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Diffusion of New Products with Heterogeneous Consumers

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Abstract. Does a new product spread faster among heterogeneous or homogeneous consumers? We analyze this question using the stochastic discrete Bass model in which consumers may differ in their individual external influence rates $\{p_j\}$ and in their individual internal influence rates $\{q_j\}$. When the network is complete and the heterogeneity is only manifested in $\{p_j\}$ or only in $\{q_j\}$, it always slows down the diffusion, compared with the corresponding homogeneous network. When, however, consumers are heterogeneous in both $\{p_j\}$ and $\{q_j\}$, heterogeneity slows down the diffusion in some cases but accelerates it in others. Moreover, the dominance between the heterogeneous and homogeneous adoption levels is global in time in some cases but changes with time in others. Perhaps surprisingly, global dominance between two networks is not always preserved under "additive transformations", such as adding an identical node to both networks. When the network is not complete, the effect of heterogeneity depends also on its spatial distribution within the network.

Keywords: marketing • bass model • heterogeneity • agent-based model • stochastic models • discrete models • diffusion in networks • analysis

1. Introduction

The study of the diffusion of innovations started in the sociology literature (De Tarde [6]) and expanded over the years (Rogers [20]). More generally, diffusion in social networks has attracted the attention of researchers in physics, mathematics, biology, computer science, social sciences, economics, and management science, as it concerns the spreading of "items" ranging from diseases and computer viruses to rumors, information, opinions, technologies, and innovations (Albert et al. [1], Anderson and May [2], Jackson [15], Pastor-Satorras and Vespignani [18], Strang and Soule [21]). In marketing, diffusion of new products plays a key role, with applications in retail service, industrial technology, and agriculture and in educational, pharmaceutical, and consumer-durables markets (Mahajan et al. [16]).

The first quantitative model of the diffusion of new products was proposed in 1969 by Bass [3]. In this model, we consider a population of size M and denote by $n = n(t)$ the number of individuals who adopted the product by time t. Each of the $(M - n)$ nonadopters may adopt the product because of *external influences* by mass media at a constant rate of p and because of *internal influences* by individuals who already adopted the product at the rate of $\frac{q}{M}n$. Thus, the rate of internal influences increases linearly with the number of adopters. The individual adoption rate of each nonadopter is the sum of the nonadopter's external and internal adoption rates, that is,

$$p + \frac{q}{M}n(t). \tag{1}$$

Therefore, the rate of change of the number of adopters is

$$n'(t) = (M - n(t))\Big(p + \frac{q}{M}n(t)\Big). \tag{2}$$

The Bass Model (2) inspired a huge amount of follow-up research; in 2004, it was chosen as one of the most cited papers in the 50-year history of *Management Science* (Hoppe [14]). From a modeling perspective, it is a *compartmental model*. Thus, the population is divided into two *compartments* (groups), adopters and nonadopters; Equation (2) provides the rate at which individuals move between these two compartments. Most of the extensions of the Bass model have also been compartmental models, given by a deterministic ordinary differential equation (ODE) or ODEs. As a result, they are relatively easy to analyze. Compartmental models, however, make two implicit assumptions whose validity is highly questionable:

Assumption 1. *All individuals within the population are equally likely to influence each other. In other words, the underlying social network is a complete graph.*

Assumption 2. *All individuals within the population are homogeneous, that is, they all have the same p and q.*

To check the consequences of these assumptions, one needs to go back to the more fundamental discrete model for the stochastic adoption of each individual in the population (Rand and Rust [19]). For example, the discrete analogue of the compartmental Model (2) is, compare to (1),

$$\text{Probability}\left(\begin{matrix} j \text{ adopts in} \\ (t, t+\Delta t) \end{matrix} \,\middle|\, \begin{matrix} j \text{ did not adopt} \\ \text{by time t} \end{matrix}\right) = \left(p_j + \frac{q_j}{M_j} N_j(t)\right)\Delta t, \tag{3}$$

where p_j and q_j are the rates of external and internal influences on j, M_j is the number of peers (the *degree*) of j, and $N_j(t)$ is the number of adopters at time t among his or her M_j peers.

Discrete stochastic Bass models are considerably harder to analyze than compartmental models. They enable us, however, to relax the assumptions of a complete network and of homogeneity. *Most of the analysis of the discrete Bass model so far has been concerned with the role of the network structure.* Niu [17] showed that as $M \to \infty$, the discrete Bass model on a *homogeneous complete network* approaches the compartmental Bass Model (2). Fibich and Gibori [10] analyzed the discrete Bass model on *Cartesian networks*. Fibich et al. [12] analyzed the effect of *boundary conditions* in Cartesian networks. Fibich [7, 8] analyzed the discrete *Bass-SIR model*, in which adopters eventually recover and no longer influence others to adopt, on various networks. Fibich and Levin [11] analyzed the *percolation* of new products on various networks, from which a fraction of the nodes is randomly removed.

All of the above studies analyzed the discrete Bass model on *homogeneous* networks, that is, when all individuals have the same p and q. Goldenberg et al. [13] studied numerically the discrete Bass model on complete networks with heterogeneous consumers and observed that heterogeneity has a small effect on the aggregate diffusion. To the best of our knowledge, *analysis* of the effect of heterogeneity in the discrete Bass model was only done in Fibich et al. [9]. In that study, Fibich, Gavious, and Solan used the *averaging principle* to estimate the quantitative difference between heterogeneous and homogeneous networks. Specifically, they showed that if the network is translation-invariant and the heterogeneity is mild, the difference between the aggregate diffusion in the heterogeneous and the corresponding homogeneous networks scales as ϵ^2, where ϵ is the level of heterogeneity of $\{p_j\}$ and $\{q_j\}$.

Several studies used compartmental models to study the effect of heterogeneity. Bulte and Joshi [4] divided the population into two groups: the influentials with $p = p_1$ and $q = q_1$ and the imitators with $p = 0$ and $q = q_2$. Their numerical results revealed that heterogeneity in p and q can change the qualitative behavior of the diffusion (Bulte and Joshi [4]). Chaterjee and Eliashberg [5] constructed a compartmental diffusion model that allowed for heterogeneity in consumers' initial perceptions and price hurdles. Although their study did not directly analyze heterogeneity in p and q within the framework of the discrete Bass model, it showed that heterogeneity can alter the qualitative behavior of aggregate adoption (Chatterjee and Eliashberg [5].)

This paper provides the first-ever analysis of the qualitative effect of heterogeneity in the stochastic discrete Bass model. We show that heterogeneity in p and q can speed up or slow down the diffusion compared with the corresponding homogeneous network. This result is surprising because heterogeneity only in p or only in q always slows down the diffusion. In some cases, the dominance between the heterogeneous and homogeneous networks is global in time; in others, the dominance flips after some time. When the network is not complete, the effect of heterogeneity also depends on the way in which it is spatially distributed in the network.

The methodological contribution of this paper consists of several novel analytical tools: the *master equations* for heterogeneous networks, *explicit expressions for the first three derivatives* of the expected adoption on heterogeneous networks at $t = 0$, and a cumulative distribution function *(CDF) dominance condition* for comparing the diffusion on two networks. We also use the *dominance principle* for heterogeneous networks, which was introduced in Fibich et al. [12].

From a more general perspective, the vast majority of models in marketing and in economics assume that all individuals are homogeneous. This assumption is made not because it is believed to hold but simply because heterogeneous models are typically an order of magnitude harder to analyze than their homogeneous counterparts. This paper thus adds to the relatively thin literature on heterogeneous models in marketing and in economics.

The paper is organized as follows. In Section 2, we introduce the heterogeneous discrete Bass model. We then review some results for homogeneous complete networks (Section 2.1), for one-sided and two-sided homogeneous circular networks (Section 2.2), and the *dominance principle* for heterogeneous networks (Section 2.3). In Section 3, we introduce several novel analytic tools for the heterogeneous discrete Bass model. Thus, in Section 3.1 we derive the *master equations for the heterogeneous Bass model*. This linear system of ODEs can be solved analytically to yield an explicit expression for the expected fraction of adopters in any heterogeneous network; we provide explicit expressions for $M = 2$ and $M = 3$. In Section 3.2, we derive *explicit expressions for the first three derivatives* at $t = 0$ of the adoption in heterogeneous networks. These expressions allow us to analyze the *initial diffusion dynamics* on heterogeneous and homogeneous networks. In Section 3.3, we introduce the *CDF dominance condition*. This condition allows us to compare the adoption levels of two different networks by comparing the CDFs of the times of the mth adoptions in both networks for $m = 1, \ldots, M$. We use this tool throughout the paper to compare the diffusion in heterogeneous and homogeneous networks.

In Section 4, we compare heterogeneous complete networks with their homogeneous counterparts that have the same number of nodes, the same average p, and the same average q. When the heterogeneity is only in p, it always slows down the diffusion (Section 4.1). This is also the case for networks that are heterogeneous only in q (Section 4.2), provided that the heterogeneity in q is *mild*, that is, that a nonadopter is equally influenced by all other adopters, see (3). In Section 4.3 we consider networks that are heterogeneous in both p and q. When the heterogeneities in p and q are positively correlated, heterogeneity slows down the diffusion. When the heterogeneities in p and q are not positively correlated, however, the diffusion in the heterogeneous case can be slower than, faster than, or equal to that in the homogeneous case.

Consider two networks for which the adoption in the first is lower than in the second for all times. Will this global-in-time dominance be preserved if we increase the p_j values of all the nodes in both networks by the same amount or if we add an identical node to both networks? In Section 5, we show that this is indeed the case when there is a node-wise and edge-wise dominance between the two networks but not necessarily in other cases. We also show that the dominance between heterogeneous and homogeneous networks is not necessarily global in time but rather can flip with time. In Section 6, we consider heterogeneous periodic one-dimensional networks (circles). In Sections 6.1 and 6.2, we derive the master equations for heterogeneous one-sided and two-sided circle, respectively. We explicitly solve these equations for any M in the one-sided case and for $M = 2$ and $M = 3$ in the two-sided case. We then show that the adoption in a heterogeneous two-sided circle can be higher or lower than that in the corresponding heterogeneous one-sided circle (Section 6.3). This is different from the homogeneous case, where diffusion on a two-sided circle is identical to that on the corresponding one-sided circle (Fibich and Gibori [10]).

