stochastic processes on social networks. In this paper, we introduce the "funnel theorems," which relate the susceptibility probability of a node of indegree L to the product of its susceptibility probabilities on L modified networks, in which the node under consideration is only influenced by one incoming edge at a time. We then illustrate how the "funnel theorems" can be used to analyze the spreading dynamics on various network structures, both with and without cycles.

I. INTRODUCTION

Mathematical models for the spreading of epidemics have been around for a century.[1] For example, in the Susceptible–Infected (SI) model, the epidemics starts from a few infected individuals and progresses as infected individuals transmit it to susceptible ones. Mathematical models for the spreading of innovations is a younger problem—the first model was introduced in 1969 by Bass.[2] In this model, individuals adopt a new product because of external influences by mass media and internal influences by individuals who have already adopted the product (peer effect, word of mouth).

For many years, the spreading of epidemics and innovations were only analyzed using compartmental models, which are typically given by one or several deterministic ordinary differential equations. Such models implicitly assume that all the individuals within the population are equally likely to influence each other, i.e., that the underlying social network is a homogeneous complete graph. In more recent years, research on the spreading of epidemics and innovations has gradually shifted to network models, in which the adoption/infection event by each individual is stochastic.[3,4] These network models allow for heterogeneity among individuals and for implementing a network structure where individuals can only be influenced by their peers.

The stochastic spreading of new products and epidemics on networks has been extensively studied using numerical simulations, see e.g., Refs. 3–9. These simulations are relatively straightforward to implement, even for spreading models that are considerably more complex than the unidirectional Bass or SI models. Because these models are stochastic, each simulation run yields a different outcome. Consequently, numerical studies typically report results averaged over a sufficiently large number of simulations to ensure statistical reliability.

The analysis of stochastic spreading of new products and epidemics on networks is more challenging. Existing research in this area has primarily focused on two key quantities: the probability of adoption/infection at individual nodes and the expected adoption/infection level across the network. The starting point of the analysis is usually the master (Kolmogorov) equations, which are $2^M - 1$ coupled deterministic linear ODEs for the susceptibility probabilities of all the subsets of the nodes, where M is the number of network nodes. To be able to solve this exponentially large system explicitly, one needs to reduce the number of ODEs significantly. The common approach to do that has been to employ some approximation (mean-field, pairwise model,[10,11] etc.). More often than not, there has been no rigorous error bound for the accuracy of the approximation.

The goal of this study is to develop a novel analytic tool for obtaining explicit solutions of the master equations that are exact. At present, there are three analytic approaches for solving the master equations explicitly, without making any approximation. The first is based on *utilizing symmetries* of the network to reduce the number of master equations without making any approximation. This approach was applied to homogeneous and inhomogeneous complete networks and circles.[12–15] The second approach is based on the *indifference principle*.[16] This analytic tool simplifies the explicit calculation by replacing the original network with a simpler one. The indifference principle has been used to compute the susceptibility probabilities of nodes on bounded and unbounded lines and on percolation lines.[16,17] The third approach is to identify networks on which there is an exact closure at the level of triplets, such as undirected graphs with no cycles,[18] and infinite configuration model networks with Poisson-type distributions.[19]

In this paper, we introduce a new approach, which is based on the *funnel theorems*. Choose some node j and partition the remaining $M-1$ nodes into L disjoint subsets of nodes, denoted by $\{A_l\}_{l=1}^{L}$. The funnel theorems provide the sign and magnitude of the difference between the susceptibility probability of j in the original network and the product of its susceptibility probabilities in L modified networks in which j can only be influenced by edges arriving from A_l, where $l = 1, \ldots, L$ (see Fig. 1 for an illustration). In general, *the susceptibility probabilities of j in the modified networks are easier to compute since the indegree of j is lower than in the original network*. For example, application of the funnel theorem to interior nodes of undirected lines reduces this problem to that of nodes on directed lines, which is an easier task. In particular, one can let $L = \text{indegree}(j)$, so that from each set, A_l emerges exactly one incoming edge to j. Since $\text{indegree}(j) = 1$ in the modified networks, the susceptibility probability of j can be related to that of the tail node of the incoming edge (see, e.g., see Theorem 4.1 in Ref. 20).

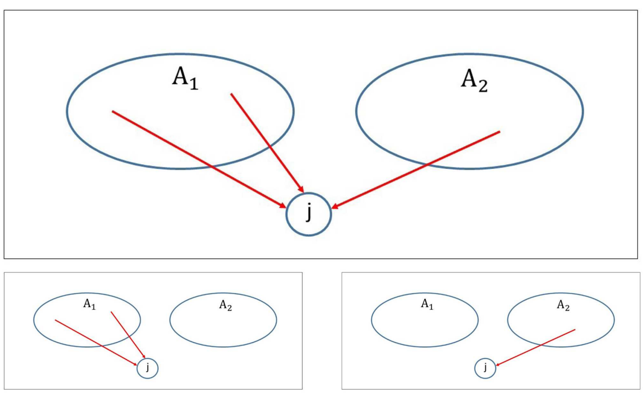

FIG. 1. The funnel theorems (for the case of $L = 2$ disjoint sets of nodes) relate the susceptibility probability of a node j that can be influenced by incoming edges from the sets A_1 and A_2 (top row), to the product of its susceptibility probabilities in the modified networks $\mathcal{N}^{A_1,p_j,I_j^0}$ and $\mathcal{N}^{A_2,p_j,I_j^0}$, where j can be influenced by incoming edges only from A_1 or only from A_2, respectively (bottom row).

The funnel relation is an equality if j is a vertex cut or, more generally, if j is a *funnel node*. This is the case, e.g., for any node on an undirected network that does not lie on a cycle. When j is not a funnel node, however, the funnel relation is a strict inequality. This is the case, e.g., for any node in an undirected network that lies on a cycle. For such situations, the funnel theorems provide lower and upper bounds for the difference between the susceptibility probability of j in the original network and the product of its susceptibility probability on the L modified networks.

The unknowns in the master equations are the susceptibility probability of sets of nodes. An alternative approach for analyzing the spreading dynamics is the edge-based compartmental model (EBCM) of Voltz[21] and Miller *et al.*,[22] in which the unknowns are the fractions of edges with nodes at certain states. In general, this method yields approximate solutions. In the infinite-population limit on configuration models, however, it was rigorously proved by Decreusefond *et al.*[23] and Jacobsen *et al.*[24] that this method yields exact solutions.

This paper is organized as follows. Section II presents a unified model for the Bass and SI models on networks. Section III is a theoretical review of the master equations and the indifference principle. Section IV presents the main result of this paper—the funnel theorems. The power of the funnel theorems is illustrated in Sec. V, where we use the funnel equality to easily obtain novel explicit exact expressions for the adoption/infection probability of nodes that are a vertex cut among L identical networks and for interior nodes on bounded lines. We also show how the funnel theorems can be used to compute the exact expected adoption/infection level on sparse d-regular networks and Erdős–Renyí networks, which have numerous cycles. In this case, the use of the funnel equality is more involved. Moreover, one needs to use the lower and upper bounds of the funnel theorems to show that the effect of cycles vanishes on infinite networks.

The second part of this paper is devoted to proving the funnel theorems. Let $\{\Omega_l\}_{l=1}^{L}$ be disjoint subsets of the nodes. The key ingredient in the proof of the funnel theorems is an estimate of the sign and magnitude of the difference $[S_{\cup_{l=1}^{L}\Omega_l}] - \prod_{l=1}^{L}[S_{\Omega_l}]$ between the probability that all the nodes in $\cup_{l=1}^{L}\Omega_l$ are susceptible and the product of the L probabilities that all the nodes in each Ω_l are susceptible. In Sec. VI, we show that this difference is always nonnegative and find the necessary and sufficient condition for it to be equal to zero. For the case where this difference is positive, we also obtain an upper bound for its magnitude. The funnel theorems are proved in Sec. VII. Section VIII concludes with a discussion and open questions.

II. THE BASS/SI MODEL ON NETWORKS

The Bass model describes the adoption process of an innovative new product in a population. When the product is first introduced into the market, all the individuals are susceptible (non-adopters). As time goes on, individuals gradually adopt the product due to external influences by mass media and internal influences by individuals who already adopted the product (peers effect, word of mouth). The SI model describes the spreading of infectious diseases within a population. In this model, some individuals are initially infected (the "patient zero" cases), and all subsequent infections occur through interactions between infected and susceptible individuals. In both models, once an individual becomes an adopter/infected, they remain so at all later times. In particular, they remain "contagious" forever. The difference between the two models is in the external influences: In the SI model, they occur at $t = 0$ and in the Bass model at $t > 0$.

It is convenient to unify these two models into a single model, the Bass/SI model on networks, as follows. Consider M individuals, denoted by $\mathcal{M} := \{1, \ldots, M\}$. Let $X_j(t)$ denote the state of individual j at time t so that

$$X_j(t) = \begin{cases} 1 & \text{if } j \text{ is adopter/infected at time } t, \\ 0 & \text{if } j \text{ is susceptible at time } t, \end{cases} \qquad j \in \mathcal{M}.$$

The initial conditions at $t = 0$ are stochastic so that

$$X_j(0) = X_j^0 \in \{0, 1\}, \quad j \in \mathcal{M}, \tag{2.1a}$$

where

$$\mathbb{P}\left(X_j^0 = 1\right) = I_j^0, \quad \mathbb{P}\left(X_j^0 = 0\right) = 1 - I_j^0, \quad I_j^0 \in [0, 1], \quad j \in \mathcal{M}, \tag{2.1b}$$

and

$$\text{the random variables } \left\{X_j^0\right\}_{j\in\mathcal{M}} \text{ are independent.} \tag{2.1c}$$

So long as j is susceptible, its adoption/infection rate at time t is

$$\lambda_j(t) = p_j + \sum_{k\in\mathcal{M}} q_{k\to j} X_k(t), \quad j \in \mathcal{M}. \tag{2.1d}$$

Here, p_j is the rate of external influences on j, and $q_{k\to j}$ is the rate of internal influences (*peer effects*) by k on j, provided that k is already an adopter/infected. Once j adopts the product/becomes infected, it remains so at all later times (i.e., the only admissible transition is $X_j = 0 \to X_j = 1$). Hence, as $\Delta t \to 0$,

$$\mathbb{P}(X_j(t + \Delta t) = 1 \mid \mathbf{X}(t)) = \begin{cases} \lambda_j(t)\Delta t & \text{if } X_j(t) = 0, \\ 1 & \text{if } X_j(t) = 1, \end{cases} \quad j \in \mathcal{M}, \tag{2.1e}$$

where $\mathbf{X}(t) := \{X_j(t)\}_{j\in\mathcal{M}}$ is the state of the network at time t, and

$$\text{the random variables } \{X_j(t + \Delta t) \mid \mathbf{X}(t)\}_{j\in\mathcal{M}} \text{ are independent.} \tag{2.1f}$$

We assume that all the nodes have a positive probability to be initially susceptible, that the external and internal influence rates are non-negative, and that any node can adopt externally, either at $t = 0$ or at $t > 0$, i.e.,

$$0 \le I_j^0 < 1, \quad p_j \ge 0, \quad q_{k\to j} \ge 0, \quad I_j^0 + p_j > 0, \quad k, j \in \mathcal{M}. \tag{2.1g}$$

In the Bass model, there are no adopters when the product is first introduced into the market, and so, $I_j^0 \equiv 0$ and $p_j > 0$. In the SI model, there are only internal influences for $t > 0$, and so, $p_j = 0$ and $I_j^0 > 0$.[25]

The internal adoption rates $\{q_{k\to j}\}$ induce a *directed weighted graph* on the nodes $\mathcal{M}$ so that the directed edge $k \to j$ exists if and only if $q_{k\to j} > 0$, and its weight is given by $q_{k\to j}$. We denote the network that corresponds to (2.1) by $\mathcal{N} = \mathcal{N}(\mathcal{M}, \{p_j\}, \{q_{k\to j}\}, \{I_j^0\})$.

III. THEORY REVIEW

The starting point of nearly all of the analytic theory of the Bass/SI model (2.1) on networks is the master equations. Let $\emptyset \ne \Omega \subset \mathcal{M}$ be a nontrivial subset of the nodes, and let $\Omega^c := \mathcal{M} \setminus \Omega$ denote the complementary set. Let

$$X_\Omega(t) := \max_{k\in\Omega} X_k(t). \tag{3.1}$$

Thus, $X_\Omega = 0$ if none of the nodes in Ω are adopters at time t, and $X_\Omega = 1$ if at least one of the nodes in Ω is an adopter. Let

$$S_\Omega(t) := \{X_\Omega(t) = 0\}, \quad [S_\Omega](t) := \mathbb{P}(S_\Omega(t)) \tag{3.2}$$

denote the event that all nodes in Ω are susceptible at time t and the probability of this event, respectively. To simplify the presentation, we introduce the notations

$$S_k := S_{\{k\}}, \quad S_{\Omega_1,\ldots,\Omega_L} := S_{\cup_{l=1}^L \Omega_l}.$$

Thus, for example, $S_{\Omega,k} := S_{\Omega\cup\{k\}}$ and $S_{m_1,m_2,m_3} := S_{\{m_1,m_2,m_3\}}$. We also denote the sum of the external influences on the nodes in Ω and the sum of the internal influences by the node $k \in \Omega^c$ on the nodes in Ω by

$$p_\Omega := \sum_{m\in\Omega} p_m, \quad q_{k\to\Omega} := \sum_{m\in\Omega} q_{k\to m},$$

respectively. We then have

Theorem 3.1 (Ref. 13): *The master equations for the Bass/SI model (2.1) are*

$$\frac{d[S_\Omega]}{dt} = -\left(p_\Omega + \sum_{k\in\Omega^c} q_{k\to\Omega}\right)[S_\Omega] + \sum_{k\in\Omega^c} q_{k\to\Omega}[S_{\Omega,k}], \tag{3.3a}$$

subject to the initial conditions

$$[S_\Omega](0) = \left[S_\Omega^0\right], \quad \left[S_\Omega^0\right] := \prod_{m\in\Omega} \left(1 - I_m^0\right), \tag{3.3b}$$

for all $\emptyset \ne \Omega \subset \mathcal{M}$.

The quantities of most interest are the susceptibility probabilities $\{[S_j](t)\}$ of the nodes and the expected susceptibility level $[S](t) := \frac{1}{M}\sum_{j=1}^M [S_j](t)$ in the network. Solving the master equations (3.3a) and (3.3b) for $\{[S_j](t)\}$ requires knowing the susceptibility probabilities $\{[S_{k,j}]\}$ of all pairs of nodes. The master equations for $\{[S_{k,j}](t)\}$, in turn, involve the susceptibility probabilities $\{[S_{m,k,j}](t)\}$

of all the triplets of nodes, etc. Therefore, to close the system of equations for $\{[S_j](t)\}$, one needs to solve $2^M - 1$ master equations for the susceptibility probabilities of all the nontrivial subsets of M modes. Consequently, an explicit exact solution of the master equations is, in general, not feasible.