The analytic tools developed in this study can also be applied to homogeneous networks. Indeed, in Fibich and Gibori [10], the authors conjectured that diffusion on infinite homogeneous Cartesian networks becomes faster as the dimension of the network increases. In Section 7, we prove this conjecture for small times. When the network is not complete, the effect of heterogeneity depends also on the way in which it is distributed among the nodes. To illustrate this, in Section 8 we consider two heterogeneous one-sided circles that have the same nodes but differ in the way in which the nodes are distributed in the circle. We explicitly compute the aggregate adoption for both networks as $M \to \infty$. We obtain different expressions, which show that the adoption indeed depends also on the spatial distribution of the heterogeneity. Finally, in Section 9 we show that as we vary the level of heterogeneity, its effect varies continuously and monotonically (at least for weak heterogeneity). In addition, the effect of the variance of the parameters is much smaller than that of their mean.

1.1. Emerging Picture

This paper contains numerous results. In order to see the wood for the trees, it is useful to summarize some unifying themes:

1. When the network is heterogeneous only in p_j or only in q_j, heterogeneity always slows the diffusion for all times.
2. When the heterogeneity is both in p_j and q_j, the qualitative effect of heterogeneity is more complex:
   a. When the heterogeneities in p and q are positively correlated, heterogeneity always slows the diffusion for all times.
   b. When, however, the heterogeneities in p and q are not positively correlated, heterogeneity can accelerate or slow down the diffusion. Moreover, the dominance between the heterogeneous and homogeneous networks is global in time for some cases but changes with time for others.
3. Global dominance between two networks is not necessarily preserved under "additive transformations," such as increasing all the $\{p_j\}$ of both networks by the same amount or adding an identical node to both networks.
4. When a network is heterogeneous and not complete, the effect of heterogeneity on the aggregate diffusion depends also on the spatial distribution of the heterogeneity among the nodes.

2. The Heterogeneous Discrete Bass Model

We begin by introducing the diffusion model that is analyzed in this study. A new product is introduced at time $t = 0$ to a network with M potential consumers. We denote by $X_j(t)$ the state of consumer j at time t, so that

$$X_j(t) = \begin{cases} 1, & \text{if consumer } j \text{ adopts the product by time } t, \\ 0, & \text{otherwise.} \end{cases}$$

Because all consumers are nonadopters at $t = 0$,

$$X_j(0) = 0, \qquad j = 1, \ldots, M. \tag{4}$$

Once a consumer adopts the product, it remains an adopter for all time. The underlying social network is represented by a weighted directed graph, where the weight of the edge from node i to node j is $q_{i,j} \geq 0$ and $q_{i,j} = 0$ if there is no edge from i to j. Thus, if i already adopted the product and $q_{i,j} > 0$, his or her rate of internal influence on consumer j to adopt is $q_{i,j}$. In addition, consumer j experiences an external influence to adopt, at the rate of p_j. Hence, as $dt \longrightarrow 0$,

$$\mathrm{Prob}(X_j(t+dt) = 1) = \begin{cases} 1, & \text{if } X_j(t) = 1, \\ \left(p_j + \displaystyle\sum_{\substack{k=1 \\ k \neq j}}^{M} q_{k,j} X_k(t)\right) dt, & \text{if } X_j(t) = 0. \end{cases} \tag{5}$$

If j is a nonadopter, the maximal internal influence that can be exerted on j, which occurs when all of his or her peers are adopters, is denoted by

$$q_j := \sum_{\substack{k=1 \\ k \neq j}}^{M} q_{k,j}. \tag{6}$$

We assume that any individual can be influenced by at least one other individual, that is, that

$$q_j > 0, \qquad j = 1, \ldots, M. \tag{7}$$

We also denote by

$$q^k := \sum_{\substack{j=1 \\ j \neq k}}^{M} q_{k,j} \tag{8}$$

the sum of the internal influences that k exerts on his or her peers.

We mostly consider a *milder form of heterogeneity* in q, where $\{q_j\}_{j=1}^M$ can be heterogeneous but each individual is equally influenced by any of his or her peers, that is,

$$q_{i,j} = \begin{cases} \dfrac{q_j}{\mathrm{degree}(j)}, & \text{if } i \text{ influences } j, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Therefore, the network structure is preserved under mild heterogeneity. For example, in the case of a mildly heterogeneous complete network, (5) reads

$$\mathrm{Prob}(X_j(t+dt) = 1) = \begin{cases} 1, & \text{if } X_j(t) = 1, \\ \left(p_j + \dfrac{q_j}{M-1} N(t)\right) dt, & \text{if } X_j(t) = 0, \end{cases}$$

where $N(t) := \sum_{j=1}^M X_j(t)$ is the number of adopters at time t.

Our main goal is to compute the effect of the heterogeneity in $\{p_j\}$ and $\{q_{i,j}\}$ (or $\{q_j\}$) on the expected number of adopters

$$n(t) := \mathbf{E}\left[\sum_{j=1}^{M} X_j(t)\right] = \mathbf{E}[N(t)], \tag{10}$$

or equivalently on the expected fraction of adopters

$$f(t) := \frac{1}{M} n(t). \tag{11}$$

2.1. Homogeneous Complete Networks

When the network is complete and homogeneous, then

$$p_j \equiv p, \qquad q_{i,j} = \frac{q}{M-1}, \qquad i,j = 1,\ldots,M, \qquad i \neq j. \tag{12}$$

In that case, (5) reads

$$\mathrm{Prob}(X_j(t+dt)=1) = \begin{cases} 1, & \text{if } X_j(t)=1, \\ \left(p + \frac{q}{M-1}N(t)\right)dt, & \text{if } X_j(t)=0. \end{cases} \tag{13}$$

Niu [17] proved that as $M \to \infty$, the expected fraction of adopters in (13) approaches the solution of the compartmental Bass model (Bass [3])

$$f'(t) = (1-f(t))(p+qf(t)), \qquad f(0)=0. \tag{14}$$

This equation can be solved explicitly, yielding the Bass Equation (Bass [3])

$$f_{\mathrm{Bass}}(t;p,q) = \frac{1-e^{-(p+q)t}}{1+\frac{q}{p}e^{-(p+q)t}}. \tag{15}$$

2.2. Homogeneous Circles

Let us denote by $f^{\text{1-sided}}_{\text{circle}}(t;p,q,M)$ the expected fraction of adopters in a homogeneous one-sided circle with M nodes where each individual is only influenced by his left neighbor (see Figure 1(a)), that is,

$$p_j \equiv p, \qquad q_{i,j} = \begin{cases} q, & \text{if } (j-i) \bmod M = 1, \\ 0, & \text{if } (j-i) \bmod M \neq 1, \end{cases} \qquad j,i = 1,\ldots,M.$$

Similarly, denote by $f^{\text{2-sided}}_{\text{circle}}(t;p,q^R,q^L,M)$ the expected fraction of adopters in a homogeneous two-sided circle with M nodes where each individual can be influenced by his or her left and right neighbors (see Figure 1(b)), that is,

$$p_j \equiv p, \qquad q_{i,j} = \begin{cases} q^L & \text{if } (j-i) \bmod M = 1, \\ q^R & \text{if } (i-j) \bmod M = 1, \\ 0 & \text{if } |j-i| \bmod M \neq 1, \end{cases} \qquad j,i = 1,\ldots,M.$$

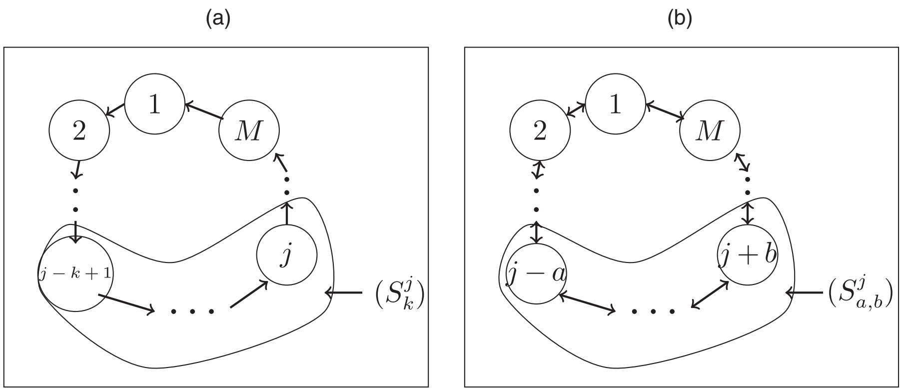
In Fibich and Gibori [10], the authors proved that the diffusion on one-sided and two-sided homogeneous circles are identical

$$f^{\text{1-sided}}_{\text{circle}}(t;p,q,M) \equiv f^{\text{2-sided}}_{\text{circle}}(t;p,q^R,q^L,M), \qquad 0 \le t < \infty, \tag{16}$$

provided that the maximal internal influence experienced by each node is identical in both cases, that is, that

$$q = q^L + q^R. \tag{17}$$

Figure 1. (a) A one-sided circle with the nonadopters chain (S^j_k). (b) A two-sided circle with the nonadopters chain $(S^j_{a,b})$.



In addition, they explicitly computed the diffusion on infinite homogeneous circles:

$$\lim_{M\to\infty} f^{\text{1-sided}}_{\text{circle}}(t;p,q,M) = f_{1D}(t;p,q) := 1 - e^{-(p+q)t+q\frac{1-e^{-pt}}{p}}. \tag{18}$$

2.3. Dominance Principle

A useful tool for comparing the diffusion in two networks is the dominance principle. Let us begin with the following definition:

Definition 1 (Node-Wise and Edge-Wise Dominance). Consider the heterogeneous discrete Bass Model (5) on networks A and B with M nodes, with external parameters $\{p_i^A\}$ and $\{p_i^B\}$, and internal parameters $\{q_{i,j}^A\}$ and $\{q_{i,j}^B\}$, respectively. We say that $A \preceq B$ if

$$p_j^A \le p_j^B \quad \text{for all } j \qquad \text{and} \qquad q_{i,j}^A \le q_{i,j}^B \quad \text{for all } i \ne j.$$

We say that $A \prec B$ if at least one of these M^2 inequalities is strict.

Lemma 1 (Dominance Principle (Fibich et al. [12])). *If $A \preceq B$, then $f_A(t) \le f_B(t)$ for $t > 0$. If $A \prec B$, then $f_A(t) < f_B(t)$ for $t > 0$.*

3. Analytic Tools

In this section, we introduce several novel analytical tools. These tools will be later used to analyze the effect of heterogeneity in the heterogeneous discrete Bass Model (5).

3.1. Master Equations

In order to analytically compute the expected number of adopters, we derive the master equations for a general heterogeneous network with M nodes as follows. Let $(S^{m_1},\dots,S^{m_n})(t)$ denote the event that at time t, nodes $\{m_1,\dots,m_n\}$ are nonadopters, where $1 \le n \le M$, $m_i \in \{1,\dots,M\}$, and $m_i \ne m_j$ if $i \ne j$. Let $[S^{m_1},\dots,S^{m_n}](t)$ denote the probability that such an event occurs.

Lemma 2. *The master equations for the heterogeneous discrete Bass Model (5) are*

$$\frac{d}{dt}[S^{m_1},\dots,S^{m_n}](t) = -\left(\sum_{i=1}^n p_{m_i} + \sum_{j=n+1}^M \sum_{i=1}^n q_{l_j,m_i}\right)[S^{m_1},\dots,S^{m_n}](t) + \sum_{j=n+1}^M \left(\sum_{i=1}^n q_{l_j,m_i}\right)[S^{m_1},\dots,S^{m_n},S^{l_j}](t), \tag{19a}$$

for any $\{m_1,\dots,m_n\} \subsetneq \{1,\dots,M\}$, where $\{l_{n+1},\dots,l_M\} = \{1,\dots,M\}\backslash\{m_1,\dots,m_n\}$ and

$$\frac{d}{dt}[S^1,S^2,\dots,S^M](t) = -\left(\sum_{i=1}^M p_i\right)[S^1,S^2,\dots,S^M](t), \tag{19b}$$

subject to the initial conditions

$$[S^{m_1},\dots,S^{m_n}](0) = 1, \qquad \forall\{m_1,\dots,m_n\} \subset \{1,\dots,M\}. \tag{19c}$$

Proof. See Appendix A. □

The master Equations (19) constitute a linear system of $2^M - 1$ differential equations for all possible subsets $\{m_1,\dots,m_n\} \subset \{1,\dots,M\}$. These equations can be solved explicitly as follows. By (19b)–(19c),

$$[S^1,\dots,S^M](t) = e^{-(\Sigma_{j=1}^M p_j)t}. \tag{20}$$

Proceeding to solve (19) backward from $n = M-1$ to $n = 1$ gives (Appendix B)

$$[S^k](t) = \sum_{n=1}^M \sum_{\{m_1=k,m_2,\dots,m_n\}} c_{k(m_1,\dots,m_n)} e^{-(\Sigma_{i=1}^n p_{m_i} + \Sigma_{j=n+1}^M \Sigma_{i=1}^n q_{l_j,m_i})t}, \tag{21}$$

where $c_{k(m_1,\ldots,m_n)}$ are constants. Once we solve for $\{[S^k]\}_{k=1}^M$, the expected fraction of adopters in the network is given by (see (10) and (11)),

$$f\Big(t;\{p_i\},\{q_{i,j}\}\Big) = 1 - \frac{1}{M}\sum_{k=1}^{M}[S^k](t). \tag{22}$$

For example, the master Equations (19) for $M = 2$ read

$$\frac{d}{dt}[S^i](t) = -(p_i + q_{i-1,i})[S^i](t) + q_{i-1,i}[S^1,S^2](t), \qquad i = 1,2, \tag{23a}$$

$$\frac{d}{dt}[S^1,S^2](t) = -(p_1+p_2)[S^1,S^2](t), \tag{23b}$$

$$[S^1](0) = [S^2](0) = [S^1,S^2](0) = 1. \tag{23c}$$

Solving this system for $M = 2$ yields

$$[S^1](t) = a_1 e^{-(p_1+q_{2,1})t} + b_1 e^{-(p_1+p_2)t}, \qquad [S^2](t) = a_2 e^{-(p_2+q_{1,2})t} + b_2 e^{-(p_1+p_2)t}.$$

Therefore, by (22),

$$f(t) = 1 - \frac{1}{2}\sum_{j=1}^{2}\Big[a_j e^{-(p_j+q_{j-1,j})t} - b_j e^{-(p_1+p_2)t}\Big], \quad a_j = \frac{p_{j-1}}{p_{j-1}-q_{j-1,j}}, \qquad b_j = \frac{q_{j-1,j}}{p_{j-1}-q_{j-1,j}}. \tag{24}$$

Similarly, when $M = 3$,

$$f(t) = 1 - \frac{1}{3}\sum_{j=1}^{3}\Big[a_j e^{-(p_j+q_{j-1,j}+q_{j+1,j})t} - b_j e^{-(p_j+p_{j+1}+q_{j-1,j}+q_{j+2,j+1})t} + c_j e^{-(p_1+p_2+p_3)t}\Big], \tag{25a}$$

where

$$\begin{aligned} a_j &= 1 + \left(1 + \frac{q_{j-1,j}+q_{j+2,j+1}}{p_{j-1}-q_{j-1,j}-q_{j+2,j+1}}\right)\frac{q_{j+1,j}}{p_{j+1}+q_{j+2,j+1}-q_{j+1,j}} \\ &\quad + \left(1 + \frac{q_{j-2,j-1}+q_{j+1,j}}{p_{j+1}-q_{j-2,j-1}-q_{j+1,j}}\right)\frac{q_{j-1,j}}{p_{j-1}+q_{j-2,j-1}-q_{j-1,j}} \\ &\quad - \frac{q_{j+1,j}\dfrac{q_{j-1,j}+q_{j+2,j+1}}{p_{j-1}-q_{j-1,j}-q_{j+2,j+1}} + q_{j-1,j}\dfrac{q_{j-2,j-1}+q_{j+1,j}}{p_{j+1}-q_{j-2,j-1}-q_{j+1,j}}}{p_{j+1}+p_{j-1}-q_{j-1,j}-q_{j+1,j}}, \end{aligned} \tag{25b}$$

$$b_j = \left(1 + \frac{q_{j-1,j}+q_{j+2,j+1}}{p_{j-1}-q_{j-1,j}-q_{j+2,j+1}}\right)\left(\frac{q_{j+1,j}}{p_{j+1}+q_{j+2,j+1}-q_{j+1,j}} + \frac{q_{j,j+1}}{p_j+q_{j-1,j}-q_{j,j+1}}\right), \tag{25c}$$

$$c_j = \frac{q_{j+1,j}\dfrac{q_{j-1,j}+q_{j+2,j+1}}{p_{j-1}-q_{j-1,j}-q_{j+2,j+1}} + q_{j-1,j}\dfrac{q_{j-2,j-1}+q_{j+1,j}}{p_{j+1}-q_{j-2,j-1}-q_{j+1,j}}}{p_{j+1}+p_{j-1}-q_{j-1,j}-q_{j+1,j}}. \tag{25d}$$

Remark 1. The subscripts of $q_{i,j}$ in (23)–(25) are modulo M and in $\{1,\ldots,M\}$. For example, if $i = 1$, then "$i-1 = M$," and if $i = M$, then "$i+2 = 2$", etc.

Remark 2. The master Equations (19) hold for heterogeneous (and homogeneous) networks *with any structure*. For example, we can have a one-sided circle by setting $q_{i,j} = 0$ for $(j-i) \bmod M \neq 1$ (Section 6.1), a two-sided circle by setting $q_{i,j} = 0$ for $|j-i| \bmod M \neq 1$ (Section 6.2), a D-dimensional Cartesian structure by setting $q_{i,j}$ as in Equation (49) (Section 7), etc.

3.2. Initial Dynamics

We can use the master Equations (19) to analyze the initial dynamics, by deriving explicit expressions for $f'(0)$, $f''(0)$, and $f'''(0)$. We begin with the most general heterogeneity:

Lemma 3. *Consider the heterogeneous discrete Bass model (5). Then*

$$f'(0) = \frac{1}{M}\sum_{j=1}^{M} p_j, \qquad f''(0) = \frac{1}{M}\left(\sum_{i=1}^{M} p_i q^i - \sum_{i=1}^{M} p_i^2\right), \tag{26}$$

where q^i is defined in (8). In addition, if the heterogeneity in q is mild (see (9)), then

$$f''(0) = \frac{1}{M(M-1)}\left[\sum_{j=1}^{M} q_j \sum_{i=1}^{M} p_i - \sum_{j=1}^{M} q_j p_j\right] - \frac{1}{M}\sum_{i=1}^{M} p_i^2. \tag{27}$$

Proof. Substituting $t = 0$ in (19a) and using (19c) gives

$$\frac{d}{dt}[S^{m_1}, \ldots, S^{m_n}](0) = -\left(\sum_{i=1}^{n} p_{m_i} + \sum_{j=n+1}^{M}\sum_{i=1}^{n} q_{l_j,m_i}\right) + \sum_{j=n+1}^{M}\left(\sum_{i=1}^{n} q_{l_j,m_i}\right) = -\sum_{i=1}^{n} p_{m_i}. \tag{28}$$

Hence, by (11) and (22), we get Equation (26) for $f'(0)$. Differentiating (19a) and using (28) gives the equation for $f''(0)$. Substituting (9) in (26) gives (27). □

These explicit expressions allow us to determine on which network the diffusion is initially faster. The expressions for the derivatives become simpler when the heterogeneity is just in q:[1]

Corollary 1. *Consider a network of size M which is homogeneous in p and heterogeneous in q. Then (26) reads*

$$f'(0) = p, \qquad f''(0) = p\left(\frac{1}{M}\sum_{j=1}^{M} q_j - p\right). \tag{29}$$

If, in addition, the heterogeneity in q is mild (see (9)), then

$$f'''(0) = p^3 + \frac{p}{M}\left(\frac{M-2}{(M-1)^2}\left(\sum_{i=1}^{M} q_i\right)^2 - \frac{2M-3}{(M-1)^2}\sum_{i=1}^{M} q_i^2 - 4p\sum_{i=1}^{M} q_i\right). \tag{30}$$

Proof. See Appendix C. □

Finally, we consider the homogeneous case:

Corollary 2. *Consider a complete homogeneous network with p, q, and M, see (12). Then*

$$f'(0) = p, \qquad f''(0) = p(q-p), \qquad f'''(0) = p\left(p^2 - 4pq + \frac{M-3}{M-1}q^2\right). \tag{31}$$

Proof. This follows from Corollary 1. □

3.3. CDF Dominance Condition

In this section, we derive a sufficient condition for the adoption in network A to be slower than in network B. Let A be a network with M nodes. For a specific realization of the discrete model (5), let t_i^A denote the time between the $(i-1)$ th and ith adoptions, where $i = 1, \ldots, M$ and $t_0^A := 0$. Therefore, the time of the mth adoption is

$$T_m^A := t_0 + t_1 + \cdots + t_m, \qquad m = 0, 1, \ldots, M,$$

and the number of adopters at time t is given by

$$N_A(t) = \max\{m \in \{0, \ldots, M\} : T_m^A \le t\}. \tag{32}$$

The expected number of adopters in network A is $n_A(t) = \mathbf{E}[N_A(t)]$.