For *networks with an inherent symmetry or structure*, it is sometimes possible to obtain an exact reduced system of master equations that can be solved explicitly. This is the case, e.g., with complete and circular networks.[12–14] Another approach for obtaining an exact explicit solution is to use the *indifference principle*. Let us first recall

Definition 3.1 (influential edge[16]): Consider the Bass/SI model (2.1). Let $\Omega \subset \mathcal{M}$. A directed edge $k \to m$ is said to be "influential to Ω" if $k \in \Omega^c$, and if either $m \in \Omega$ or there is a path from m to Ω, which does not go through the node k. Any edge that is not "influential to Ω" is called "non-influential to Ω."

We then have

Theorem 3.2 (indifference principle[16]): *Consider the Bass/SI model (2.1). Let $\emptyset \neq \Omega \subset \mathcal{M}$. Then, $[S_\Omega]$ remains unchanged if we delete or add edges that are non-influential to Ω.*

The indifference principle is a powerful tool that enables us to add or delete noninfluential edges so that the value of $[S_\Omega]$ will remain unchanged, but its calculation on the modified network becomes simpler. For example, it can be used to compute the exact susceptibility level on percolation lines.[17]

IV. THE FUNNEL THEOREMS

The funnel theorems are a novel analytic tool that extends the range of networks for which exact explicit solutions can be obtained. To introduce these theorems, we begin with a few definitions.

Definition 4.1 (partition of nodes): Let $L \geq 2$, $j \in \mathcal{M}$ and $\emptyset \neq A_l \subset \mathcal{M} \setminus \{j\}$ for $l = 1, \ldots, L$. We say that "$\{A_1, \ldots, A_L, \{j\}\}$ is a partition of $\mathcal{M}$" if $A_1 \cup \cdots \cup A_L \cup \{j\} = \mathcal{M}$ and the sets $\{A_l\}_{l=1}^L$ are mutually disjoint.

Figure 2 illustrates a partition into $L = 3$ disjoint sets.

Consider the Bass/SI model (2.1) on the network $\mathcal{N}$. Let $\mathcal{N}^{A_l, p_j, I_j^0}$ denote the modified network in which j experiences external influences and internal influences from its peers in A_l (see Fig. 1):

Definition 4.2 $\left(\mathcal{N}^{A_l, p_j, I_j^0}\right.$ and $\left.\left[S_j^{A_l, p_j, I_j^0}\right]\right)$: Let $j \in \mathcal{M}$ and $A_l \subset \mathcal{M} \setminus \{j\}$. The network $\mathcal{N}^{A_l, p_j, I_j^0}$ is obtained from $\mathcal{N}$ by deleting all the internal influences on j by nodes that are not in A_l, i.e., by setting $q_{k \to j}^{A_l, p_j, I_j^0} := 0$ for $k \in \mathcal{M} \setminus A_l$. The susceptibility probability of j in the network $\mathcal{N}^{A_l, p_j, I_j^0}$ is denoted by $\left[S_j^{A_l, p_j, I_j^0}\right] := [S_j]\left(t; \mathcal{N}^{A_l, p_j, I_j^0}\right)$.

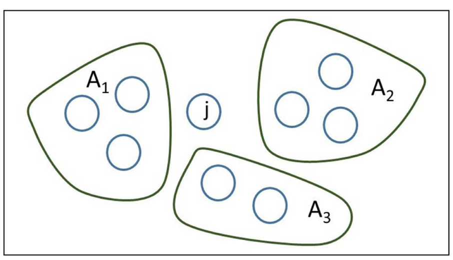

FIG. 2. A partition $\{A_1, A_2, A_3, \{j\}\}$.

If the node $j \in \mathcal{M}$ is isolated (i.e., has a zero indegree), it can only adopt due to external influences, and its susceptibility probability is denoted by

$$[S_j^{\text{isolated}}] := \left[S_j^{A_l = \emptyset, p_j, I_j^0}\right].$$

A. Lower bound

We first show that the susceptibility probability $[S_j]$ of any node j is always greater than or equal to the product of the susceptibility probabilities $\prod_{l=1}^L \left[S_j^{A_l, p_j, I_j^0}\right]$, divided by $[S_j^{\text{isolated}}]^{L-1}$:

Theorem 4.1: *Consider the Bass/SI model (2.1). Let $j \in \mathcal{M}$, and let $\{A_1, \ldots, A_L, \{j\}\}$ be a partition of $\mathcal{M}$. Then,*

$$[S_j] \geq \frac{\prod_{l=1}^L \left[S_j^{A_l, p_j, I_j^0}\right]}{\left[S_j^{\text{isolated}}\right]^{L-1}}, \quad t \geq 0 \text{ (\textbf{funnel inequality})}, \tag{4.1}$$

where

$$\left[S_j^{\text{isolated}}\right] = \left(1 - I_j^0\right) e^{-p_j t}. \tag{4.2}$$

Proof. See Sec. VII. □

Note that in (4.1), we needed to divide $\prod_{l=1}^L \left[S_j^{A_l, p_j, I_j^0}\right]$ by $[S_j^{\text{isolated}}]^{L-1}$ since each of the L terms in the product includes the external influences on j.

In order to determine the conditions under which the funnel inequality becomes an equality, we introduce some more definitions.

Definition 4.3 (influential node): Let $\emptyset \neq \Omega \subset \mathcal{M}$. We say that "node m is influential to Ω" if $m \in \Omega$, or if $m \in \Omega^c$, and there is a finite simple path from m to Ω.

Definition 4.4 (funnel node): Let $\{A_1, \ldots, A_L, \{j\}\}$ be a partition of $\mathcal{M}$. A node j is called a "funnel node of $\{A_l\}_{l=1}^L$ in the network $\mathcal{N}$" if there is no node in $\mathcal{M} \setminus \{j\}$, which is influential to j both in $\mathcal{N}^{A_l}$ and in $\mathcal{N}^{A_{\tilde{l}}}$ for some $l \neq \tilde{l}$.

Recall also the following terminology from graph theory:

Definition 4.5 (vertex cut): Let $\{A_1, \ldots, A_L, \{j\}\}$ be a partition of $\mathcal{M}$. A node j is called a "vertex cut between $\{A_l\}_{l=1}^L$" if removing j from the network makes the sets $\{A_l\}_{l=1}^L$ disconnected from each other.

For example, the node j is a vertex cut of $\{A_1, A_2, A_3\}$ in Fig. 3(a), but not in Figs. 3(b) and 3(c). Any node that is a vertex cut is also a funnel node:

Lemma 4.1: *Let $\{A_1, \ldots, A_L, \{j\}\}$ be a partition of $\mathcal{M}$. If node j is a vertex cut between $\{A_l\}_{l=1}^L$, then j is a funnel node.*

Proof. Let j be a vertex cut between $\{A_l\}_{l=1}^L$. If node $m \in A_l$ is influential to j, then m cannot be influential to j in $\mathcal{N}^{A_{\tilde{l}}}$ since in $\mathcal{N}^{A_{\tilde{l}}}$, we removed all the edges from A_l to j, and there is no sequence of edges (influential or not) from m to $A_{\tilde{l}}$. □

Therefore, for example, j is a funnel node in Fig. 3(a). The converse statement, however, is not true; i.e., j can be a funnel node even if the sets A_l and $A_{\tilde{l}}$ are directly connected. Indeed, this is the case if for any $m \in A_l$ and $\tilde{m} \in A_{\tilde{l}}$ such that the edge $m \to \tilde{m}$ exists,

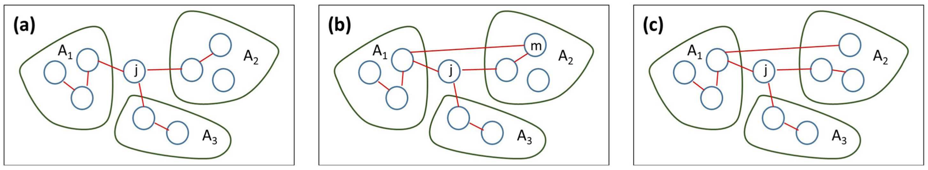

FIG. 3. (a) j is a vertex cut of $\{A_1, A_2, A_3\}$. Therefore, it is also a funnel node of $\{A_1, A_2, A_3\}$. (b) j is not a funnel node of $\{A_1, A_2, A_3\}$ since the mode m is influential to j in $\mathcal{N}^{A_1}$ and in $\mathcal{N}^{A_2}$. (c) j is a funnel node, but not a vertex cut, of $\{A_1, A_2, A_3\}$.

either the node m is non-influential to j in $\mathcal{N}^{A_l}$, or the node $\widetilde{m}$ is non-influential to j in $\mathcal{N}^{A_{\widetilde{l}}}$; see, e.g., Fig. 3(c).

Theorem 4.2: *Assume the conditions of Theorem 4.1.*

- *If j is a funnel node of $\{A_l\}_{l=1}^L$, then*

$$[S_j] = \frac{\prod_{l=1}^{L}\left[S_j^{A_l,p_j,I_j^0}\right]}{\left[S_j^{\text{isolated}}\right]^{L-1}}, \quad t \geq 0 \quad (\textbf{funnel equality}). \tag{4.3}$$

- *If, however, j is not a funnel node of $\{A_l\}_{l=1}^L$, then*

$$[S_j] > \frac{\prod_{l=1}^{L}\left[S_j^{A_l,p_j,I_j^0}\right]}{\left[S_j^{\text{isolated}}\right]^{L-1}}, \quad t > 0 \quad (\textbf{strict funnel inequality}). \tag{4.4}$$

Proof. See Sec. VII. □

Intuitively, the event that j remains susceptible occurs if and only if the following $L+1$ events occur: the "external event" that j remains susceptible under the external influences and the L "internal events" that j remains susceptible under the internal influences by edges arriving from A_l, where $l = 1, \ldots, L$. If j is a funnel node, these $L+1$ events are independent, and so, we have the funnel equality. If j is not a funnel node, however, some of the L "internal events" are positively correlated. Consequently, the probability that all the L "internal events" occur is larger than the product of their individual probabilities.

Corollary 4.1: *Assume the conditions of Theorem 4.1. Let $j \in \mathcal{M}$. If for any $m \in \mathcal{M}\setminus\{j\}$, there is at most one finite path from m to j, then the funnel equality (4.3) holds.*

Proof. If there exists node m, which is influential to j in $\mathcal{N}^{A_l}$ and in $\mathcal{N}^{A_{\widetilde{l}}}$, then there are two different paths leading from m to j. Therefore, there is no such node m. Hence, j is a funnel node of $\{A_1\}_{l=1}^L$, and therefore, the result follows from Theorem 4.2. □

Corollary 4.2: *Assume the conditions of Theorem 4.1. If the network is undirected and contains no cycles, the funnel equality (4.3) holds for all $j \in \mathcal{M}$.*

Proof. This follows from Corollary 4.1. □

B. Upper bound

Theorems 4.1 and 4.2 provide a lower bound for $[S_j] - \frac{\prod_{l=1}^{L}\left[S_j^{A_l,p_j,I_j^0}\right]}{[S_j^{\text{isolated}}]^{L-1}}$. We can also derive an upper bound for this difference. To simplify the presentation, we assume that

1. All the nodes have the same weight and initial condition, i.e.,

$$p_j \equiv p, \quad I_j^0 \equiv I^0, \quad j \in \mathcal{M}. \tag{4.5a}$$

2. The network is undirected, and all the edges have the same weight, i.e.,

$$q_{k\to j} = q_{j\to k} \in \{0, q\}, \quad k, j \in \mathcal{M}. \tag{4.5b}$$

3. The parameters satisfy; see (2.1g),

$$q > 0, \quad p \geq 0, \quad 0 \leq I^0 < 1, \quad p + I^0 > 0. \tag{4.5c}$$

Theorem 4.3: *Assume the conditions of Theorem 4.1. Let (4.5) hold. Assume that there are $N_j \geq 1$ cycles $\{C_n\}_{n=1}^{N_j}$ in which j is connected to A_l on one side and to $A_{\widetilde{l}}$ on the other side, where $l \neq \widetilde{l}$ (A_l and $A_{\widetilde{l}}$ may be different for each cycle). Then,*

$$0 < [S_j] - \frac{\prod_{l=1}^{L}\left[S_j^{A_l,p_j,I_j^0}\right]}{\left[S_j^{\text{isolated}}\right]^{L-1}} < \left[S_j^{\text{isolated}}\right]\sum_{n=1}^{N_j} E(t; K_n + 1), \quad t > 0, \tag{4.6}$$

where K_n is the number of nodes of C_n, and $E(t;K)$ satisfies the bound

$$E(t;K) \leq 2(1 - I^0)e^{-(p+q)t}\left(\frac{eqt}{\left\lfloor\frac{K+1}{2}\right\rfloor}\right)^{\left\lfloor\frac{K+1}{2}\right\rfloor}, \quad 0 < t < \frac{1}{q}\left\lfloor\frac{K+1}{2}\right\rfloor. \tag{4.7a}$$

We also have the global-in-time bound

$$E(t;K) \leq 2(1 - I^0)\left(\frac{q}{p+q}\right)^{\left\lfloor\frac{K+1}{2}\right\rfloor}, \quad t \geq 0. \tag{4.7b}$$

Proof. See Sec. VII. □

On undirected networks, j is a funnel node of $\{A_l\}_{l=1}^L$ if and only if there is no cycle in which j is connected to A_l on one side and to $A_{\widetilde{l}}$ on the other side, where $l \neq \widetilde{l}$. The difference between $[S_j]$ and $\frac{\prod_{l=1}^{L}\left[S_j^{A_l,p_j,I_j^0}\right]}{[S_j^{\text{isolated}}]^{L-1}}$, thus, arises from the presence of nodes that can lead to the adoption/infection of j through two sequences of peer-effect events: one whose final node before j is from A_l and the other whose final node before j is from $A_{\widetilde{l}}$, where $l \neq \widetilde{l}$. Since the effect of an infected node on the state of another node decays super-exponentially with their distance, see Ref. 26, so does the difference between $[S_j]$ and $\frac{\prod_{l=1}^{L}\left[S_j^{A_l,p_j,I_j^0}\right]}{[S_j^{\text{isolated}}]^{L-1}}$.

V. APPLICATIONS OF THE FUNNEL THEOREMS

The funnel theorems enable us to obtain explicit exact expressions for the susceptibility probability of nodes on various networks. Intuitively, this is because the indegree of the node j on the modified networks $\{\mathcal{N}^{A_l,p_j,I_j^0}\}_{l=1}^{L}$ is lower than in the original network.