Let B be a different network with M nodes and define t_i^B, T_m^B, $N_B(t)$, and $n_B(t)$ in a similar manner. We now show that the adoption in A is slower than in B, if $\{t_i^A\}$ and $\{t_i^B\}$ satisfy a certain CDF dominance condition:

Theorem 1. *Let $\{t_i^A(\omega_i)\}_{i=1}^M$ and $\{t_i^B(\omega_i)\}_{i=1}^M$ be two sequences of independent nonnegative random variables that satisfy the CDF dominance condition*

$$F_{t_i^A}(\tau) \le F_{t_i^B}(\tau), \qquad 1 \le i \le M, \qquad \tau \ge 0, \tag{33a}$$

such that there exists at least one index $1 \le j \le M$ for which

$$F_{t_j^A}(\tau) < F_{t_j^B}(\tau), \qquad \tau > 0. \tag{33b}$$

Then the expected number of adopters in A is less than that in B, that is,

$$n_A(t) < n_B(t), \qquad 0 < t < \infty.$$

Proof. See Appendix D. □

4. Heterogeneity in Complete Networks

In this section, we consider the qualitative effect of heterogeneity in complete networks. To do that, we compare the adoption in a heterogeneous network with that in a homogeneous network that has the same number of nodes M, the same average $\{p_j\}$, and the same average $\{q_{i,j}\}$ or $\{q_j\}$. Thus, we compare $f^{\text{het}}(t;\{p_i\}_{i=1}^M,\{q_{i,j}\}_{i,j=1}^M)$ or $f^{\text{het}}(t;\{p_i\}_{i=1}^M,\{q_j\}_{j=1}^M)$ with $f^{\text{hom}}(t;p,q,M)$, where

$$p = \frac{1}{M}\sum_{i=1}^M p_i, \qquad q = \frac{1}{M}\sum_{i=1}^M \sum_{\substack{j=1\\ j\neq i}}^M q_{i,j} = \frac{1}{M}\sum_{j=1}^M q_j. \tag{34}$$

4.1. Heterogeneity in *p*

We begin with complete networks that are heterogeneous in p and homogeneous in q, that is,

$$q_{i,j} \equiv \frac{q}{M-1}, \qquad \forall j \neq i. \tag{35}$$

When the network has just two nodes, we can use the master Equations (23) for $M = 2$ to show that heterogeneity in p always slows down the diffusion:

Lemma 4. *Consider a heterogeneous network with $M = 2$, $p_1 \neq p_2$, and $q_{1,2} = q_{2,1} \equiv q$ and let $p = \frac{p_1+p_2}{2}$. Then*

$$f^{\text{het}}(t;\{p_1,p_2\},q) < f^{\text{hom}}(t;p,q,M=2), \qquad 0 < t < \infty.$$

Proof. See Appendix E. □

One could try to generalize this result to any network size M by induction on the network size M. To do that, we need a result that global dominance between two networks is preserved when we add a new identical node to both networks. As Lemma 10 in Section 5 will show, however, this is not always the case. Therefore, we take a different approach and generalize Lemma 4 to any network size M by making use of the *CDF dominance condition*:

Theorem 2. *Consider a complete graph with M nodes that are heterogeneous in $\{p_i\}_{i=i}^M$ and homogeneous in q (see (35)). Let $p = \frac{1}{M}\sum_{i=1}^M p_i$. Then*

$$f^{\text{het}}(t;\{p_i\}_{i=1}^M,q) < f^{\text{hom}}(t;p,q,M), \qquad 0 < t < \infty.$$

Proof. Let t_k^{hom} denote the time between the $(k-1)$ th and kth adoptions in the homogeneous network, where $k = 1,\dots,M$. Let $F_k^{\text{hom}}(\tau) := \text{Prob}(t_k^{\text{hom}} \le \tau)$ denote the cumulative distribution function of t_k^{hom}. Let t_k^{het} and $F_k^{\text{het}}(\tau)$ be defined similarly for the heterogeneous network. We now introduce two auxiliary lemmas:

Lemma 5. *Consider a complete graph with M nodes that are heterogeneous in $\{p_i\}_{i=1}^M$ and in $\{q_i\}_{i=1}^M$ where q_i is the influence exerted on (and not by) node i. Furthermore, assume that the p_i values and q_i values are positively correlated, that is, $p_1 \le p_2 \le \cdots \le p_M$ and $q_1 \le q_2 \le \cdots \le q_M$. Let $p = \frac{1}{M}\sum_{i=1}^M p_i$ and $q = \frac{1}{M}\sum_{i=1}^M q_i$. Let t_1^{hom} denote the time until the first adoption and t_k^{hom} denote the time between the $(k-1)$th and kth adoptions in the homogeneous network for $k = 2,\dots,M$. Let $F_k^{\text{hom}}(t) := \text{Prob}(t_k^{\text{hom}} \le t)$ denote the cumulative distribution function of t_k^{hom} for $k = 1,\dots,M$. Let t_k^{het} and $F_k^{\text{het}}(t)$ be defined similarly for the heterogeneous network. Then $F_1^{\text{het}}(t) = F_1^{\text{hom}}(t)$ for $t \ge 0$, and*

$$F_k^{\text{het}}(t) < F_k^{\text{hom}}(t), \qquad k = 2,\dots,M, \qquad 0 < t < \infty. \tag{36}$$

Proof. See Appendix F. □

Lemma 6. *The random variables (t_i) are independent.*

Proof. See Appendix F. □

By Lemma 5, (in the special case where $q_1 = \cdots = q_M = q$), $F_1^{\text{het}}(\tau) \equiv F_1^{\text{hom}}(\tau)$ for $\tau \geq 0$ and

$$F_k^{\text{het}}(\tau) < F_k^{\text{hom}}(\tau), \qquad k = 2, \ldots, M, \qquad 0 < \tau < \infty.$$

In addition, by Lemma 6, the random variables $\{t_k^{\text{hom}}\}$ and the random variables $\{t_k^{\text{het}}\}$ are independent. Hence, the conditions of Theorem 1 are satisfied, and so Theorem 2 follows by (11). □

Remark 3. We could also prove Theorem 2 for small times using the explicit expressions for $f'(0)$ and $f''(0)$; see Appendix G.

The results of this section show that *in complete networks, heterogeneity in* $\{p_j\}$ *always slows down the diffusion*. This follows from the convexity of the model. Indeed, as the proof of Lemma 5 shows, the times between consecutive adoptions follow an exponential distribution, for which the CDF is convex in $\{p_j\}$.

To further motivate this result, consider a heterogeneous network with $M = 2$ nodes, where $p_1 = 2p$, $p_2 = 0$, and $q_1 = q_2 = q$ and its homogeneous counterpart with $p_1 = p_2 = p$ and $q_1 = q_2 = q$. In both the homogeneous and the heterogeneous cases, the first adoption occurs at a rate of $2p$. In the homogeneous case, however, the second adoption occurs at a rate of $p + q$, whereas in the heterogeneous case it occurs at a rate of q. Hence, the adoption is faster in the homogeneous network. More generally, consider an $M = 2$ heterogeneous network with $p_1 = p + \varepsilon$, $p_2 = p - \varepsilon$, and $q_1 = q_2 = q$, where $0 < \varepsilon \leq p$. The first adoption occurs at a rate of $2p$ in both the heterogeneous and homogeneous networks. For the homogeneous network, the second adoption is at the rate of $p + q$. In the heterogeneous network, because $p_1 > p_2$, then in the majority of cases node 1 is the first to adopt, in which case the second adoption is by node 2 at a rate of $p - \varepsilon + q$. In the minority of cases, node 2 is the first to adopt, in which case the second adoption is by node 1 at a rate of $p + \varepsilon + q$. Therefore, the overall adoption in the heterogeneous case is slower. This intuition can be generalized to networks with M nodes. The first adoption always occurs at the same rate for the homogeneous and heterogeneous cases, but subsequent adoptions are slower in the heterogeneous case.

4.2. Mild Heterogeneity in *q*

We now consider complete networks that are mildly heterogeneous in q but homogeneous in p, that is,

$$p_i \equiv p, \qquad i = 1, \ldots, M.$$

Theorem 3. *Consider a complete graph with M nodes that are mildly heterogeneous in* $\{q_i\}_{i=i}^M$ *(see (9)) and homogeneous in* p. *Let* $q = \frac{1}{M}\sum_{i=1}^M q_i$. *Then*

$$f^{\text{het}}(t; p, \{q_i\}_{i=1}^M) < f^{\text{hom}}(t; p, q, M), \qquad 0 < t < \infty.$$

Proof. Let t_k^{hom}, $F_k^{\text{hom}}(\tau)$, t_k^{het}, and $F_k^{\text{het}}(\tau)$ be defined as in the proof of Theorem 2. By Lemma 5, in the special case where $p_1 = p_2 = \ldots p_M$, $F_1^{\text{het}}(\tau) \equiv F_1^{\text{hom}}(\tau)$ for $\tau \geq 0$, and

$$F_k^{\text{het}}(\tau) < F_k^{\text{hom}}(\tau), \qquad k = 2, \ldots, M, \qquad 0 < \tau < \infty.$$

Therefore, the proof is the same as for Theorem 2. □

Remark 4. We can also prove Theorem 3 for small times using the explicit expressions for $f'(0)$, $f''(0)$, and $f'''(0)$; see Appendix G.

Remark 5. Theorem 3 does *not* extend to the case of a general heterogeneity $\{q_{i,j}\}$. For example, if $M = 3$, then a complete network is also a two-sided circle; so, by (16), if $q^L \neq q^R$ and $q^L + q^R \equiv q$, then

$$f^{\text{het}}\left(t; p, \{q_{i,j}\}\right) \equiv f_{\text{circle}}^{\text{2-sided}}\left(t; p, q^R, q^L, M = 3\right) \equiv f_{\text{circle}}^{\text{2-sided}}\left(t; p, \frac{q}{2}, \frac{q}{2}, M = 3\right) \equiv f^{\text{hom}}(t; p, q).$$

The question of whether $f^{\text{het}} \leq f^{\text{hom}}$ when the heterogeneity in q is not mild is currently open.

In summary, the results in this section show that *in complete networks, mild heterogeneity in* $\{q_j\}$ *always slows down the diffusion*.

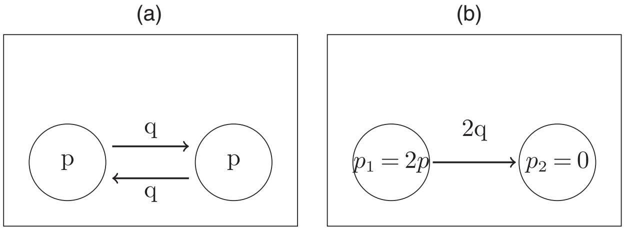
4.3. Heterogeneity in *p* and *q*

So far, we saw that one-dimensional heterogeneity (i.e., just in p or just in q) always slows down the adoption. We now show that this is not always the case when the network is heterogeneous in both p and q.

4.3.1. Positive Correlation Between $\{p_j\}$ and $\{q_j\}$.

When $\{p_j\}$ and $\{q_j\}$ are *positively correlated*, heterogeneity always slows down the adoption:

Figure 2. Networks used in Lemma 7. (a) Homogeneous. (b) Heterogeneous.



Theorem 4. *Consider a complete graph with M nodes that are heterogeneous in $\{p_i\}_{i=1}^M$ and in $\{q_i\}_{i=1}^M$, such that (9) holds. Furthermore, assume that $\{p_i\}$ and $\{q_i\}$ are positively correlated, so that $p_1 \le p_2 \le \cdots \le p_M$ and $q_1 \le q_2 \le \cdots \le q_M$. Let $p = \frac{1}{M}\sum_{i=1}^M p_i$ and $q = \frac{1}{M}\sum_{i=1}^M q_i$. Then*

$$f^{\text{het}}\Big(t; \{p_i\}_{i=1}^M, \{q_i\}_{i=1}^M\Big) < f^{\text{hom}}(t; p, q, M), \qquad 0 < t < \infty.$$

Proof. Let t_k^{hom}, $F_k^{\text{hom}}(\tau)$, $t_k^{\text{het.}}$, and $F_k^{\text{het.}}(\tau)$ be defined as in the proof of Theorem 2. In Lemma 5, we prove the CDF dominance condition, $F_1^{\text{het}}(\tau) \equiv F_1^{\text{hom}}(\tau)$ for $\tau \ge 0$ and

$$F_k^{\text{het}}(\tau) < F_k^{\text{hom}}(\tau), \qquad k = 2, \ldots, M, \qquad 0 < \tau < \infty.$$

The rest of the proof is the same as for Theorem 2. □

Remark 6. We can prove Theorem 4 for small times using the explicit expressions for $f'(0)$ and $f''(0)$; see Appendix G.

4.3.2. Nonpositive Correlation Between $\{p_j\}$ and $\{q_j\}$.

When $\{p_j\}$ and $\{q_j\}$ are not positively correlated, the effect of the heterogeneity can become more diverse, even for networks with $M = 2$. In particular, f^{het} *can be higher than* f^{hom} *for all times*:

Lemma 7. *Let B be a heterogeneous network with $M = 2$, $\{p_1, p_2\} = \{2p, 0\}$, and $\{q_1, q_2\} = \{0, 2q\}$ and let A be the corresponding homogeneous network with $M = 2$, $p_1 = p_2 = p$, and $q_1 = q_2 = q$; see Figure 2. Then for $0 < t < \infty$,*

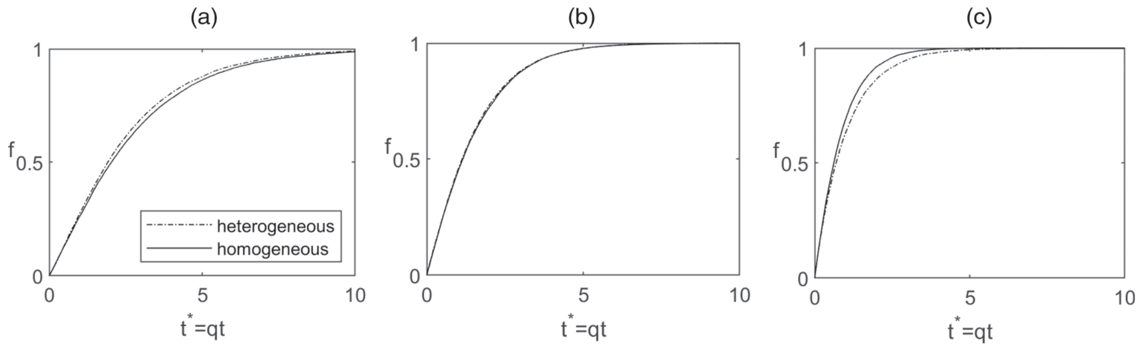
$$\begin{cases} f^B(t) > f^A(t), & \text{if } q > p, \\ f^B(t) = f^A(t), & \text{if } q = p, \\ f^B(t) < f^A(t), & \text{if } q < p. \end{cases}$$

Proof. This follows from the master Equation (23) for $M = 2$; see Appendix H. □

Remark 7. We can also prove Lemma 7 for small times using the explicit expressions for $f'(0)$ and $f''(0)$; see Appendix G.

Lemma 7 is confirmed in Figure 3 using simulations of (5). The intuition behind this result is as follows. In both networks, the first adoption occurs at the rate of $2p$. The second adoption occurs at a rate of $2q$ in the heterogeneous case

Figure 3. Fractional adoption in the homogeneous (solid) and heterogeneous (dash-dot) networks shown in Figure 2. Plot shows the average of 10^4 simulations of (5). (a) $p = \frac{q}{2}$. (b) $p = q$. (c) $p = 2q$.



and at the rate of $p+q$ in the homogeneous case. Therefore, by Theorem 1, $f^{\text{het}}(t)$ is (globally in-time) greater than, equal to, or less than $f^{\text{hom}}(t)$ if q is greater than, equal to, or less than p, respectively. We can use this intuition to show that for any M, $f^{\text{het}}(t)$ can be (globally in-time) greater than, equal to, or less than $f^{\text{hom}}(t)$:

Lemma 8. *Consider two complete networks with M nodes: a homogeneous network with $p_i = p$ and $q_{i,j} = \frac{q}{M-1}$ for all nodes, and a heterogeneous network with $p_1 = Mp$, $p_2 = p_3 = \cdots = p_M = 0$, $q_{1,2} = q_{1,3} = \cdots = q_{1,M} = \frac{2q}{M-1}$, $q_{2,1} = q_{3,1} = \cdots = q_{M,1} = 0$, and $q_{i,j} = \frac{q}{M-1}$ when $i \neq 1$ and $j \neq 1$; see Figure 4. Then for $0 < t < \infty$,*

$$\begin{cases} f^{\text{het}}(t) > f^{\text{hom}}(t), & \text{if } \dfrac{q}{M-1} > p, \\ f^{\text{het}}(t) = f^{\text{hom}}(t), & \text{if } \dfrac{q}{M-1} = p, \\ f^{\text{het}}(t) < f^{\text{hom}}(t), & \text{if } \dfrac{q}{M-1} < p. \end{cases}$$

Proof. The first adoption in both networks occurs at the rate of Mp. The second adoption occurs at a rate of $2q$ in the heterogeneous case and $(M-1)p+q$ in the homogeneous case. The n th adoption occurs at a rate of $n(M-n+1)\frac{q}{M-1}$ in the heterogeneous case and $(M-n+1)\left(p+(n-1)\frac{q}{M-1}\right)$ in the homogeneous case. Hence, the result follows from Theorem 1. □

5. Loss of Global Dominance Under Additive Transformations

Let A and B be two networks with M nodes such that the expected adoption in A is slower than in B for all times, that is,

$$f_A(t) < f_B(t), \qquad 0 < t < \infty.$$

Consider the following two additive transformations of these networks:

Transformation 1 (T1): $(A,B) \longrightarrow (A',B')$, where A' and B' are obtained by adding Δp to all nodes in both networks, that is,

$$p_j^{A'} = p_j^A + \Delta p, \qquad p_j^{B'} = p_j^B + \Delta p, \qquad j = 1, \ldots, M. \tag{37}$$