A. Vertex cut between identical networks

The simplest application of the funnel theorem is when the node is a vertex cut between L identical networks:

Lemma 5.1: *Consider the Bass/SI model (2.1). Let the node j be a vertex cut between L identical networks. Denote by $[S_j](t;L)$ the susceptibility probability of j in this network and by $[S_j](t;L=1)$ the susceptibility probability of j when it belongs to only one of these networks. Then,*

$$[S_j](t;L) = \frac{[S_j]^L(t;L=1)}{\left[S_j^{\text{isolated}}\right]^{L-1}}. \tag{5.1}$$

Proof. By Lemma 4.1, j is a funnel node of the L identical networks. Therefore, the funnel equality holds, see Thoerem 4.2, and so the result follows from (4.3). □

Thus, if we know the susceptibility probability of j in some network, we can write an explicit expression for the susceptibility probability of j when it is a vertex cut of L such networks.

Example 5.1: *Consider the Bass/SI model on the homogeneous infinite line where each node can be influenced by its two adjacent nodes; i.e.,*

$$I_j^0 \equiv I^0, \quad p_j \equiv p, \quad q_{k\to j} = \frac{q}{2}\mathbb{1}_{|j-k|=1}, \quad k,j \in \mathbb{Z}. \tag{5.2}$$

The susceptibility probability of each node on the line is identical and is given by $[S_j^{\mathbb{Z}}] = [S^{1D}]$, where

$$[S^{1D}](t;p,q,I^0) := \begin{cases} 1-(1-I^0)e^{-(p+q)t+q(1-I^0)\frac{1-e^{-pt}}{p}} & \text{if } p>0, \\ 1-(1-I^0)e^{-qI^0t} & \text{if } p=0, \end{cases} \tag{5.3}$$

see Refs. 12, 14, and 15.

We then have

Corollary 5.1: *Consider the Bass/SI model (2.1). Let the node $\mathbf{0}^L$ be the intersection point of the L identical infinite lines (5.2). Then, the susceptibility probability at the intersection node is*

$$[S_{\mathbf{0}^L}](t) = [S^{1D}](t;p,Lq,I^0), \tag{5.4}$$

where $[S^{1D}]$ is given by (5.3).

Proof. By (5.1),

$$[S_{\mathbf{0}^L}](t) = \frac{[S^{1D}]^L(t;p,q,I^0)}{[S_j^{\text{isolated}}]^{L-1}}.$$

Substitution of (4.2) and (5.3) proves the result. □

This result is illustrated in Fig. 4.

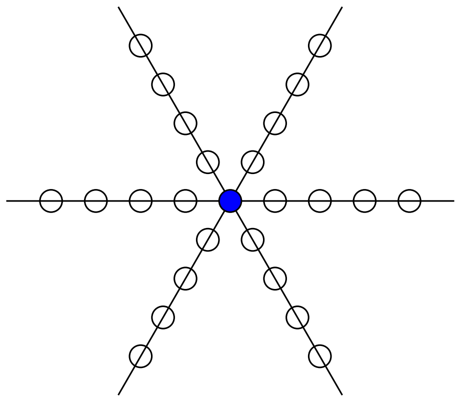

FIG. 4. The susceptibility probability at the intersection node of three infinite lines is given by (5.4) with $L=3$.

B. Vertex cut between non-identical networks

We can also apply the funnel theorem for a node, which is a vertex cut between nonidentical networks.

Example 5.2: *Consider the Bass/SI model (2.1) on the homogeneous bounded line $[1,\ldots,M]$, where each node can be influenced by its two adjacent nodes so that*

$$I_j^0 \equiv I^0, \quad p_j \equiv p, \quad q_{k\to j} = \frac{q}{2}\mathbb{1}_{|j-k|=1}, \quad k,j \in \{1,\ldots,M\}. \tag{5.5}$$

Let $\left[S_j^{[1,\ldots,M]}\right](t;p,q,I^0)$ denote the susceptibility probability of the node j. The susceptibility probability of the boundary nodes can be calculated using the indifference principle,[16]

$$\left[S_1^{[1,\ldots,M]}\right] = \left[S_M^{[1,\ldots,M]}\right] = [S^{\text{circle}}]\left(t;p,\frac{q}{2},I^0,M\right), \tag{5.6}$$

where $[S^{\text{circle}}](t;p,q,I^0,M)$ is the expected susceptibility level in the Bass/SI model on a circle with the same parameters (i.e., when we add the edge $1 \leftrightarrow M$ between the boundary nodes). An exact explicit expression for $[S^{\text{circle}}]$ has been obtained by utilizing translation invariance.[12,16]

We now use the funnel theorem to obtain a novel explicit expression for the susceptibility probability of the interior nodes, in terms of the known expression for $[S^{\text{circle}}]$:

Lemma 5.2: *Consider the Bass/SI model (2.1) on the bounded line (5.5). Then,*

$$[S_j^{[1,\ldots,M]}](t;p,q,I^0) = \frac{[S^{\text{circle}}]\left(t;p,\frac{q}{2},I^0,j\right)[S^{\text{circle}}]\left(t;p,\frac{q}{2},I^0,M+1-j\right)}{(1-I^0)e^{-pt}}, \quad j=1,\ldots,M.$$

Proof. Let $1<j<M$, $A_1 := \{1,\ldots,j-1\}$, and $A_2 := \{j+1,\ldots,M\}$. Then, $\{A_1,A_2,\{j\}\}$ is a partition of the nodes, and j is a *vertex cut* between A_1 and A_2. Therefore, by the funnel equality (4.3),

$$\left[S_j^{[1,\ldots,M]}\right] = \frac{\left[S_j^{A_1,p_j,I_j^0}\right]\left[S_j^{A_2,p_j,I_j^0}\right]}{\left[S_j^{\text{isolated}}\right]}. \tag{5.7}$$

As far as j is concerned, the network $\mathcal{N}^{A_1,p_j,I_j^0}$ is identical to the network $[1,\ldots,j]$. Therefore,

$$[S_j^{A_1,p_j,I_j^0}] = [S_j^{[1,\ldots,j]}] = [S^{\rm circle}]\left(t; p, \frac{q}{2}, I^0, j\right);$$

see (5.6), Similarly,

$$[S_j^{A_2,p_j,I_j^0}] = [S_j^{[j,\ldots,M]}] = [S^{\rm circle}]\left(t; p, \frac{q}{2}, I^0, M+1-j\right).$$

Since $[S_j^{\rm isolated}]$ is given by (4.2), the result follows. □

This expression is considerably simpler than the one that was derived in Lemma 4.6 of Ref. 16.

C. Sparse random networks with numerous loops

Moving on to more challenging applications of the funnel theorems, let us consider d-regular networks, where the degree of each node is equal to d. Assume that the weight and initial condition of all the nodes are p and I^0, respectively, and the weight of all the edges is $\frac{q}{d}$. Although large d-regular networks have numerous cycles, one can use the funnel theorems to compute explicitly and exactly the expected susceptibility level as $M \to \infty$:

Theorem 5.1 (Ref. 27): *With probability one with respect to the distribution of graphs, the expected susceptibility level in the Bass/SI model on infinite random d-regular networks is the solution of the equation*

$$\frac{d[S]}{dt} = -[S]\left(p + q\left(1 - \left(\frac{[S]}{e^{-pt}(1-I^0)}\right)^{-\frac{2}{d}}[S]\right)\right), \qquad [S](0) = 1 - I^0. \tag{5.8}$$

Sketch of proof. A rigorous proof of Theorem 5.1 is presented in Ref. 27. Here, we only provide a sketch of the derivation, in order to highlight the use of the funnel theorems. Let $j \in \mathcal{M}$, let $\{k_1,\ldots,k_d\}$ denote the d neighbors of j, and let $\{A_1,\ldots,A_d,\{j\}\}$ be a partition of $\mathcal{M}$ such that $k_l \in A_l$ for $l = 1,\ldots,d$. Assume first that there are no cycles in the network. Then, the funnel equality holds (Corollary 4.2), and therefore,

$$[S_j] = \frac{\prod_{l=1}^d \left[S_j^{A_l,p,I^0}\right]}{\left[S_j^{\rm isolated}\right]^{d-1}} = \frac{y^d(t)}{((1-I^0)e^{-pt})^{d-1}}, \tag{5.9a}$$

where $y(t)$ is the susceptibility probability of a degree-one node in an otherwise infinite d-regular network. By the indifference principle,[16] the reduced master equation for $y(t)$ is

$$\frac{dy}{dt} = -\left(p + \frac{q}{d}\right)y + \frac{q}{d}(1-I^0)e^{-pt}z(t), \tag{5.9b}$$

where $z(t)$ is the susceptibility probability of a node of degree $d-1$ in an otherwise infinite d-regular network. Finally, by the funnel equality, the same derivation as of (4.3), only with $d-1$ instead of d, gives that

$$z(t) = \frac{y^{d-1}(t)}{((1-I^0)e^{-pt})^{d-2}}. \tag{5.9c}$$

Combining Eqs. (5.9a), (5.9b), and (5.9c), we get (5.8).

Let us now justify why we could neglect the effect of cycles and use the funnel equality in (4.3) and in (5.9c). On d-regular networks, the number of cycles of length K increases exponentially with K. The upper bound of the funnel equality shows, however, that the error introduced by using the funnel equality when j lies on a circle of length K decays to zero at a rate that is super-exponential in K (Theorem 4.3). As a result, the overall error introduced by using the funnel equality goes to zero as $M \to \infty$. □

Let us further consider sparse Erdős–Rényi (ER) networks such that for any two nodes $k, j \in \mathcal{M}$, the edge between k and j exists with probability $\frac{\lambda}{M}$, independently of all other edges, where $0 < \lambda < \infty$. Assume also that the weight and initial condition of all the nodes are p and I^0, respectively, and the weight of all the edges is $\frac{q}{\lambda}$. The funnel theorems can be used in the same fashion, though with more technical details, to compute explicitly and exactly the susceptibility level on infinite sparse ER networks:

Theorem 5.2 (Ref. 27): *With probability one with respect to the distribution of graphs, the expected susceptibility level in the Bass/SI model (2.1) on sparse infinite ER networks is given by*

$$[S^{\rm ER}(t; p, q, \lambda, I^0)] = (1-I^0)e^{-pt-\lambda(1-y(t))}, \quad t \geq 0, \tag{5.10a}$$

where $y(t)$ is the solution of the equation

$$\frac{dy}{dt} = \frac{q}{\lambda}\left(-y + (1-I^0)e^{-pt-\lambda(1-y)}\right), \quad t \geq 0,\ y(0) = 1. \tag{5.10b}$$

See Ref. 27 for more details.

VI. LOWER AND UPPER BOUNDS FOR $[S_{\cup_{l=1}^L \Omega_l}] - \prod_{l=1}^L [S_{\Omega_l}]$

In Sec. VII, we shall see that the proof of the funnel theorems relies on knowing the sign and magnitude of the difference $[S_{\cup_{l=1}^L \Omega_l}] - \prod_{l=1}^L [S_{\Omega_l}]$. It is reasonable to assume that the susceptibility probabilities of $L \geq 2$ disjoint sets of nodes are uncorrelated when the sets are "disconnected," and positively correlated otherwise.[28] Indeed, we have

Theorem 6.1: *Consider the Bass/SI model (2.1). Let $\Omega_1,\ldots,\Omega_L \subset \mathcal{M}$ such that $\Omega_l \cap \Omega_{\tilde{l}} = \emptyset$ for any $l \neq \tilde{l}$. Then,*

$$\left[S_{\cup_{l=1}^L \Omega_l}\right] \geq \prod_{l=1}^L [S_{\Omega_l}], \quad t \geq 0. \tag{6.1}$$

In addition,

1. *If there exist $l \neq \tilde{l}$ and a node in $\mathcal{M}$ that is influential to both Ω_l and $\Omega_{\tilde{l}}$, then*

$$\left[S_{\cup_{l=1}^L \Omega_l}\right] > \prod_{l=1}^L [S_{\Omega_l}], \quad t > 0. \tag{6.2}$$

2. *If, however, for any $l \neq \tilde{l}$, there is no node in $\mathcal{M}$ that is influential to both Ω_l and $\Omega_{\tilde{l}}$, then*

$$\left[S_{\cup_{l=1}^L \Omega_l}\right] = \prod_{l=1}^L [S_{\Omega_l}], \quad t \geq 0. \tag{6.3}$$

Proof. See Sec. VI B. □

We can also derive an upper bound for $[S_{\cup_{l=1}^{L}\Omega_l}] - \prod_{l=1}^{L}[S_{\Omega_l}]$ by utilizing the spatiotemporal estimates for the correlation between the susceptibility probabilities of two nodes that were derived in Ref. 26.

Theorem 6.2: *Consider the Bass/SI model (2.1) on an undirected network such that (4.5) holds. Let $\Omega_1, \ldots, \Omega_L \subset \mathcal{M}$ such that $\Omega_l \cap \Omega_{\tilde{l}} = \emptyset$ for any $l \neq \tilde{l}$. Denote by $\{\Gamma_n\}_{n=1}^{N_L}$ the N_L distinct simple paths that connect between pairs of sets in $\{\Omega_1, \ldots, \Omega_L\}$ such that the interior nodes of $\{\Gamma_n\}_{n=1}^{N_L}$ are in $\mathcal{M}\backslash\bigcup_{l=1}^{L}\Omega_l$. Let $N_L \geq 1$. Then,*

$$0 < \left[S_{\cup_{l=1}^{L}\Omega_l}\right] - \prod_{l=1}^{L}[S_{\Omega_l}] < \sum_{n=1}^{N_L} E(t; K_n), \quad t > 0, \tag{6.4}$$

where K_n is the number of nodes of the path Γ_n (including the two boundary nodes in $\cup_{l=1}^{L}\Omega_l$), and $E(t; K_n)$ satisfies the bounds (4.7a) and (4.7b).

Proof. See Sec. VI C. □

Indeed, since the effect of an infected node on the state of another node decays super-exponentially with their distance, see Ref. 26, so does the correlation between their states.