Transformation 2 (T2): $(A,B) \longrightarrow (A'',B'')$, where A'' and B'' are obtained by adding an identical $(M+1)$ node to A and B, so that

$$p_{M+1}^{A''} = p_{M+1}^{B''} = p_{M+1}, \tag{38a}$$

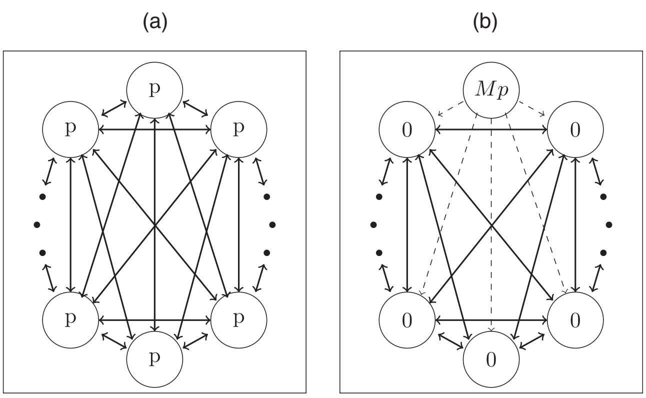
$$q_{i,M+1}^{A''} = q_{i,M+1}^{B''} = q_{M+1}^{\text{in}}, \qquad q_{M+1,i}^{A''} = q_{M+1,i}^{B''} = q_{M+1}^{\text{out}}, \qquad i = 1, \ldots, M. \tag{38b}$$

It is natural to ask the following:[2]

Question. *Is the global dominance between two networks preserved under the "additive" transformations (37) and (38), that is, is it true that $f_{A'}(t) < f_{B'}(t)$ and $f_{A''}(t) < f_{B''}(t)$ for $0 < t < \infty$?*

In some cases, global dominance is indeed preserved:

Figure 4. Networks used in Lemma 8. A solid edge indicates an internal influence of $\frac{q}{M-1}$, and a dashed edge indicates an internal influence of $\frac{2q}{M-1}$. (a) Homogeneous network. (b) Heterogeneous network.



Lemma 9. *Let $A \prec B$ (see Definition 1). Then*

$$f_A(t) < f_B(t), \qquad 0 < t < \infty,$$

and for any Δp, p_{M+1}, q^{in}_{M+1}, and q^{out}_{M+1},

$$f_{A'}(t) < f_{B'}(t), \quad f_{A''}(t) < f_{B''}(t), \qquad 0 < t < \infty.$$

Proof. If $A \prec B$, then $A' \prec B'$ and $A'' \prec B''$. Hence, the result follows from the *dominance principle* (Lemma 1). □

This lemma may seem to suggest that global dominance is indeed preserved under the additive transformations (37) and (38). This, however, is not always the case. Indeed, global dominance can be lost under a uniform addition of Δp to all nodes (T1) or under the addition of an identical node (T2):

Lemma 10. *There exist two networks A and B of size M such that $f_A(t) < f_B(t)$ for $0 < t < \infty$, but*

$$f^{A'}(t) > f^{B'}(t), \qquad t \gg 1, \tag{39a}$$

and

$$f^{A''}(t) > f^{B''}(t), \qquad t \gg 1. \tag{39b}$$

Proof. Let B be a heterogeneous network with $M = 2$, $\{p_1, p_2\} = \{2p, 0\}$, and $\{q_1, q_2\} = \{0, 2q\}$; let A be the corresponding homogeneous network with $p_1 = p_2 = p$ and $q_1 = q_2 = q$; and let $p < q$. Then by Lemma 7, $f^A(t) < f^B(t)$ for $0 < t < \infty$; see Figure 5(a).

Let A' and B' be the networks obtained from A and B when we increase all the $\{p_j\}$ s by Δp; see (37). By (24), $f^{A'}(t) = 1 - \frac{p+\Delta p}{p-q+\Delta p} e^{-(p+\Delta p+q)t} + \frac{q}{p-q+\Delta p} e^{-(2p+2\Delta p)t}$, and $f^{B'} = 1 - \frac{1}{2}\left[e^{-(2p+\Delta p)t} + \frac{2p+\Delta p}{2p-2q+\Delta p} e^{-(2q+\Delta p)t} - \frac{2q}{2p-2q+\Delta p} e^{-(2p+2\Delta p)t}\right]$. Let $\Delta p > q - p$. Then for $t \gg 1$, $f^{A'}(t) \approx 1 - \frac{p+\Delta p}{p-q+\Delta p} e^{-(p+\Delta p+q)t}$ and $f^{B'} \approx 1 - \frac{1}{2} e^{-(2p+\Delta p)t}$. Therefore, because $q > p$, we have (39a).

If we add an identical $(M+1)$ st node to A and B with $p_{M+1} = \infty$ and $q^{\mathrm{out}}_{M+1} \equiv \Delta p$, then A'' and B'' are equivalent to A' and B', and so (39b) holds. By continuity, the result also holds for finite but sufficiently large values of p_{M+1}. □

The above calculations show that the dominance between a homogeneous and heterogeneous network is not always global in time but rather can change with time:

Corollary 3. *There exists a heterogeneous network and a corresponding homogeneous network for which $f^{\mathrm{het}}(t) - f^{\mathrm{hom}}(t)$ changes its sign in $0 < t < \infty$.*

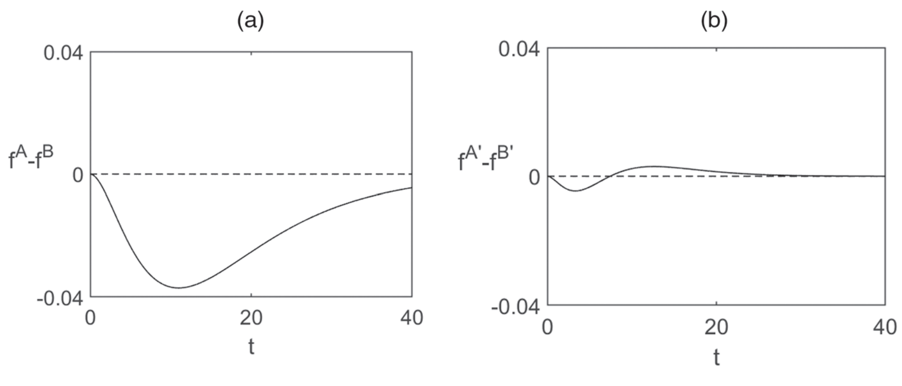
Proof. By (26) and (31), $(f^{A'})'(0) = (f^{B'})'(0) = p + \Delta p$ and $(f^{A'})''(0) - (f^{B'})''(0) = p(p-q) < 0$. Therefore,

$$f^{A'}(t) < f^{B'}(t), \qquad 0 < t \ll 1. \tag{40}$$

The result follows from (39a) and (40). □

Thus, the flip of the dominance as $(A, B) \to (A', B')$ occurs for $t \gg 1$ (see (39a)) but not for $t \ll 1$ (see (40)), as is illustrated in Figure 5. Indeed, the uniform increase of $\{p_i\}$ by Δp does not change the dominance between the heterogeneous and homogeneous networks during the initial dynamics because, by Lemma 3, the initial

Figure 5. (a) Difference between f^A and f^B from Lemma 7. (b) Difference between $f^{A'}$ and $f^{B'}$ from Lemma 10. Here $p = 0.05$, $q = 0.15$, and $\Delta p = 0.15$.



dynamics are determined by the mean and variance of $\{p_i\}$, each of which is equally affected by a uniform shift by Δp in the heterogeneous and homogeneous cases (Appendix G). This uniform increase, however, can affect the dominance later on. Indeed, as $\Delta p \to \infty$, A' and B' become homogeneous in p. However, B' is heterogeneous in q, and so its diffusion is slower than in the homogeneous case A' (Theorem 3).

6. Heterogeneity in 1D Networks

In Section 4, we analyzed the diffusion in complete networks. We now consider the opposite type of networks, namely, structured sparse networks where each node is only connected to one or two nodes.

6.1. One-Sided Circle

Assume that M consumers are located on a one-sided circle, such that each node can only be influenced by its left neighbor (Figure 1(a)). Thus, if $(j-i) \bmod M \neq 1$, then $q_{i,j} = 0$. In this case, (5) reads

$$\mathrm{Prob}(X_j(t+dt)=1) = \begin{cases} 1, & \text{if } X_j(t)=1, \\ (p_j + q_j X_{j-1}(t))\,dt, & \text{if } X_j(t)=0, \end{cases} \tag{41}$$

where $q_j = q_{j-1,j}$; see (6). Let $(S_k^j)(t) := (S^{j-k+1}, S^{j-k+2}, \ldots, S^j)(t)$ denote the event that the chain of k nodes that ends at node j are all nonadopters at time t (see Figure 1(a)), and let $[S_k^j](t)$ denote the probability of that event. We proceed to derive the master equations for $[S_k^j](t)$:

Lemma 11. *Consider the heterogeneous discrete Bass model (41) on a one-sided circle. For any j, $1 \le j \le M$, the M master equations for $\{[S_k^j]\}_{k=1}^M$ are*[3]

$$\frac{d}{dt}[S_k^j](t) = -\left(\left(\sum_{i=j-k+1}^{j} p_i\right) + q_{j-k+1}\right)[S_k^j](t) + q_{j-k+1}[S_{k+1}^j](t), \qquad k = 1, \ldots, M-1, \tag{42a}$$

and[4]

$$\frac{d}{dt}[S_M^j](t) = \left(-\sum_{i=1}^{M} p_i\right)[S_M^j](t), \tag{42b}$$

subject to the initial conditions

$$[S_k^j](0) = 1, \qquad k = 1, \ldots, M. \tag{42c}$$

Proof. See Appendix I. □

Equations (42) for $\{[S_k^j](t)\}_{k=1}^M$ are decoupled from those for $\{[S_k^i](t)\}_{k=1}^M$ for $i \neq j$. This allows us to solve them explicitly for any M and thus to obtain the fractional adoption on a one-sided heterogeneous circle:

Theorem 5. *Consider the heterogeneous discrete Bass model (41) on the one-sided circle. Then*

$$f_{\text{circle}}^{\text{1-sided}}(t) = 1 - \frac{1}{M}\sum_{j=1}^{M}[S_1^j](t), \qquad [S_1^j](t) = \sum_{k=1}^{M} c_k^j v_k^j(1) e^{\lambda_k^j t}, \tag{43}$$

where[3]

$$\lambda_k^j = \begin{cases} \left(-\sum_{i=j-k+1}^{j} p_i\right) - q_{j-k+1}, & k = 1, \ldots, M-1, \\ -\sum_{i=1}^{M} p_i, & k = M, \end{cases} \tag{44a}$$

$$v_k^j(1) = \begin{cases} 1, & k = 1, \\ \prod_{m=j-k+2}^{j} \frac{-q_m}{\left(\sum_{i=j-k+1}^{m-1} p_i\right) + q_{j-k+1} - q_m}, & k = 2, \ldots, M-1, \\ \prod_{m=j-M+2}^{j} \frac{-q_m}{\left(\sum_{i=j-M+1}^{m-1} p_i\right) - q_m}, & k = M, \end{cases} \tag{44b}$$

$c_M^j = 1$, *and for* $k = M-1, \dots, 1$,

$$c_k^j = 1 - \prod_{m=j+1-(M-1)}^{j+1-k} \frac{-q_m}{\left(\sum_{i=j-M+1}^{m-1} p_i\right) - q_m} - \sum_{l=k+1}^{M-1} \left(\prod_{m=j-l+2}^{j-k+1} \frac{-q_m}{\left(\sum_{i=j-l+1}^{m-1} p_i\right) + q_{j-l+1} - q_m} \right) c_l^j. \tag{44c}$$

Proof. See Appendix J. □

As expected, when M = 2, (43)–(44) reduce to (24); when $M = 3$, it reduces to (25) with $q_{j+1,j} = q_{j+2,j+1} = 0$.