A. Auxiliary results

Before proving Theorem 6.1, some auxiliary results will be needed. We first note some consequences of the master equations:

Lemma 6.1: *$[S_{\Omega}^{0}] > 0$ for all $\emptyset \neq \Omega \subset \mathcal{M}$.*

Proof. This follows from (2.1g) and (3.3b). □

Lemma 6.2: *Let $\emptyset \neq \Omega \subset \mathcal{M}$. Then,*

$$0 < \left[S_{\Omega}^{0}\right] e^{-\left(p_{\Omega} + \sum_{k\in\Omega^c} q_{k\to\Omega}\right)t} \leq [S_{\Omega}] \leq \left[S_{\Omega}^{0}\right] e^{-p_{\Omega}t} < 1, \quad t > 0. \tag{6.5}$$

Proof. The master equation (3.3a) can be rewritten as

$$\frac{d[S_{\Omega}]}{dt} = -p_{\Omega}[S_{\Omega}] - \sum_{k\in\Omega^c} q_{k\to\Omega}\left([S_{\Omega}] - [S_{\Omega,k}]\right). \tag{6.6}$$

If the event $S_{\Omega,k}$ occurs, the event S_{Ω} occurs as well. Therefore,

$$[S_{\Omega}] \geq [S_{\Omega,k}]. \tag{6.7}$$

In addition, we have that $q_{k\to\Omega} \geq 0$ and $[S_{\Omega,k}] \geq 0$. Therefore, from Eq. (6.6), we have that

$$-p_{\Omega}[S_{\Omega}] \geq \frac{d[S_{\Omega}]}{dt} \geq -\left(p_{\Omega} + \sum_{k\in\Omega^c} q_{k\to\Omega}\right)[S_{\Omega}].$$

In addition, $[S_{\Omega}^{0}] > 0$; see Lemma 6.1. Therefore, the result follows. □

Lemma 6.3: *Let $\emptyset \neq \Omega \subsetneq \mathcal{M}$ and $k \in \Omega^c$. Then, $[S_{\Omega,k}] < [S_{\Omega}]$ for $t > 0$.*

Proof. By the law of the sum of probability,

$$[S_{\Omega}] - [S_{\Omega,k}] = [S_{\Omega} \cap I_k] \geq [S_{\mathcal{M}_{-k}} \cap I_k] = [S_{\mathcal{M}_{-k}}] - [S_{\mathcal{M}}],$$

where $[S_{\Omega} \cap I_k] := \mathbb{P}(X_{\Omega} = 0, X_k = 1)$ and $\mathcal{M}_{-k} := \mathcal{M}\backslash\{k\}$. Therefore, it is sufficient to prove that $y(t) := [S_{\mathcal{M}_{-k}}] - [S_{\mathcal{M}}] > 0$ for $t > 0$. From the master equations (3.3a) and (3.3b), we have that

$$[S_{\mathcal{M}}] = e^{-p_{\mathcal{M}}t} \prod_{m\in\mathcal{M}} (1 - I_m^0)$$

and

$$\frac{dy}{dt} = -(p_{\mathcal{M}_{-k}} + q_{k\to\mathcal{M}_{-k}})y + p_k[S_{\mathcal{M}}], \quad y(0) = I_k^0 \prod_{m\in\mathcal{M}_{-k}} \left(1 - I_m^0\right). \tag{6.8}$$

Therefore, by (2.1g), $[S_{\mathcal{M}}] > 0$ for $t > 0$ and $y(0) \geq 0$. Furthermore, by (2.1g), either $p_k > 0$ or $y(0) > 0$. Therefore, it follows from (6.8) that $y(t) > 0$ for $t > 0$. □

The following result is immediate:

Lemma 6.4: *Let $\emptyset \neq \Omega_1, \Omega_2 \subset \mathcal{M}$ such that $\Omega_1 \cap \Omega_2 = \emptyset$. Then, there exists a node in $\mathcal{M}$, which is influential to both Ω_1 and Ω_2 if and only if at least one of the following conditions hold:*

1. *There exists a path from Ω_1 to Ω_2 or from Ω_2 to Ω_1.*
2. *There exists a node $m \notin \Omega_1 \cup \Omega_2$ from which there exist a path to Ω_1 and a path to Ω_2.*

Corollary 6.1: *Let $\emptyset \neq \Omega_1, \Omega_2 \subset \mathcal{M}$ such that $\Omega_1 \cap \Omega_2 = \emptyset$. Let the network be undirected. Then, there exists a node that is influential to both Ω_1 and Ω_2 if and only if there exists a path between Ω_1 and Ω_2.*

We also have

Lemma 6.5: *Let $\emptyset \neq \Omega_1, \Omega_2 \subset \mathcal{M}$ such that $\Omega_1 \cap \Omega_2 = \emptyset$. Denote*

$$Q_{\Omega_1,\Omega_2} := [S_{\Omega_1,\Omega_2}] - [S_{\Omega_1}][S_{\Omega_2}]. \tag{6.9}$$

Then, $Q_{\Omega_1,\Omega_2}(t)$ satisfies the equation

$$\begin{aligned}\frac{dQ_{\Omega_1,\Omega_2}}{dt} &+ \left(p_{\Omega_1\cup\Omega_2} + \sum_{m\notin\Omega_1\cup\Omega_2} q_{m\to\Omega_1\cup\Omega_2}\right) Q_{\Omega_1,\Omega_2} \\ &= \sum_{m\notin\Omega_1\cup\Omega_2} \left(q_{m\to\Omega_1} Q_{\Omega_1\cup\{m\},\Omega_2} + q_{m\to\Omega_2} Q_{\Omega_1,\Omega_2\cup\{m\}}\right) \\ &\quad + \sum_{m\in\Omega_2} q_{m\to\Omega_1}\left([S_{\Omega_1}] - [S_{\Omega_1,m}]\right)[S_{\Omega_2}] \\ &\quad + \sum_{m\in\Omega_1} q_{m\to\Omega_2}\left([S_{\Omega_2}] - [S_{\Omega_2,m}]\right)[S_{\Omega_1}],\end{aligned} \tag{6.10a}$$

subject to the initial condition

$$Q_{\Omega_1,\Omega_2}(0) = 0. \tag{6.10b}$$

Proof. Using (6.9) and the master equation (3.3a), we have that

$$
\begin{aligned}
\frac{dQ_{\Omega_1,\Omega_2}}{dt} &= \frac{d[S_{\Omega_1,\Omega_2}]}{dt} - [S_{\Omega_1}]\frac{d[S_{\Omega_2}]}{dt} - [S_{\Omega_2}]\frac{d[S_{\Omega_1}]}{dt} \\
&= -\left(p_{\Omega_1\cup\Omega_2} + \sum_{m\notin\Omega_1\cup\Omega_2} q_{m\to\Omega_1\cup\Omega_2}\right)[S_{\Omega_1,\Omega_2}] + \sum_{m\notin\Omega_1\cup\Omega_2} q_{m\to\Omega_1\cup\Omega_2}[S_{\Omega_1\cup\Omega_2,m}] \\
&\quad + [S_{\Omega_1}]\left(p_{\Omega_2} + \sum_{m\notin\Omega_2} q_{m\to\Omega_2}\right)[S_{\Omega_2}] - [S_{\Omega_1}]\sum_{m\notin\Omega_2} q_{m\to\Omega_2}[S_{\Omega_2,m}] \\
&\quad + [S_{\Omega_2}]\left(p_{\Omega_1} + \sum_{m\notin\Omega_1} q_{m\to\Omega_1}\right)[S_{\Omega_1}] - [S_{\Omega_2}]\sum_{m\notin\Omega_1} q_{m\to\Omega_1}[S_{\Omega_1,m}] \\
&= -\left(p_{\Omega_1\cup\Omega_2} + \sum_{m\notin\Omega_1\cup\Omega_2} q_{m\to\Omega_1\cup\Omega_2}\right)\left(Q_{\Omega_1,\Omega_2} - [S_{\Omega_1}][S_{\Omega_2}]\right) + \sum_{m\notin\Omega_1\cup\Omega_2} q_{m\to\Omega_1}\left(Q_{\Omega_1\cup\{m\},\Omega_2} - [S_{\Omega_1,m}][S_{\Omega_2}]\right) \\
&\quad + \sum_{m\notin\Omega_1\cup\Omega_2} q_{m\to\Omega_2}\left(Q_{\Omega_1,\Omega_2\cup\{m\}} - [S_{\Omega_1}][S_{\Omega_2,m}]\right) + [S_{\Omega_1}]\left(p_{\Omega_2} + \sum_{m\notin\Omega_2} q_{m\to\Omega_2}\right)[S_{\Omega_2}] - [S_{\Omega_1}]\sum_{m\notin\Omega_2} q_{m\to\Omega_2}[S_{\Omega_2,m}] \\
&\quad + [S_{\Omega_2}]\left(p_{\Omega_1} + \sum_{m\notin\Omega_1} q_{m\to\Omega_1}\right)[S_{\Omega_1}] - [S_{\Omega_2}]\sum_{m\notin\Omega_1} q_{m\to\Omega_1}[S_{\Omega_1,m}],
\end{aligned}
$$

which leads to (6.10a). The initial condition follows from the independence of the initial conditions of nodes; see (2.1c). □

B. Proof of Theorem 6.1

We now turn to the proof of Theorem 6.1.

Lemma 6.6: *Let $\emptyset \neq \Omega_1, \Omega_2 \subset \mathcal{M}$ such that $\Omega_1 \cap \Omega_2 = \emptyset$. Then, $Q_{\Omega_1,\Omega_2}(t) \geq 0$ for $t \geq 0$.*

Proof. We proceed by backward induction on the size of $\Omega_1 \cup \Omega_2$. Consider the induction base where $\Omega_1 \cup \Omega_2 = \mathcal{M}$. Then, Eq. (6.10) for Q_{Ω_1,Ω_2} reduces to

$$
\begin{aligned}
\frac{dQ_{\Omega_1,\Omega_2}}{dt} + c_{\Omega_1,\Omega_2} Q_{\Omega_1,\Omega_2} &= \sum_{k\in\Omega_2} q_{k\to\Omega_1}\left([S_{\Omega_1}] - [S_{\Omega_1,k}]\right)[S_{\Omega_2}] \\
&\quad + \sum_{j\in\Omega_1} q_{j\to\Omega_2}\left([S_{\Omega_2}] - [S_{\Omega_2,j}]\right)[S_{\Omega_1}],
\end{aligned}
\tag{6.11a}
$$

where $c_{\Omega_1,\Omega_2} := p_{\Omega_1\cup\Omega_2} + \sum_{m\notin\Omega_1\cup\Omega_2} q_{m\to\Omega_1\cup\Omega_2} \geq 0$, subject to

$$
Q_{\Omega_1,\Omega_2}(0) = 0. \tag{6.11b}
$$

By Lemma 6.3, $[S_{\Omega_1}] - [S_{\Omega_1,k}] > 0$ and $[S_{\Omega_2}] - [S_{\Omega_2,j}] > 0$. In addition, $q_{k\to\Omega_1} \geq 0$ and $q_{j\to\Omega_2} \geq 0$. Therefore, we have that

$$
\frac{dQ_{\Omega_1,\Omega_2}}{dt} + c_{\Omega_1,\Omega_2} Q_{\Omega_1,\Omega_2} \geq 0, \quad Q_{\Omega_1,\Omega_2}(0) = 0. \tag{6.12}
$$

This differential inequality implies that $Q_{\Omega_1,\Omega_2}(t) \geq 0$ for $t \geq 0$.

Assume by induction that $Q_{\Omega_1,\Omega_2}(t) \geq 0$ for $t \geq 0$ for all Ω_1, Ω_2 for which $|\Omega_1 \cup \Omega_2| = n+1$. Consider Ω_1, Ω_2 for which $|\Omega_1 \cup \Omega_2| = n$. Then, the right-hand side of Eq. (6.10a) is nonnegative. Therefore, the differential inequality (6.12) holds, and so, $Q_{\Omega_1,\Omega_2} \geq 0$ for $t \geq 0$. □

Lemma 6.7: *Consider the Bass/SI model (2.1). Let $\emptyset \neq \Omega_1, \Omega_2 \subset \mathcal{M}$ such that $\Omega_1 \cap \Omega_2 = \emptyset$. Then,*

$$
[S_{\Omega_1\cup\Omega_2}] \geq [S_{\Omega_1}][S_{\Omega_2}], \quad t \geq 0. \tag{6.13}
$$

In addition,

1. *If there exists a node that is influential to both Ω_1 and Ω_2, then*

$$
[S_{\Omega_1\cup\Omega_2}] > [S_{\Omega_1}][S_{\Omega_2}], \quad t > 0. \tag{6.14}
$$

2. *If, however, there is no node that is influential to both Ω_1 and Ω_2, then*

$$
[S_{\Omega_1\cup\Omega_2}] = [S_{\Omega_1}][S_{\Omega_2}], \quad t \geq 0. \tag{6.15}
$$

Proof. Inequality (6.6) is Lemma 6.6. To prove (6.14) and (6.15), we proceed by backward induction on the size of $\Omega_1 \cup \Omega_2$.

Consider the induction base where $\Omega_1 \cup \Omega_2 = \mathcal{M}$. Then, Q_{Ω_1,Ω_2} satisfies Eq. (6.11). Therefore, since the right-hand side of (6.12) is non-negative, see (6.12), then $Q_{\Omega_1,\Omega_2} > 0$ is and only if the right-hand side of (6.11) is positive. By Lemma 6.3, $[S_{\Omega_1}] - [S_{\Omega_1,k}] > 0$ and $[S_{\Omega_2}] - [S_{\Omega_2,j}] > 0$ for all j and k. Hence, $Q_{\Omega_1,\Omega_2}(t)$ is positive for all $t > 0$ if and only if there exist $j \in \Omega_1$ and $k \in \Omega_2$ such that either $q_{j,k} > 0$ or $q_{k\to j} > 0$ so that the inhomogeneous term in the ODE (6.11a) is positive and is identically zero otherwise. This proves the theorem for $\Omega_1 \cup \Omega_2 = \mathcal{M}$ since in this case, the only relevant paths, see Lemma 6.4, are directed edges from Ω_1 to Ω_2 or from Ω_2 to Ω_1.

Now, assume by induction that the lemma holds for all Ω_1, Ω_2 for which $|\Omega_1 \cup \Omega_2| = n+1$. Consider Ω_1, Ω_2 for which $|\Omega_1 \cup \Omega_2| = n$. Since $[S_{\Omega_1}] - [S_{\Omega_1,k}]$ and $[S_{\Omega_2}] - [S_{\Omega_2,j}]$ are both positive (Lemma 6.3) and $Q_{\Omega_1\cup\{m\},\Omega_2}$ and $Q_{\Omega_1,\Omega_2\cup\{m\}}$ are nonnegative by Lemma 6.6, Eq. (6.10) implies that $Q_{\Omega_1,\Omega_2} > 0$ for $t > 0$ if and only if at least one of the following three conditions holds and is identically zero otherwise:

C1. For some $j \in \Omega_1$ and $k \in \Omega_2$, either $q_{j\to k} > 0$ or $q_{k\to j} > 0$.
C2. For some $m \notin \Omega_1 \cup \Omega_2$ and $j \in \Omega_1$, $q_{m\to j} > 0$ and $Q_{\Omega_1\cup\{m\},\Omega_2} > 0$.
C3. For some $m \notin \Omega_1 \cup \Omega_2$ and $k \in \Omega_2$, $q_{m\to k} > 0$ and $Q_{\Omega_1,\Omega_2\cup\{m\}} > 0$.

Therefore, to finish the proof, we need to show that at least one of the conditions C1–C3 holds if and only if there exist a path of the claimed forms in Lemma 6.4.

We first show if any of conditions C1–C3 holds, there exists a path of the claimed form:

- Assume that Condition C1 holds. Then, there exists a single-edge path from Ω_1 to Ω_2 or from Ω_2 to Ω_1.
- Assume that Condition C2 holds. Then, there is an edge from m to $j \in \Omega_1$. In addition, since $Q_{\Omega_1\cup\{m\},\Omega_2} > 0$, then by the induction hypothesis,
  D1. there is a path from $\Omega_1 \cup \{m\}$ to Ω_2, or
  D2. there is a path from Ω_2 to $\Omega_1 \cup \{m\}$, or
  D3. there is a node $\tilde{m} \notin \Omega_1 \cup \{m\} \cup \Omega_2$ and paths from $\tilde{m}$ to $\Omega_1 \cup \{m\}$ and to Ω_2.
  Now,
  (i) If Condition D1 holds, there is either a path from Ω_1 to Ω_2, or there are paths from m to Ω_1 and to Ω_2.
  (ii) If Condition D2 holds, there is a path from Ω_2 to Ω_1, which may or may not go through m.
  (iii) If Condition D3 holds, there is a node $\tilde{m}$ from which there are paths to Ω_1 (which may or may not go through m) and to Ω_2.
  Hence, when Condition C2 holds, there exists a path of the claimed form.
- The proof for Condition C3 is the same as for Condition C2.

To finish the proof, we now show if there exists a path of the claimed form, then at least one of conditions C1–C3 holds.

- If there is a single-edge path between Ω_1 and Ω_2, then Condition C1 holds.
- Assume that there is a path with $L \geq 2$ edges from Ω_1 to Ω_2. Denote by m the next to last node in the path. Then, $m \notin \Omega_1 \cup \Omega_2$, and the path without the last edge is a path from Ω_1 to $\Omega_2 \cup \{m\}$. Since $|\Omega_1 \cup \Omega_2 \cup \{m\}| = |\Omega_1 \cup \Omega_2| + 1$, then by the induction assumption, $Q_{\Omega_1,\Omega_2\cup\{m\}} > 0$. In addition, $q_{m\to k} > 0$ for some $k \in \Omega_2$. Therefore, Condition C3 holds. Similarly, if there is a path with $L > 1$ edges from Ω_2 to Ω_1, then Condition C2 holds.
- Finally, suppose that there is some node $\tilde{m} \notin \Omega_1 \cup \Omega_2$ and paths from $\tilde{m}$ to Ω_1 and to Ω_2. Since the case of a path from Ω_1 to Ω_2 or from Ω_2 to Ω_1 has already been considered, we may assume that the path from $\tilde{m}$ to Ω_1 contains no element of Ω_2 and vice versa. Also, by truncating the paths at the first node reached of the desired set, we may assume that no node of either path except the last belongs to $\Omega_1 \cup \Omega_2$. Let m be the next to last node of the path to Ω_1; note that m might be $\tilde{m}$. Since the path continues from m to Ω_1, then $q_{m\to j} > 0$ for some $j \in \Omega_1$. Moreover, the path from $\tilde{m}$ to m is a path from $\tilde{m}$ to $\Omega_1 \cup \{m\}$, so there exist paths from $\tilde{m}$ to $\Omega_1 \cup \{m\}$ and from $\tilde{m}$ to Ω_2. Therefore, by the induction hypothesis, $Q_{\Omega_1\cup\{m\},\Omega_2} > 0$. Hence, condition C2 holds.

□

Proof of Theorem 6.1. We proceed by induction on L. The induction base $L = 2$ is Lemma 6.7. Assume that Theorem 6.1 holds for L. To prove Theorem 6.1 for $L+1$, let us denote $\widetilde{\Omega}_1 := \bigcup_{l=1}^{L} \Omega_l$ and $\widetilde{\Omega}_2 := \Omega_{L+1}$. Then,

$$[S_{\cup_{l=1}^{L+1}\Omega_l}] = [S_{\widetilde{\Omega}_1,\widetilde{\Omega}_2}] \geq [S_{\widetilde{\Omega}_1}][S_{\widetilde{\Omega}_2}] = [S_{\cup_{l=1}^{L}\Omega_l}][S_{\Omega_{L+1}}] \geq \prod_{l=1}^{L+1}[S_{\Omega_l}],$$

where the first inequality follows from Lemma 6.7 and the second from the induction assumption. By Lemma 6.7, the first inequality is an equality if and only there is no node, which is influential to both $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$. By the induction assumption, the second inequality is an equality if and only if for any $l, \tilde{l} \in \{1, \ldots, L\}$ where $l \neq \tilde{l}$, there is no node in $\mathcal{M}$, which is influential to both Ω_l and $\Omega_{\tilde{l}}$. Therefore, Theorem 6.1 follows for $L+1$. □

C. Proof of Theorem 6.2

We first prove Theorem 6.2 for two sets that are connected by a *single path*:

Lemma 6.8: *Consider the Bass/SI model (2.1) on an undirected network such that (4.5) holds. Let $\emptyset \neq \Omega_1, \Omega_2 \subset \mathcal{M}$ such that $\Omega_1 \cap \Omega_2 = \emptyset$. If there is a unique simple path Γ between Ω_1 and Ω_2, then*

$$[S_{\Omega_1,\Omega_2}] - [S_{\Omega_1}][S_{\Omega_2}] < E(t;K), \quad t > 0, \tag{6.16}$$

where K is the number of nodes of the path Γ, and $E(t;K)$ satisfies the bounds (4.7a) and (4.7b).

Proof. Let $t > 0$. Denote by $\mathcal{N}^-$ the network obtained by deleting the *central edge* of Γ. If K is even, we delete the $\frac{K}{2}$th edge. If K is odd, we delete either the $\frac{K-1}{2}$th or the $\left(\frac{K-1}{2}+1\right)$th edge. In this network, there is no node that is influential to Ω_1 and to Ω_2; see Corollary 6.1. Hence, by Lemma 6.7,

$$[S^-_{\Omega_1,\Omega_2}] = [S^-_{\Omega_1}][S^-_{\Omega_2}],$$

where $[S^-_{\Omega}]$ denotes the susceptibility probability of Ω in $\mathcal{N}^-$. Since the deleted edge is influential to Ω_1 and Ω_2, it follows from the *dominance principle*, see Ref. 16, that

$$[S_{\Omega_1}] < [S^-_{\Omega_1}], \quad [S_{\Omega_2}] < [S^-_{\Omega_2}], \quad [S_{\Omega_1,\Omega_2}] < [S^-_{\Omega_1,\Omega_2}].$$

Therefore,

$$\begin{aligned}
&[S_{\Omega_1,\Omega_2}] - [S_{\Omega_1}][S_{\Omega_2}] \\
&\quad < [S^-_{\Omega_1,\Omega_2}] - [S_{\Omega_1}][S_{\Omega_2}] \\
&\quad = [S^-_{\Omega_1,\Omega_2}] - [S^-_{\Omega_1}][S^-_{\Omega_2}] + [S^-_{\Omega_1}][S^-_{\Omega_2}] - [S_{\Omega_1}][S_{\Omega_2}] \\
&\quad = [S^-_{\Omega_1}][S^-_{\Omega_2}] - [S_{\Omega_1}][S_{\Omega_2}] \\
&\quad = ([S^-_{\Omega_1}] - [S_{\Omega_1}])[S^-_{\Omega_2}] + ([S^-_{\Omega_2}] - [S_{\Omega_2}])[S_{\Omega_1}].
\end{aligned}$$

Since $0 < [S^-_{\Omega_2}], [S_{\Omega_1}] < 1$, see (6.5), we have that

$$[S_{\Omega_1,\Omega_2}] - [S_{\Omega_1}][S_{\Omega_2}] < \left([S^-_{\Omega_1}] - [S_{\Omega_1}]\right) + \left([S^-_{\Omega_2}] - [S_{\Omega_2}]\right). \tag{6.17a}$$

Denote by m_1 and m_2 the nodes of the deleted central edge, which are connected in $\mathcal{N}^-$ to Ω_1 and to Ω_2, respectively, and by $\mathcal{N}^+$ the network that is obtained by transferring the two directional weights of the deleted edge to the nodes m_1 and m_2, i.e., by setting

$$q^+_{m_2\to m_1} = q^+_{m_1\to m_2} = 0, \quad p^+_{m_1} = p^+_{m_2} = p + q.$$

Denote the probabilities in $\mathcal{N}^+$ by $[S^+_\Omega]$. By the dominance principle, $[S_{\Omega_1}] > [S^+_{\Omega_1}]$ and $[S_{\Omega_2}] > [S^+_{\Omega_2}]$. Hence,

$$\left[S^-_{\Omega_1}\right] - [S_{\Omega_1}] < \left[S^-_{\Omega_1}\right] - \left[S^+_{\Omega_1}\right], \quad \left[S^-_{\Omega_2}\right] - [S_j] < \left[S^-_{\Omega_2}\right] - \left[S^+_{\Omega_2}\right]. \tag{6.17b}$$

Combining inequalities (6.17), we obtain (6.16) with

$$E(t;K) := \left([S^-_{\Omega_1}] - [S^+_{\Omega_1}]\right) + \left([S^-_{\Omega_2}] - [S^+_{\Omega_2}]\right).$$

Next, we derive the bound (4.7a) for $E(t;K)$. Denote by $i_1 \in \Omega_1$ and $i_2 \in \Omega_2$ the end nodes of the path Γ. The difference $[S^-_{\Omega_1}] - [S^+_{\Omega_1}]$ is only due to realizations in which i_1 adopts because of an *adoption path* from m_1 to Ω_1 in $\mathcal{N}^+$, but not in $\mathcal{N}^-$. Therefore,

$$[S^-_{\Omega_1}] - [S^+_{\Omega_1}] < [S^-_{i_1}] - [S^+_{i_1}].$$

Denote by $\widetilde{\mathcal{N}}^+$ and $\widetilde{\mathcal{N}}^-$ the networks obtained from $\mathcal{N}^+$ and $\mathcal{N}^-$ by keeping only the nodes and edges from m_1 to i_1 and denote the probabilities in $\widetilde{\mathcal{N}}^\pm$ by $[\widetilde{S}^\pm_\Omega]$. Then,

$$[S^-_{i_1}] - [S^+_{i_1}] < [\widetilde{S}^-_{i_1}] - [\widetilde{S}^+_{i_1}].$$

The networks $\widetilde{\mathcal{N}}^+$ and $\widetilde{\mathcal{N}}^-$ are the homogeneous and heterogeneous one-sided lines with $p_1 = p$ and $\bar{p}_1 := p + q$ that were defined in Lemma 15 of Ref. 26, and the number of nodes in these lines is either $\lfloor \frac{K}{2} \rfloor$ or $\lfloor \frac{K}{2} \rfloor + 1$. Therefore, by Eq. (34) in Ref. 26,

$$\begin{aligned}
&\left[S^-_{\Omega_1}\right] - \left[S^+_{\Omega_1}\right] < \left[S^-_{i_1}\right] - \left[S^+_{i_1}\right] \\
&\qquad < \left[\widetilde{S}^-_{i_1}\right] - \left[\widetilde{S}^+_{i_1}\right] < (1 - I^0)e^{-(p+q)t}\left(\frac{eqt}{\lfloor \frac{K}{2} \rfloor}\right)^{\lfloor \frac{K}{2} \rfloor}, \\
&\qquad \left\lfloor \frac{K}{2} \right\rfloor > qt.
\end{aligned} \tag{6.17c}$$

The same bound also holds for $[S^-_{\Omega_2}] - [S^+_{\Omega_2}]$. Therefore, we obtain (4.7a).

Finally, to prove the globally uniform upper bound (4.7b), we note that, by Lemma 6.2,

$$\left[S^-_{\Omega_1}\right] - \left[S^+_{\Omega_1}\right] < (1 - I^0)e^{-pt}, \quad t > 0. \tag{6.18}$$

As in the proof of Corollary 3 in Ref. 26 from inequalities (6.17c) and (6.18), it follows that

$$\left[S^-_{\Omega_1}\right] - \left[S^+_{\Omega_1}\right] < (1 - I^0)\left(\frac{q}{p+q}\right)^{\lfloor \frac{K}{2} \rfloor}, \quad t > 0. \tag{6.19}$$

The same bound also holds for $[S^-_{\Omega_2}] - [S^+_{\Omega_2}]$. Therefore, we have (4.7b). □

Next, we consider two disjoint sets that are connected by N paths:

Lemma 6.9: *Consider the Bass/SI model (2.1) on an undirected network such that (4.5) holds. Let $\emptyset \neq \Omega_1, \Omega_2 \subset \mathcal{M}$ such that $\Omega_1 \cap \Omega_2 = \emptyset$. If there are $N \geq 2$ distinct simple paths $\{\Gamma_n\}_{n=1}^N$ between Ω_1 and Ω_2 such that their interior nodes are in $\mathcal{M}\setminus(\Omega_1 \cup \Omega_2)$, then*

$$[S_{\Omega_1,\Omega_2}] - [S_{\Omega_1}][S_{\Omega_2}] < \sum_{n=1}^{N} E(t;K_n), \quad t > 0, \tag{6.20}$$

where K_n is the number of nodes of the path Γ_n, and $E(t;K_n)$ satisfies the bounds (4.7a) and (4.7b).