6.2. Two-Sided Circle

Consider now a circle with M nodes where each node can be influenced by its left and right neighbors. Thus, if $|j-i| \bmod M \neq 1$, then $q_{i,j} = 0$ (Figure 1(b)). In this case, (5) reads

$$\text{Prob}(X_j(t+dt) = 1) = \begin{cases} 1, & \text{if } X_j(t) = 1, \\ [p_j + q_{j-1,j} X_{j-1}(t) + q_{j+1,j} X_{j+1}(t)]dt, & \text{if } X_j(t) = 0. \end{cases} \tag{45}$$

Let $(S_{m,n}^j)(t) := (S^{j-m}, \dots, S^j, \dots, S^{j+n})(t)$ denote the event that the $m+n+1$ nodes $\{j-m, \dots, j, \dots, j+n\}$ are all non-adopters at time t (see Figure 1(b)), and let $[S_{m,n}^j](t; M)$ denote the probability of that event. We proceed to derive the master equations for $[S^j] = [S_{0,0}^j]$:

Lemma 12. *Consider the heterogeneous discrete Bass model (45) on the two-sided circle. For any* $1 \le j \le M$ *and any* $0 \le m+n \le M-2$, *the master equations for* $\{[S_{m,n}^j]\}$ *are*[5]

$$\begin{aligned} \frac{d}{dt}[S_{m,n}^j](t) = &-\left(\left(\sum_{i=j-m}^{j+n} p_i\right) + q_{j-m-1,j-m} + q_{j+n+1,j+n}\right)[S_{m,n}^j](t) + q_{j+n+1,j+n}[S_{m,n+1}^j](t) \\ &+ q_{j-m-1,j-m}[S_{m+1,n}](t), \end{aligned} \tag{46a}$$

and

$$\frac{d}{dt}[S_M](t) = \left(-\sum_{i=1}^{M} p_i\right)[S_M](t), \tag{46b}$$

subject to the initial condition

$$[S_{m,n}^j](0) = 1, \qquad 0 \le m+n \le M-1, \tag{46c}$$

where $[S_M](t) := [S_{M-1,0}^j](t) = [S_{M-2,1}^j](t) = \cdots = [S_{0,M-1}^j](t)$.

Proof. See Appendix K. □

Remark 8. As in the one-sided case, the $\frac{M(M-1)}{2} + 1$ equations for $\{[S_{m,n}^j](t)\}_{m,n}$ are decoupled from those for $\{[S_{m,n}^i](t)\}_{m,n}$ for $i \neq j$.

Theorem 6. *Consider the heterogeneous discrete Bass Model (45) on the two-sided circle. Then*

$$f_{\text{circle}}^{\text{2-sided}}(t) = 1 - \frac{1}{M}\sum_{j=1}^{M}[S_{0,0}^j](t), \tag{47}$$

where $\{[S_{0,0}^j](t)\}_{j=1}^M$ *can be determined from (46).*

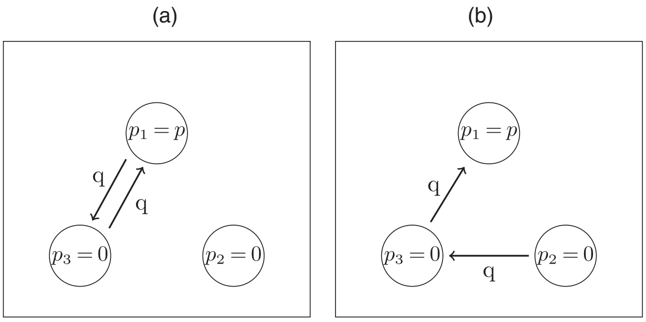
Unlike the one-sided case, we have not found a way to explicitly solve for $[S_{0,0}^j](t)$ for a general M (see Appendix L). We did, however, obtain explicit solutions of (46) for $M = 2$ and $M = 3$ (Appendix L). As expected, the resulting expressions for $f_{\text{circle}}^{\text{2-sided}}$ identify with (24) and (25), respectively.

6.3. Comparison of One-Sided and Two-Sided Circles

Recall that on homogeneous circles, one-sided and two-sided diffusion are identical if $q = q^R + q^L$; see (16). To extend this condition to the heterogeneous case, we interpret it as saying that the sum of the incoming $q_{k,j}$ into each node is identical in both networks, that is,

$$q_j^{\text{1-sided}} = q_{j-1,j}^{\text{2-sided}} + q_{j+1,j}^{\text{2-sided}}, \qquad j = 1, \dots, M. \tag{48}$$

Figure 6. Heterogeneous networks that satisfy (48) for which $f^{\text{2-sided}}_{\text{circle}}(t) > f^{\text{1-sided}}_{\text{circle}}(t)$. (a) Two-sided network. (b) One-sided network.



In light of (16), is it true that in the heterogeneous case $f^{\text{1-sided}}_{\text{circle}}(t; p, \{q_j\}) \equiv f^{\text{2-sided}}_{\text{circle}}(t; , p, \{q_{j\pm1,j}\})$ when (48) holds?

When $M = 2$, heterogeneous one-sided and two-sided circles are by definition identical, and so $f^{\text{1-sided}}_{\text{circle}} \equiv f^{\text{2-sided}}_{\text{circle}}$. When $M > 2$ however, this is no longer the case:

Lemma 13. *Consider a one-sided and a two-sided heterogeneous circles with $M \geq 3$ nodes, for which (48) holds. Then $f^{\text{1-sided}}_{\text{circle}}(t)$ can be (globally in-time) larger or smaller than $f^{\text{2-sided}}_{\text{circle}}(t)$.*

Proof. Consider the following $M = 3$ circles with $p_1 = p$ and $p_2 = p_3 = 0$.

1. Let $q_{1,3} = q_{3,1} = q$ and $q_{2,3} = q_{3,2} = q_{2,1} = q_{1,2} = 0$ (Figure 6). By (48), in the one-sided case $q_3 = q_1 = q$ and $q_2 = 0$. Hence, node 1 adopts at the same rate in both networks, node 2 adopts in neither case, and node 3 only adopts in the two-sided circle. Therefore, $f^{\text{2-sided}}_{\text{circle}}(t)$ is larger than $f^{\text{1-sided}}_{\text{circle}}(t)$ for all t.

2. Let $q_{2,3} = q_{3,2} = q$ and $q_{1,3} = q_{3,1} = q_{2,1} = q_{1,2} = 0$ (Figure 7). By (48), in the one-sided case $q_3 = q_2 = q$ and $q_1 = 0$. Hence, node 1 adopts at the same rate in both cases, but nodes 2 and 3 only adopt in the one-sided case. Therefore, $f^{\text{1-sided}}_{\text{circle}}(t)$ is larger than $f^{\text{2-sided}}_{\text{circle}}(t)$ for all t. □

Intuitively, the overall impact of an edge depends not only on its own weight but also on the node that it originates from. Thus, generally speaking, an edge that originates from a node with $p_i = 0$ has a weaker effect than an equal-weight edge that originates from a node with $p_i > 0$. Hence, one can utilize this insight to construct networks for which $f^{\text{1-sided}}_{\text{circle}}(t)$ is higher or lower than $f^{\text{2-sided}}_{\text{circle}}(t)$.

7. D-Dimensional Homogeneous Cartesian Networks

Let $f_D(t; p, q)$ denote the fraction of adopters on the infinite D-dimensional Cartesian *homogeneous* network, where nodes are labeled by their D-dimensional coordinate vector $\mathbf{i} = (i_1, \ldots, i_D)$,

$$p_{\mathbf{i}} \equiv p, \qquad q_{\mathbf{i},\mathbf{j}} = \begin{cases} \dfrac{q}{2D}, & \text{if } \|\mathbf{i}-\mathbf{j}\|_1 = 1, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathbf{i},\mathbf{j} \in \mathbb{Z}^D, \tag{49}$$

and $\|\mathbf{i}-\mathbf{j}\|_1 := \sum_{k=1}^{D} |i_k - j_k|$. Thus, each node can be influenced by its $2D$ nearest neighbors at the rate of $\frac{q}{2D}$. See, for example, Figure 8 for $M = 2$.

Figure 7. Same as Figure 6 with heterogeneous networks for which $f^{\text{2-sided}}_{\text{circle}}(t) < f^{\text{1-sided}}_{\text{circle}}(t)$.