Proof. Denote the end nodes of the path Γ_n by $i_{1,n} \in \Omega_1$ and $i_{2,n} \in \Omega_2$. *Assume first that the N paths are disjoint*, i.e., that do not share interior nodes (they may share, however, the end nodes $\{i_{1,n}\}$ and $\{i_{2,n}\}$). Denote by $\mathcal{N}^-$ the network obtained by deleting the N central edges of $\{\Gamma_n\}_{n=1}^N$. Then, as in the proof of Lemma 6.8, see (6.17a),

$$[S_{\Omega_1,\Omega_2}] - [S_{\Omega_1}][S_{\Omega_2}] < \left(\left[S^-_{\Omega_1}\right] - [S_{\Omega_1}]\right) + \left(\left[S^-_{\Omega_2}\right] - [S_{\Omega_2}]\right). \tag{6.21a}$$

For $n = 1, \ldots, N$, denote by $m_{1,n}$ and $m_{2,n}$ the nodes of the deleted central edge of Γ_n, which are connected in $\mathcal{N}^-$ to Ω_1 and to Ω_2, respectively. Denote by $\mathcal{N}^+$ the network obtained by transferring the $2n$ directional weights of the deleted edges to the $2n$ nodes of these edges, i.e.,

$$q^+_{m_{2,n}\to m_{1,n}} = q^+_{m_{1,n}\to m_{2,n}} = 0, \quad p^+_{m_{1,n}} = p^+_{m_{2,n}} = p + q, \quad n = 1, \ldots, N.$$

As in the proof of Lemma 6.8, see (6.17b),

$$\left[S^-_{\Omega_1}\right] - [S_{\Omega_1}] < \left[S^-_{\Omega_1}\right] - \left[S^+_{\Omega_1}\right]. \tag{6.21b}$$

The difference between $[S^-_{\Omega_1}]$ and $[S^+_{\Omega_1}]$ is due to realizations in which Ω_1 adopts because of one of the N adoption paths from $m_{1,n}$ to Ω_1 in $\mathcal{N}^+$, but not in $\mathcal{N}^-$. Therefore, it is bounded by the sum of the individual differences in $[S_{\Omega_1}]$ due to each of these N paths, i.e.,

$$\left[S^-_{\Omega_1}\right] - \left[S^+_{\Omega_1}\right] \leq \sum_{n=1}^{N} \left([S^-_{\Omega_1}] - [S^{+,n}_{\Omega_1}]\right), \tag{6.21c}$$

where $[S^{+,n}_{\Omega_1}]$ is the susceptibility probability of Ω_1 in the network $\mathcal{N}^{+,n}$, that is obtained from $\mathcal{N}^-$ by setting

$$q^+_{m_{2,n}\to m_{1,n}} = q^+_{m_{1,n}\to m_{2,n}} = 0, \quad p^{+,n}_{m_{1,n}} = p^{+,n}_{m_{2,n}} = p + q. \tag{6.22}$$

Combining inequalities (6.21), and noting that (6.21b) and (6.21c) also hold for Ω_2, we obtain

$$[S_{\Omega_1,\Omega_2}] - [S_{\Omega_1}][S_{\Omega_2}] < \sum_{n=1}^{N} E(t;K_n), \tag{6.23}$$
$$E(t;K_n) := \left([S^-_{\Omega_1}] - [S^{+,n}_{\Omega_1}]\right) + \left([S^-_{\Omega_2}] - [S^{+,n}_{\Omega_2}]\right).$$

Let us now derive the bounds (4.7a) and (4.7b) for $E(t;K_n)$. By (6.17c) and (6.22),

$$[S^-_{\Omega_1}] - [S^{+,n}_{\Omega_1}] < (1-I^0)e^{-(p+q)t}\left(\frac{eqt}{\left\lfloor\frac{K_n}{2}\right\rfloor}\right)^{\left\lfloor\frac{K_n}{2}\right\rfloor}, \quad \left\lfloor\frac{K_n}{2}\right\rfloor > qt. \tag{6.24}$$

The same bound also holds for $[S^-_{\Omega_2}] - [S^{+,n}_{\Omega_2}]$. Therefore, we obtain (4.7a). Finally, as in the proof of Lemma 6.8, for all $t > 0$, we have that

$$[S^-_{\Omega_1}] - [S^{+,n}_{\Omega_1}] < (1-I^0)e^{-pt}, \quad t > 0.$$

From this inequality and (6.24), it follows that

$$[S^-_{\Omega_1}] - [S^{+,n}_{\Omega_1}] < (1-I^0)\left(\frac{q}{p+q}\right)^{\left\lfloor\frac{K_n}{2}\right\rfloor}, \quad t > 0.$$

The same bound also holds for $[S^-_{\Omega_2}] - [S^{+,n}_{\Omega_2}]$. Hence, we obtain (4.7b).

Consider now the case where the paths $\{\Gamma_n\}_{n=1}^N$ *are not disjoint.* Note that in this case, $\{\Gamma_n\}_{n=1}^N$ refers to all the possible paths between Ω_1 and Ω_2. Thus, for example, two paths that intersect at a single node are counted as four different paths. Similarly, if two paths merge into a single path, then separate, then merge into a single path, they are also counted as four paths. Without loss of generality, we can assume that the paths are arranged in order of increasing length so that $K_1 \le K_2 \cdots \le K_N$. We construct the network $\mathcal{N}^-$ iteratively, as follows. For $n = 1, \dots, N$, if after the $n-1$th iteration all the edges of the path Γ_n still exist, we delete the central edge of Γ_n. At the end of this iterative process, all the N paths are disconnected, and so the sets Ω_1 and Ω_2 are disjoint in $\mathcal{N}^-$. Therefore, $[S^-_{\Omega_1,\Omega_2}] = [S^-_{\Omega_1}][S^-_{\Omega_2}]$, and so (6.21a) holds. As before, let $\mathcal{N}^+$ be obtained from $\mathcal{N}^-$ by increasing the weights of the nodes of the deleted edges from p to $p+q$. Then, (6.21b) holds. We claim that the bound (6.21c) also holds, and therefore, that (6.23) holds.

To prove that (6.21c) still holds, we first note that, as in the case of disjoint paths, the difference $[S^-_{\Omega_1}] - [S^+_{\Omega_1}]$ is only due to the adoption paths that start from the nodes of the deleted edges and reach Ω_1 in $\mathcal{N}^-$ (and in $\mathcal{N}^+$). Unlike the case of disjoint paths, however, in the networks $\mathcal{N}^-$ and $\mathcal{N}^+$, there can be more than one adoption path from a node of a deleted edge to Ω_1. Moreover, these adoption paths can intersect or even share edges. Nevertheless, since the overall contribution to $[S^-_{\Omega_1}] - [S^+_{\Omega_1}]$ from these adoption paths is due to realizations in which Ω_1 adopts because of one of these adoption paths in $\mathcal{N}^+$ but not in $\mathcal{N}^-$, it is still bounded by the contribution due to each of these adoption paths separately. Therefore, we now consider the separate contribution of each of these adoption paths.

Assume that in the nth iteration in the construction of $\mathcal{N}^-$, we deleted the central edge $m_{1,n} \leftrightarrow m_{2,n}$ of the path Γ_n. Denote by Γ^1_n and Γ^2_n the equal-length subpaths of Γ_n between Ω_1 and $m_{1,n}$ and between $m_{2,n}$ and Ω_2, respectively, i.e.,

$$\Gamma_n = \Gamma^1_n \leftrightarrow m_{1,n} \leftrightarrow m_{2,n} \leftrightarrow \Gamma^2_n.$$

In the network $\mathcal{N}^+$, the node $m_{1,n}$ has weight $p+q$. The contribution to the difference $[S^-_{\Omega_1}] - [S^+_{\Omega_1}]$ of the adoption path from $m_{1,n}$ to Ω_1 through Γ^1_n is bounded by the nth term in the sum (6.21c). We also need to consider, however, the possibility that in the network $\mathcal{N}^+$, the node $m_{1,n}$ is connected to Ω_1 through another subpath, which we denote by $\widetilde{\Gamma}^1$. Let us also denote the path between Ω_1 and Ω_2, which is made of $\widetilde{\Gamma}^1$ and Γ^2_n by $\Gamma_{\tilde{n}}$; i.e.,

$$\Gamma_{\tilde{n}} := \widetilde{\Gamma}^1 \leftrightarrow m_{1,n} \leftrightarrow m_{2,n} \leftrightarrow \Gamma^2_n.$$

- If $\widetilde{\Gamma}^1$ is shorter than Γ^1_n, the path $\Gamma_{\tilde{n}}$ is shorter than Γ_n. Since the subpath $\widetilde{\Gamma}^1$ exists in $\mathcal{N}^+$, the path $\Gamma_{\tilde{n}}$ exists at the beginning of the nth iteration. This, however, is in contradiction with the iterative construction of $\mathcal{N}^-$ since $\Gamma_{\tilde{n}}$ is shorter than Γ_n.
- If $\widetilde{\Gamma}^1$ is longer than Γ^1_n, the path $\Gamma_{\tilde{n}}$ is longer than Γ_n, and so $\tilde{n} > n$. At the $\tilde{n}$th iteration, the path $\Gamma_{\tilde{n}}$ does not exist (since we already deleted the edge $m_{1,n} \leftrightarrow m_{2,n}$). In the sum (6.21c), however, we accounted for the impact of deleting the central edge of $\Gamma_{\tilde{n}}$ by the term with $K_{\tilde{n}}$. This term is larger than the one needed for the impact of the node $m_{1,n}$ on $[S^-_{\Omega_1}] - [S^+_{\Omega_1}]$ through $\widetilde{\Gamma}^1$, since $\widetilde{\Gamma}^1$ is longer than Γ^2_n, and so the central edge of $\Gamma_{\tilde{n}}$ lies inside Γ^1_n (i.e., is closer to Ω_1 than $m_{1,n}$).
- If $\widetilde{\Gamma}^1$ has the same length as Γ^1_n, then we can assume without loss of generality that $\hat{n} > n$, and therefore, a similar argument holds.

Finally, we need to rule out the possibility that in the network $\mathcal{N}^+$ (in which the edge $m_{1,n} \leftrightarrow m_{2,n}$ has been deleted), the node $m_{2,n}$ is also connected to Ω_1. Indeed, assume by contradiction that $m_{2,n}$ is connected to Ω_1 in $\mathcal{N}^+$. Since there is no path between Ω_1 and Ω_2 in $\mathcal{N}^+$, this implies that there is no path between $m_{2,n}$ and Ω_2 in $\mathcal{N}^+$. Since, however, the path Γ^2_n between $m_{2,n}$ and Ω_2 exists at the end of the nth iteration, this implies that at some later iteration $\hat{n} > n$, the path Γ^2_n became disconnected because one of the edges was deleted. This deleted edge is the central edge $m_{1,\hat{n}} \leftrightarrow m_{2,\hat{n}}$ of the path

$$\Gamma_{\hat{n}} := \Gamma^1_{\hat{n}} \leftrightarrow m_{1,\hat{n}} \leftrightarrow m_{2,\hat{n}} \leftrightarrow \Gamma^2_{\hat{n}}.$$

Since the central edge of $\Gamma_{\hat{n}}$ was deleted at the $\hat{n}$th iteration, the path $\Gamma_{\hat{n}}$ existed at the beginning of the $\hat{n}$th iteration. Let $\bar{\Gamma}^2$ denote the subpath of Γ^2_n between $m_{2,\hat{n}}$ and Ω_2. Then, at the beginning of the $\hat{n}$th iteration, the path

$$\bar{\Gamma} := \Gamma^1_{\hat{n}} \leftrightarrow m_{1,\hat{n}} \leftrightarrow m_{2,\hat{n}} \leftrightarrow \bar{\Gamma}^2$$

between Ω_1 and Ω_2 also exists. The length of $\bar{\Gamma}$, however, is shorter than that of $\Gamma_{\hat{n}}$. Indeed, since $\bar{\Gamma}^2$ is a proper subpath of Γ^2_n, it is shorter than Γ^2_n, which in turn is shorter than $\Gamma^2_{\bar{n}}$ (since $n < \bar{n}$). Therefore, we reached a contradiction, since the central edge of $\bar{\Gamma}$ should have been deleted before that of $\Gamma_{\hat{n}}$, and so $\bar{\Gamma}$ could not exist at the beginning of the $\hat{n}$th iteration. □

Proof of Theorem 6.2. When $N_L \geq 1$, it follows from Lemma 6.4 that there exists a node that is influential to both Ω_1 and Ω_2. Therefore, the lower bound follows from Theorem 6.1.

For the upper bound, we proceed by induction on L. The case $L = 2$ is Lemma 6.9. Assume that (6.4) holds for L. Consider (6.4) for $L+1$. We can reorder the N_{L+1} paths among $\{\Omega_1, \dots, \Omega_{L+1}\}$ so that the first N_L paths are among $\{\Omega_1, \dots, \Omega_L\}$, and the paths between $\{\Omega_1, \dots, \Omega_L\}$ and Ω_{L+1} are enumerated from $N_L + 1$ to N_{L+1}. Therefore, by (6.20) with $\widetilde{\Omega}_1 := \bigcup_{l=1}^{L} \Omega_l$ and $\widetilde{\Omega}_2 := \Omega_{L+1}$,

$$\begin{aligned}&[S_{\Omega_1,\dots,\Omega_{L+1}}] - [S_{\Omega_1,\dots,\Omega_L}][S_{\Omega_{L+1}}]\\&\quad = [S_{\widetilde{\Omega}_1,\widetilde{\Omega}_2}] - [S_{\widetilde{\Omega}_1}][S_{\widetilde{\Omega}_2}] < \sum_{n=N_L+1}^{N_{L+1}} E(t; K_n).\end{aligned}$$

Hence, since $[S_{\Omega_{L+1}}] \leq 1$,

$$\begin{aligned}&[S_{\Omega_1,\dots,\Omega_{L+1}}] - \prod_{l=1}^{L+1}[S_{\Omega_l}]\\&= \left([S_{\Omega_1,\dots,\Omega_{L+1}}] - [S_{\Omega_1,\dots,\Omega_L}][S_{\Omega_{L+1}}]\right)\\&\quad + \left([S_{\Omega_1,\dots,\Omega_L}][S_{\Omega_{L+1}}] - \prod_{l=1}^{L+1}[S_{\Omega_l}]\right)\\&< \sum_{n=N_L+1}^{N_{L+1}} E(t; K_n) + [S_{\Omega_{L+1}}]\left([S_{\Omega_1,\dots,\Omega_L}] - \prod_{l=1}^{d}[S_{\Omega_l}]\right)\\&< \sum_{n=N_L+1}^{N_{L+1}} E(t; K_n) + \sum_{n=1}^{N_L} E(t; K_n).\end{aligned}$$

Therefore, we have (6.4). □

VII. PROVING THE FUNNEL THEOREMS

We are finally in a position to prove the funnel theorems. The adoption/infection of node j in network $\mathcal{N}$ is due to one of the following $L+1$ distinct influences:

1. Internal influences on j by edges that arrive from A_l for some $l \in \{1, \dots, L\}$.
2. External influences on j.

In order to identify the specific influence that leads to the adoption of j, we introduce

Definition 7.1 $\left(\mathcal{N}^{A_l}\right.$ **and** $\left.\left[S_j^{A_l}\right]\right)$**:** Let $j \in \mathcal{M}$ and $A_l \subset \mathcal{M} \setminus \{j\}$. The network $\mathcal{N}^{A_l}$ is obtained from $\mathcal{N}$ by deleting all the external influences on node j and all the internal influences on j by nodes that are not in A_l, i.e., by setting $I_j^{0,A_l} := 0$, $p_j^{A_l} := 0$, and $q_{k \to j}^{A_l} := 0$ for $k \in \mathcal{M} \setminus A_l$. The susceptibility probability of j in the network $\mathcal{N}^{A_l}$ is denoted by $\left[S_j^{A_l}\right](t) := [S_j]\left(t; \mathcal{N}^{A_l}\right)$.

The *funnel inequality* shows that the susceptibility probability $[S_j]$ is bounded from below by the product of the susceptibility probabilities of j due to each of the $L+1$ distinct influences:

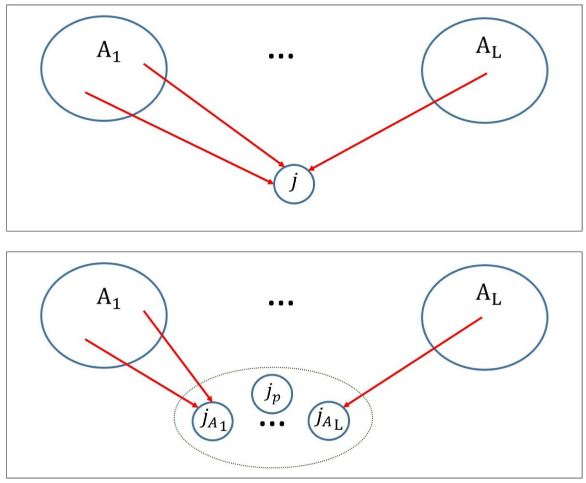

FIG. 5. In the proof of Theorem 7.1, the original network $\mathcal{N}$ (top) is replaced by the network $\mathcal{N}^+$ (bottom) in which the node j is replaced by the $L+1$ nodes $\{j_p, j_{A_1}, \dots, j_{A_L}\}$ such that j_{p,l^0} inherits the external influences on j and j_{A_l} inherits the internal influences on j from A_l for $l = 1, \dots, L$.

Theorem 7.1: *Consider the Bass/SI model (2.1). Let $\{A_1, \dots, A_L, \{j\}\}$ be a partition of $\mathcal{M}$. Then,*

$$[S_j] \geq \left[S_j^{\text{isolated}}\right] \prod_{l=1}^{L} \left[S_j^{A_l}\right], \quad t \geq 0, \tag{7.1}$$

where $[S_j^{\text{isolated}}] = (1 - I_j^0)e^{-p_j t}$.

Proof. The susceptibility probability $[S_j^{\text{isolated}}]$ for an isolated node follows from the master equations (3.3a) and (3.3b). To prove (7.1), we first note that by the indifference principle (Theorem 3.2), all the edges that emanate from j are non-influential to j. Since this holds for all the $L+2$ probabilities in (7.1), in what follows, *we can assume that no edges emanate from j.*

In principle, we need to compute the $L+2$ probabilities in (7.1) on the $L+2$ networks $\mathcal{N}$, $\mathcal{N}^{A_1}$, …, $\mathcal{N}^{A_L}$, and $\mathcal{N}^{p_j}$. We can simplify the analysis, however, by considering only two networks, as follows. Given the original network $\mathcal{N}$, we define the network $\mathcal{N}^+$ by "splitting" node j into the $L+1$ nodes $\{j_{A_1}, \dots, j_{A_L}, j_p\}$ such that (see Fig. 5):

1. j_{A_l} inherits from j the directed edges from A_l to j, i.e.,
$$[S_{j_{A_l}}^+](0) := 1, \quad p_{j_{A_l}}^+ := 0, \quad q_{k \to j_{A_l}}^+ := q_{k \to j} \mathbb{1}_{k \in A_l},$$
$$k \in \mathcal{M}, \; i = 1, \dots, K.$$
2. j_p inherits from j its weight and initial condition, i.e.,
$$[S_{j_p}^+](0) := 1 - I_j^0, \quad p_{j_p}^+ := p_j, \quad q_{k \to j_p}^+ := 0, \quad k \in \mathcal{M}.$$
3. Since no edges emanate from j in network $\mathcal{N}$, no edges emanate from $j_{A_1}, \dots, j_{A_L}$, and j_p in network $\mathcal{N}^+$.
4. The weights of the nodes $\mathcal{M} \setminus \{j\}$, and of the edges among these nodes, are the same in $\mathcal{N}$ and in $\mathcal{N}^+$.

Let $X_k^+(t)$ denote the state of node k in network $\mathcal{N}^+$, and let $[S_k^+] := \mathbb{P}(X_k^+(t) = 0)$. By construction,

$$\left[S_j^{\text{isolated}}\right] = \left[S_{j_p}^+\right], \quad \left[S_j^{A_l}\right] = \left[S_{j_{A_l}}^+\right], \quad l \in \{1, \ldots, L\}. \tag{7.2}$$

In the Appendix, we prove that

$$[S_j] = \left[S_{j_{A_1}, \ldots, j_{A_L}, j_p}^+\right], \tag{7.3}$$

where $[S_{j_{A_1}, \ldots, j_{A_L}, j_p}^+] := \mathbb{P}\left(X_{j_{A_1}}^+(t) = \cdots = X_{j_{A_L}}^+(t) = X_{j_p}^+(t) = 0\right)$. Since j_p is an isolated node in $\mathcal{N}^+$, its adoption is independent of that of $j_{A_1}, \ldots, j_{A_L}$, and so

$$\left[S_{j_{A_1}, \ldots, j_{A_L}, j_p}^+\right] = \left[S_{j_p}^+\right]\left[S_{j_{A_1}, \ldots, j_{A_L}}^+\right]. \tag{7.4}$$

Applying Theorem 6.1 to network $\mathcal{N}^+$ gives

$$\left[S_{j_{A_1}, \ldots, j_{A_L}}^+\right] \geq \prod_{l=1}^{L}\left[S_{j_{A_l}}^+\right]. \tag{7.5}$$

Combining relations (7.3)–(7.5) gives

$$[S_j] \geq \left[S_{j_p}^+\right]\prod_{l=1}^{L}\left[S_{j_{A_l}}^+\right]. \tag{7.6}$$

Substituting (7.2) in (7.6) proves (7.1). ☐

Lemma 7.1: *Consider the Bass/SI model (2.1). Let $j \in \mathcal{M}$, and let $\{A_1, \ldots, A_L, \{j\}\}$ be a partition of $\mathcal{M}$.*

- *If j is a funnel node of $\{A_1\}_{l=1}^L$, then*

$$[S_j] = \left[S_j^{\text{isolated}}\right]\prod_{l=1}^{L}\left[S_j^{A_l}\right], \quad t \geq 0 \quad \textbf{(funnel equality)}. \tag{7.7}$$

- *If, however, j is not a funnel node of A_1 and A_2, then*

$$[S_j] > \left[S_j^{\text{isolated}}\right]\prod_{l=1}^{L}\left[S_j^{A_l}\right], \quad t > 0 \quad \textbf{(strict funnel inequality)}. \tag{7.8}$$

Proof. The inequality sign in the derivation of the funnel inequality (7.1) only comes from the use of Theorem 6.1 in obtaining (7.5). By Theorem 6.1, inequality (7.5) is strict if and only if there exist $i_1, i_2 \in \{1, \ldots, L\}$ and a node $m \in \mathcal{M}$ that is influential to both j_{A_l} and to $j_{A_{\tilde{l}}}$, where $i_1 \neq i_2$. Since no edges emanate from $j_{A_1}, \ldots, j_{A_L}$, and j_p, we have that $m \in \mathcal{M}\setminus\{j\}$.

Thus, the funnel inequality is strict if and only if there exists a node $m \in \mathcal{M}\setminus\{j\}$ in $\mathcal{N}^+$, which is influential to j_{A_l} and to $j_{A_{\tilde{l}}}$. This, however, is the case if and only if there exists a node $m \in \mathcal{M}\setminus\{j\}$, which is influential to j in $\mathcal{N}^{A_l}$ and in $\mathcal{N}^{A_{\tilde{l}}}$, i.e., if j is not a funnel node of A_l and $A_{\tilde{l}}$. ☐

We can use the funnel equality to compute the combined influences from A_l and p_j:

Lemma 7.2: *Consider the Bass/SI model (2.1). Let $j \in \mathcal{M}$ and $A_l \subset \mathcal{M}\setminus\{j\}$. Then,*

$$\left[S_j^{A_l, p_j, I_j^0}\right] = \left[S_j^{A_l}\right]\left[S_j^{\text{isolated}}\right], \quad l \in \{1, \ldots, L\}, \quad t \geq 0, \tag{7.9}$$

where $[S_j^{A_l, p_j, I_j^0}] := [S_j](t; \mathcal{N}^{A_l, p_j, I_j^0})$.

Proof. Let $\widehat{\mathcal{N}}$ denote the network obtained from $\mathcal{N}^{A_l, p_j, I_j^0}$ by adding a fictitious isolated note, denoted by $M+1$. Let $\widehat{\mathcal{M}} := \{1, \ldots, M+1\}$, $B_1 := \mathcal{M}\setminus\{j\}$, and $B_2 := \{M+1\}$. Then, $\{B_1, B_2, \{j\}\}$ is a partition of $\widehat{\mathcal{M}}$, and j is a vertex cut, hence a funnel node, of B_1 and B_2 in $\widehat{\mathcal{N}}$.

Let $\widehat{X}_j$ denote the state of j in $\widehat{\mathcal{N}}$. By the funnel equality (7.7),

$$[\widehat{S}_j] = [\widehat{S}_j^{B_1}][\widehat{S}_j^{B_2}][\widehat{S}_j^{\text{isolated}}].$$

By construction,

$$[\widehat{S}_j] = [S_j^{A_l, p_j, I_j^0}], \quad [\widehat{S}_j^{B_1}] = [\widehat{S}_j^{\mathcal{M}\setminus\{j\}}] = [S_j^{A_l}],$$
$$[\widehat{S}_j^{B_2}] = [\widehat{S}_j^{\{M+1\}}] \equiv 1, \quad [\widehat{S}_j^{\text{isolated}}] = [S_j^{\text{isolated}}],$$

where $[S_j^U]$ denotes the state of j in network $\mathcal{N}^U$. Therefore, $[S_j^{A_l, p_j, I_j^0}] = [S_j^{A_l}][S_j^{\text{isolated}}]$. ☐

Proof of Theorems 4.1 and 4.2. These theorems follow from Theorem 7.1 and Lemmas 7.1 and 7.2. ☐

Proof of Theorem 4.3. The left inequality follows from (4.4). To prove the upper bound, we use the notations from the proof of Theorem 7.1. By relations (7.2)–(7.4),

$$[S_j] = [S_j^{\text{isolated}}][S_{j_{A_1}, \ldots, j_{A_L}}^+].$$

Recall that node j in network $\mathcal{N}$ is split into nodes $\{j_{A_1}, \ldots, j_{A_L}, j_{p_j}\}$ in network $\mathcal{N}^+$. Hence, the cycle C_n corresponds to a path Γ_n^+ in $\mathcal{N}^+$ between some j_{A_l} and $j_{A_{\tilde{l}}}$ that has $K_n + 1$ nodes (including j_{A_l} and $j_{A_{\tilde{l}}}$). Therefore, by Lemma 6.9,

$$[S_{j_{A_1}, \ldots, j_{A_L}}^+] - \prod_{l=1}^{L}[S_{j_{A_l}}^+] < \sum_{n=1}^{N_j} E(t; K_n + 1), \quad t > 0.$$

Multiplying this inequality by $[S_j^{\text{isolated}}]$ and using the fact that $[S_{j_{A_l}}^+] = [S_j^{A_l}]$, see (7.2), we obtain

$$[S_j] - [S_j^{\text{isolated}}]\prod_{l=1}^{L}[S_j^{A_l}] < [S_j^{\text{isolated}}]\sum_{n=1}^{N_j} E(t; K_n + 1), \quad t > 0.$$

Since $[S_j^{A_l, p_j, I_j^0}] = [S_j^{A_l}][S_j^{\text{isolated}}]$, see (7.9), the inequality (4.6) follows. ☐

VIII. DISCUSSION

The main theoretical contributions of this study are the three funnel theorems. These theorems provide a framework for calculating a node's susceptibility probability by relating it to susceptibility probabilities in related networks where the node's indegree is reduced (typically to one)—which is a more tractable problem.

Kiss *et al.*[29] derived the funnel equality (7.7) for nodes that are vertex cuts (in the SIR model). Our funnel theorems are more general in two aspects. First, we show that an equality holds not only when the node is a vertex cut, but also when the node is a funnel node, which is not a vertex cut. Second, when the node is not a funnel node, we obtain lower and upper bounds for the funnel inequality. As noted, these bounds enable us to show that the funnel equality becomes exact on some infinite sparse networks with numerous cycles.

The relation between the funnel (in)equality and the sign and magnitude of $[S_{\cup_{l=1}^{L}\Omega_l}] - \prod_{l=1}^{L}[S_{\Omega_l}]$ was not noted in previous studies. Moreover, while it has been proved that $[S_{\cup_{l=1}^{L}\Omega_l}] \geq \prod_{l=1}^{L}[S_{\Omega_l}]$, see Ref. 30, to the best of our knowledge, the necessary and sufficient condition under which this inequality is strict (Theorem 6.1), as well as the upper bound for $[S_{\cup_{l=1}^{L}\Omega_l}] - \prod_{l=1}^{L}[S_{\Omega_l}]$ (Theorem 6.2), were not obtained before. The inequality $[S_{\cup_{l=1}^{L}\Omega_l}] \geq \prod_{l=1}^{L}[S_{\Omega_l}]$ can serve as an analytic tool, beyond its role in the derivation of the funnel theorems. For example, in Theorem 3 of Ref. 31, it was used to compute the universal upper bound for the expected adoption level in the Bass model on networks.

This study only considers the Bass and SI models on networks. It is reasonable to expect that the funnel theorems can be extended to more comprehensive models in epidemiology. As noted, Kiss *et al.*[29] derived the funnel equality for nodes that are vertex cuts in the SIR model. The results of this study are likely to be extendable to the SIS model and to the Bass-SIR model.[32]

The Bass and SI models on networks only allow for pairwise interactions between individuals. In recent years, however, there has been increasing interest in modeling group interactions, which leads to the study of spreading processes on hypernetworks.[33] The master-equation methodology can be extended to the Bass and SI models on hypernetworks, enabling the derivation of explicit solutions for such systems.[34] In particular, the funnel theorems can be adapted to the Bass and SI models on hypernetworks, providing an analytic tool for analyzing the spreading dynamics in the presence of higher-order interactions.[35]

The extension of epidemiological models to non-Markovian processes is an important question that received much attention; see Ref. 3 and the numerous references therein. Whether and to what extent the results of this study can be extended to non-Markovian processes is currently an open question.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Gadi Fibich: Formal analysis (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal). **Tomer Levin:** Formal analysis (equal); Methodology (equal); Writing – original draft (equal). **Steven Schochet:** Formal analysis (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX: PROOF OF (7.3)

Let us fix $t > 0$ and $N \in \mathbb{N}$. Let $\Delta t = \frac{t}{N}$, $t^N := N\Delta t$, and $X_j^n := X_j(t^n)$. As $N \to \infty$, $\Delta t \to 0$ and $t^N \equiv t$. Then, we need to prove that

$$\lim_{N\to\infty} [S_j](t^N;\Delta t) = \lim_{N\to\infty} \left[S^+_{j_{A_1},\ldots,j_{A_L},j_p}\right](t^N;\Delta t). \tag{A.1}$$

To do this, we introduce the following implementation of the Bass/SI model (2.1):

```
Choose Δt > 0
for j = 1,...,M
    sample ω_j^0 ~ U(0,1)
     if 0 ≤ ω_j^0 ≤ I_j^0  then X_j^0 := 1  else X_j^0 := 0
end
for n = 1,2,...
    for j = 1,...,M
        if X_j^{n-1} = 1 then X_j^n := 1
        if X_j^{n-1} = 0 then
          sample ω_j^n ~ U(0,1)
          if 0 ≤ ω_j^n ≤ (p_j + Σ_{k∈M} q_{k→j} X_k^{n-1}) Δt then X_j^n := 1
          else X_j^n := 0
    end
end
```

Let us denote the outcome of this implementation by

$$\widetilde{X}_k^N := X_k(t^N;\{\boldsymbol{\omega}^n\}_{n=0}^{\infty},\Delta t), \quad k\in\mathcal{M}, \quad N=0,1,\ldots,$$

where $\boldsymbol{\omega}^n := \{\omega_k^n\}_{k\in\mathcal{M}}$. Let us also denote

$$\boldsymbol{\omega}_{-j}^n := \{\omega_k^n\}_{k\in\mathcal{M}\setminus\{j\}}, \quad \boldsymbol{\omega}^{+,n} = \{\boldsymbol{\omega}_{-j}^n,\omega_{j_{A_1}}^n,\ldots,\omega_{j_{A_L}}^n,\omega_{j_p}^n\},$$

$$\mathcal{M}^+ := (\mathcal{M}\setminus j)\cup\{j_{A_1},\ldots,j_{A_L},j_p\}.$$

The implementation of the Bass/SI model (2.1) on $\mathcal{N}^+$ is denoted by

$$\widetilde{X}_k^{+,N} := X_k^+(t^N;\{\boldsymbol{\omega}^{+,n}\}_{n=0}^{\infty},\Delta t), \quad k\in\mathcal{M}^+, \quad N=0,1,\ldots,$$

where the $L+2$ realizations $\omega_j^n,\omega_{j_{A_1}}^n,\ldots,\omega_{j_{A_L}}^n,\omega_{j_p}^n$ are independent.

Since there are no edges that emanate from the nodes $j, j_{A_1},\ldots,j_{A_L},j_p$, the sub-realizations $\{\boldsymbol{\omega}_{-j}^n\}_{n=0}^{\infty}$ completely determine $\{\widetilde{X}_k^N\}$ and $\{\widetilde{X}_k^{+,N}\}$ for all $k\in\mathcal{M}\setminus\{j\}$ and $N\in\mathbb{N}$. Hence, if we use the same $\{\boldsymbol{\omega}_{-j}^n\}_{n=0}^{\infty}$ and Δt for both networks, then

$$\widetilde{X}_k^N \equiv \widetilde{X}_k^{+,N}, \quad k\in\mathcal{M}\setminus\{j\},\ N=0,1,\ldots. \tag{A.2}$$

To compute the left-hand side of (A.1), we first note that

$$\widetilde{X}_j^N = 0 \quad\Longleftrightarrow\quad \widetilde{X}_j^n = 0, \quad n=0,\ldots,N.$$

Hence,

$$\widetilde{X}_j^N = 0 \quad\Longleftrightarrow\quad I_j^0 < \omega_j^0 \leq 1 \quad \text{and}$$

$$\omega_j^n \geq \left(p_j + \sum_{k\in\mathcal{M}\setminus\{j\}} q_{k\to j}\widetilde{X}_k(t^{n-1})\right)\Delta t, \quad n=1,\ldots,N.$$

Therefore,

$$\left[S_j \mid \{\boldsymbol{\omega}^n_{-j}\}_{n=1}^N\right](t^N;\Delta t) = [S_j^0]\prod_{n=1}^N H_j^n$$

$$H_j^n := 1 - \left(p_j + \sum_{k\in\mathcal{M}\setminus\{j\}} q_{k\to j}\widetilde{X}_k(t^{n-1})\right)\Delta t,$$

where $[S_j^0] = 1 - I_j^0$. Hence,

$$[S_j](t^N;\Delta t) = \left[S_j^0\right]\int_{[0,1]^{(M-1)\times N}}\left(\prod_{n=1}^N H_j^n\left(\left\{\boldsymbol{\omega}^n_{-j}\right\}_{n=1}^N,\Delta t\right)\right)d\boldsymbol{\omega}^1_{-j}\cdots d\boldsymbol{\omega}^N_{-j}. \quad \text{(A.3)}$$

Similarly, to compute the right-hand side of (A.1), we note that $\widetilde{X}^{+,N}_{jp} = \widetilde{X}^{+,N}_{jA_1} = \cdots = \widetilde{X}^{+,N}_{jA_L} = 0$ if and only if $\widetilde{X}^{+,0}_{jp} = \widetilde{X}^{+,0}_{jA_1} = \cdots = \widetilde{X}^{+,0}_{jA_L} = 0$, and for $n = 1,\ldots,N$,

$$\omega^n_{jp} \geq p_j\Delta t, \quad \omega^n_{jA_l} \geq \left(\sum_{k\in A_l} q_{k\to j}\widetilde{X}^{+,n-1}_k\right)\Delta t, \quad l\in\{1,\ldots,L\}.$$

Since $\widetilde{X}^{+,0}_{jA_1} = \cdots = \widetilde{X}^{+,0}_{jA_L} \equiv 0$, then $\mathbb{P}(\widetilde{X}^{+,0}_{jp} = \widetilde{X}^{+,0}_{jA_1} = \cdots = \widetilde{X}^{+,0}_{jA_L} = 0) = \mathbb{P}(\widetilde{X}^{+,0}_{jp} = 0) = [S_j^0]$. Therefore,

$$\left[S^+_{j_{A_1},\ldots,j_{A_L},j_p}\right](t^N;\Delta t) = \left[S_j^0\right]\int_{[0,1]^{(M-1)\times N}}\left(\prod_{n=1}^N H_j^{n,+}\left(\left\{\boldsymbol{\omega}^n_{-j}\right\}_{n=1}^N,\Delta t\right)\right)d\boldsymbol{\omega}^1_{-j}\cdots d\boldsymbol{\omega}^N_{-j}, \quad \text{(A.4)}$$

where

$$H_j^{n,+} := (1 - p_j\Delta t)\prod_{l=1}^L\left(1 - \Delta t\sum_{k\in A_l} q_{k\to j}\widetilde{X}^{+,n-1}_k\right).$$

By (A.2),

$$\begin{aligned} H_j^{n,+} &= (1 - p_j\Delta t)\prod_{l=1}^L\left(1 - \Delta t\sum_{k\in A_l} q_{k\to j}\widetilde{X}^{n-1}_k\right) \\ &= 1 - \left(p_j + \sum_{k\in\mathcal{M}\setminus\{j\}} q_{k\to j}\widetilde{X}^{n-1}_k\right)\Delta t + O((\Delta t)^2) \\ &= H_j^n + O((\Delta t)^2). \end{aligned}$$

Hence,

$$\prod_{n=1}^N H_j^{n,+} = (1 + O(\Delta t))\prod_{n=1}^N H_j^n. \quad \text{(A.5)}$$

Letting $N\to\infty$ and using (A.3)–(A.5) proves (A.1).

REFERENCES

[1] W. O. Kermack and A. G. McKendrick, "A contribution to the mathematical theory of epidemics," Proc. R. Soc. Lond. A **115**, 700–721 (1927).
[2] F. M. Bass, "A new product growth model for consumer durables," Manage. Sci. **15**, 215–1227 (1969).
[3] I. Z. Kiss, J. C. Miller, and P. L. Simon, *Mathematics of Epidemics on Networks* (Springer, 2017).
[4] W. Rand and R. T. Rust, "Agent-based modeling in marketing: Guidelines for rigor," Int. J. Res. Mark. **28**, 181–193 (2011).
[5] S. Bansal, B. T. Grenfell, and L. A. Meyers, "When individual behaviour matters: Homogeneous and network models in epidemiology," J. R. Soc. Interface **4**, 879–891 (2007).
[6] J. Goldenberg, S. Han, D. R. Lehmann, and J. W. Hong, "The role of hubs in the adoption process," J. Mark. **73**, 1–13 (2009).
[7] J. Goldenberg, O. Lowengart, and D. Shapira, "Zooming in: Self-emergence of movements in new product growth," Mark. Sci. **28**, 274–292 (2008).
[8] M. J. Keeling and K. T. D. Eames, "Networks and epidemic models," J. R. Soc. Interface **2**, 295–307 (2005).
[9] E. Muller and R. Peres, "The effect of social networks structure on innovation performance: A review and directions for research," Int. J. Res. Mark. **36**, 3–19 (2019).
[10] M. J. Keeling, "The effects of local spatial structure on epidemiological invasions," Proc. R. Soc. Lond. Ser. B Biol. Sci. **266**, 859–867 (1999).
[11] D. A. Rand, "Correlation equations and pair sppproximations for spatial ecologies," in *Advanced Ecological Theory: Principles and Applications* (Blackwell Science, Oxford, 1999), pp. 100–142.
[12] G. Fibich and R. Gibori, "Aggregate diffusion dynamics in agent-based models with a spatial structure," Oper. Res. **58**, 1450–1468 (2010).
[13] G. Fibich and A. Golan, "Diffusion of new products with heterogeneous consumers," Math. Oper. Res. **48**, 257–287 (2023).
[14] G. Fibich, A. Golan, and S. Schochet, "Monotone convergence of discrete Bass models," arXiv:2407.10816 (2024).
[15] G. Fibich, A. Golan, and S. Schochet, "Compartmental limit of discrete Bass models on networks," Discrete Contin. Dyn. Syst. B **28**, 3052–3078 (2023).
[16] G. Fibich, T. Levin, and O. Yakir, "Boundary effects in the discrete Bass model," SIAM J. Appl. Math. **79**, 914–937 (2019).
[17] G. Fibich and T. Levin, "Percolation of new products," Physica A **540**, 123055 (2020).
[18] K. J. Sharkey, I. Z. Kiss, R. R. Wilkinson, and P. L. Simon, "Exact equations for SIR epidemics on tree graphs," Bull. Math. Biol. **77**, 614–645 (2015).
[19] I. Z. Kiss, E. Kenah, and G. A. Rempała, "Necessary and sufficient conditions for exact closures of epidemic equations on configuration model networks," J. Math. Biol. **87**, 746–788 (2023).
[20] G. Fibich and S. Nordmann, "Exact description of SIR-Bass epidemics on 1D lattices," Discrete Contin. Dyn. Syst. **42**, 505–535 (2022).
[21] E. M. Volz, "SIR dynamics in random networks with heterogeneous connectivity," J. Math. Biol. **56**, 293–310 (2008).
[22] J. C. Miller, A. C. Slim, and E. M. Volz, "Edge-based compartmental modelling for infectious disease spread," J. R. Soc. Interface **9**, 890–906 (2012).
[23] L. Decreusefond, J.-S. Dhersin, P. Moyal, and V. C. Tran, "Large graph limit for an SIR process in random network with heterogeneous connectivity," Ann. Appl. Probab. **22**, 541–575 (2012).
[24] K. A. Jacobsen, M. G. Burch, J. H. Tien, and G. A. Rempała, "The large graph limit of a stochastic epidemic model on a dynamic multilayer network," J. Biol. Dyn. **12**, 746–788 (2018).
[25] Condition (2.1g) prevents scenarios where node j remains susceptible at all times with probability one; yet, it is classified as influential to Ω under Definition 4.3. This condition can be replaced, e.g., by the weaker condition that for any node j, there exists a node m such that $I_m^0 + p_m > 0$, and there exists a path with $0 \leq L < \infty$ edges from m to j.
[26] G. Fibich, T. Levin, and K. Gillingham, "Boundary effects in the diffusion of new products on Cartesian networks," Oper. Res. (published online 2024).
[27] G. Fibich and Y. Warman, "Explicit solutions of the SI and Bass models on sparse Erdös-Rényi and regular networks," arXiv:2411.12076 (2024).

[28]For example, let Ω_1 and Ω_2 be two disjoint sets of nodes. Assume that there is a directed path from Ω_1 to Ω_2. Then, knowing that all the nodes in Ω_1 are susceptible at time t increases the likelihood that all the nodes in Ω_2 are susceptible at time t (i.e., that $[S_{\Omega_2} \mid S_{\Omega_1}] \geq [S_{\Omega_2}]$), and knowing that all the nodes in Ω_2 are susceptible at time t increases the likelihood that all the nodes in Ω_1 are susceptible at time t (i.e., that $[S_{\Omega_1} \mid S_{\Omega_2}] \geq [S_{\Omega_1}]$). Moreover, even if there is no directed path between Ω_1 and Ω_2, but there is a node m from which there are directed paths both to Ω_1 and to Ω_2, then knowing that all the nodes in Ω_2 are susceptible increases the likelihood that m is also susceptible, hence that all the nodes in Ω_1 are susceptible. Therefore, we can expect that $[S_{\Omega_1} \mid S_{\Omega_2}] \geq [S_{\Omega_1}]$. Since $[S_{\Omega_1,\Omega_2}] = [S_{\Omega_1} \mid S_{\Omega_2}][S_{\Omega_2}]$, that is the case if and only $[S_{\Omega_1,\Omega_2}] \geq [S_{\Omega_1}][S_{\Omega_2}]$.

[29]I. Z. Kiss, C. G. Morris, F. Sélley, P. L. Simon, and R. R. Wilkinson, "Exact deterministic representation of Markovian SIR epidemics on networks with and without loops," J. Math. Biol. **70**, 437–464 (2015).
[30]E. Cator and P. Van Mieghem, "Nodal infection in Markovian susceptible-infected-susceptible and susceptible-infected-removed epidemics on networks are non-negatively correlated," Phys. Rev. E **89**, 052802 (2014).
[31]G. Fibich and T. Levin, "Universal bounds for spreading on networks," Chaos **34**, 053101 (2024).
[32]G. Fibich, "Bass-SIR model for diffusion of new products in social networks," Phys. Rev. E **94**, 032305 (2016).
[33]F. Battiston, G. Cencetti, I. Iacopini, V. Latora, M. Lucas, A. Patania, J. G. Young, and G. Petri, "Networks beyond pairwise interactions: Structure and dynamics," Phys. Rep. **874**, 1–92 (2020).
[34]G. Fibich, J. G. Restrepo, and G. Rothmann, "Explicit solutions of the Bass and susceptible-infected models on hypernetworks," Phys. Rev. E **110**, 054306 (2024).
[35]G. Fibich and G. Rothmann, "A phase transition in the susceptible-infected model on hypernetworks," arXiv:2504.11818 (2025).