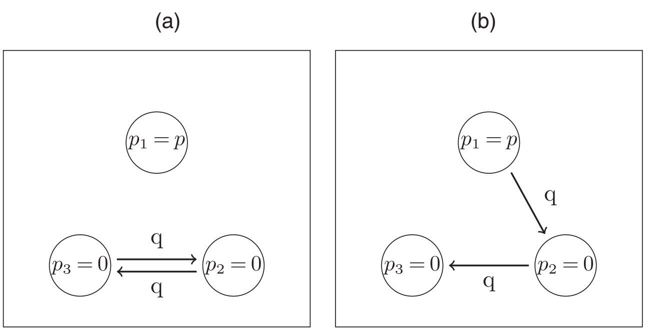
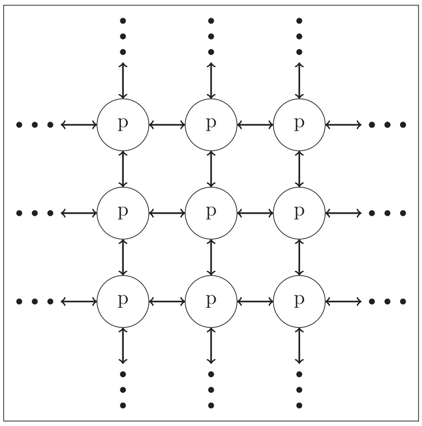


Figure 8. An infinite two-dimensional homogeneous Cartesian network. Each node has external influence of p and is influenced by its four nearest-neighbors at internal influence rates of $\frac{q}{4}$.



In Fibich and Gibori [10], the authors conjectured that for any $p,q>0$,

$$f_{1\mathrm{D}}(t;p,q) < f_{2\mathrm{D}}(t;p,q) < \cdots < f_{\mathrm{Bass}}(t;p,q), \qquad 0<t<\infty.$$

So far, this conjecture has remained open. We now prove this conjecture for the initial dynamics:

Lemma 14. *Consider the D-dimensional Cartesian networks (49). Then*

$$f_{1\mathrm{D}}(t;p,q) < f_{2\mathrm{D}}(t;p,q) < f_{3\mathrm{D}}(t;p,q) < \cdots < f_{\mathrm{Bass}}(t;p,q), \qquad 0<t\ll 1.$$

Proof. Because $f_{\mathrm{Bass}}(t;p,q) = \lim_{M\to\infty} f^{\mathrm{complete}}(t;p,q,M)$ (see Section 2.1), then by (31),

$$f'_{\mathrm{Bass}}(0) = p, \qquad f''_{\mathrm{Bass}}(0) = p(q-p), \qquad f'''_{\mathrm{Bass}}(0) = \lim_{M\to\infty} p\left(p^2 - 4pq + \frac{M-3}{M-1}q^2\right) = p(p^2-4pq+q^2).$$

In Appendix M, we show that for any dimension D,

$$f'_D(0) = p, \qquad f''_D(0) = p(q-p), \qquad f'''_D(0) = p\left(p^2 - 4pq + \frac{D-1}{D}q^2\right). \tag{50}$$

Therefore, the result follows. □

8. Distribution of Heterogeneity

When the network is not complete, the effect of heterogeneity on $f(t)$ depends also on the relative locations of the heterogeneous nodes in the network. To illustrate this, let A be a one-sided circle with M nodes that is homogeneous in q and heterogeneous in p, so that

$$q_i^A = q, \qquad p_i^A = \begin{cases} p_1, & 1\le i\le \frac{M}{2}, \\ p_2, & \frac{M}{2} < i \le M, \end{cases} \qquad i=1,\ldots,M. \tag{51}$$

Let B be a one-sided circle with M nodes that are homogeneous in q and heterogeneous in p, so that

$$q_i^B = q, \qquad p_i^B = \begin{cases} p_1, & i \text{ odd}, \\ p_2, & i \text{ even}, \end{cases} \qquad i=1,\ldots,M. \tag{52}$$

Thus, A and B have exactly the same heterogeneous nodes $\{(p_i,q_i)\}_{i=1}^M$, but their relative locations along the circle are different.

We can explicitly compute the aggregate adoptions in A and B as $M\to\infty$:

Lemma 15. *Consider the one-sided circles (51) and (52). Then*

$$\lim_{M\to\infty} f_A^{\text{het}}(t) = \frac{f_{1\text{D}}(t;p_1,q)+f_{1\text{D}}(t;p_2,q)}{2}, \tag{53}$$

where $f_{1\text{D}}$ is given by (18), and

$$\lim_{M\to\infty} f_B^{\text{het}}(t) = 1-\frac{1}{2q}e^{-qt}(\dot{V}_1(t)+\dot{U}_1(t)), \tag{54a}$$

where U_1 and *V_1 are the solution of*

$$\dot{U}_1(t) = qe^{-p_1 t}V_1(t), \qquad U_1(0)=1,$$
$$\dot{V}_1(t) = qe^{-p_2 t}U_1(t), \qquad V_1(0)=1. \tag{54b}$$

Proof. See Appendix N. □

Figure 9, (a) and (b) confirms the results of Lemma 15 numerically. In addition, Figure 9(c) shows that

$$\lim_{M\to\infty} f_A^{\text{het}}(t;p_1,p_2,q) < \lim_{M\to\infty} f_B^{\text{het}}(t;p_1,p_2,q), \qquad 0<t<\infty.$$

Therefore, in particular, the effect of heterogeneity depends also on the locations of the heterogeneous nodes along the circle. The intuition behind this inequality is as follows. Assume without loss of generality that $p_1 > p_2$. In both networks, the diffusion is limited by the rate at which the "weak" p_2 nodes adopt the product. In circle A, there is negligible interaction between the separate regions of the weak and strong nodes, and so the weak p_2 nodes adopt the product without any assistance from the p_1 nodes. In circle B, however, whenever a strong p_1 node adopts, it immediately exerts an internal influence on its adjacent weak node to adopt. Hence, the weak nodes adopt the product more quickly and so the aggregate diffusion is faster.

We can further consider the one-sided circles A, B, and C with $q_i^{A,B,C} \equiv q$ and

$$p_i^A = \begin{cases} p_1, & 1\le i<\frac{M}{3},\\ p_2, & \frac{M}{3}\le i<\frac{2M}{3},\\ p_3, & \frac{2M}{3}\le i\le M,\end{cases} \qquad p_i^B = \begin{cases} p_1, & i \bmod 3 = 1,\\ p_2, & i \bmod 3 = 2,\\ p_3, & i \bmod 3 = 0,\end{cases} \qquad p_i^C = \begin{cases} p_1, & i \bmod 3 = 1,\\ p_3, & i \bmod 3 = 2,\\ p_2, & i \bmod 3 = 0,\end{cases}$$

where $p_1 > p_2 > p_3$. Following the previous arguments, we expect that the diffusion in circle A (three separate uniform regions) is slower than in circles B and C (alternating patterns). Moreover, the diffusion in C is faster than

Figure 9. (a) Fraction of adopters f_A as a function of time in circle A with $M = 1{,}000$, $p_1 = 0.4$, $p_2 = 0.1$, and $q = 0.2$. Solid line is the explicit expression (53) and dashed line is the average of 10^4 simulations of (41). The two curves are nearly indistinguishable. (b) Same for f_B. Solid line is the explicit expression (54). (c) Comparison of f_A (dashes) and f_B (solid).

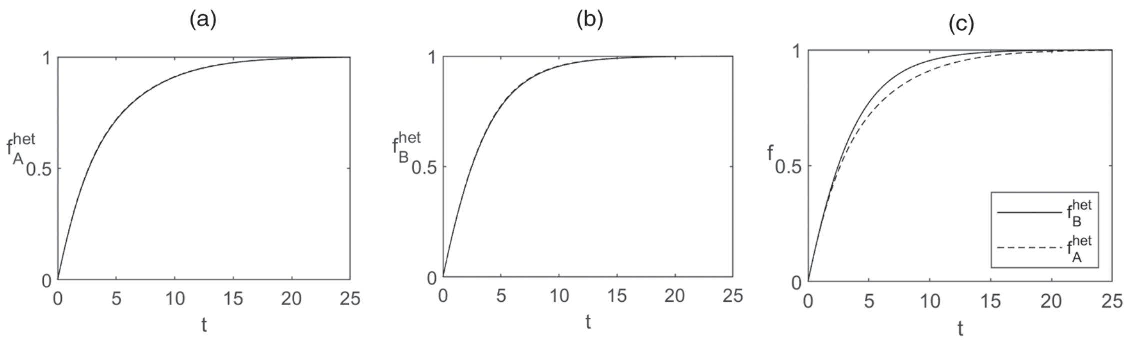
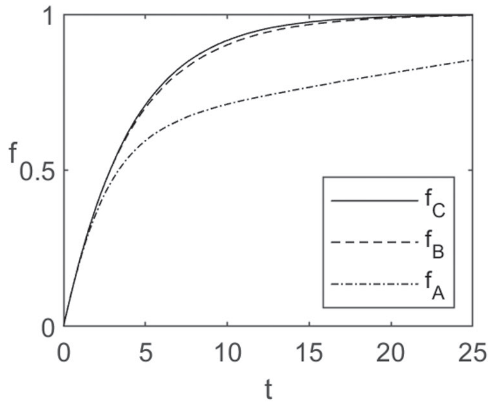


Figure 10. Comparison of f_A (dash-dot), f_B (dashes), and f_C (solid), where $M = 900$, $p_1 = 0.5$, $p_2 = 0.2$, $p_3 = 0.01$, and $q = 0.2$.



in B, because the weakest p_3 nodes are directly influenced by the strongest p_1 nodes. Figure 10 confirms these predictions numerically.

This naturally leads to the following question: For a given network structure, what is the optimal distribution of $\{p_i\}$ that maximizes the diffusion? Based on the above arguments, the weak nodes should be close to strong ones. A systematic study of this intriguing optimization problem, however, is left for a future study.

9. Level of Heterogeneity

Until now, we considered a dichotomous distinction of heterogeneity versus homogeneity. We now briefly consider the quantitative effect of varying the level of heterogeneity on $f^{\text{hom}}(t) - f^{\text{het}}(t)$. We consider vertex-transitive networks, that is, networks that are structured exactly the same about any node (e.g., circles, infinite D-dimensional Cartesian networks, complete networks).

Lemma 16 (Fibich [8], Fibich et al. [9]). *Let $\varepsilon > 0$ and $h_{j,p},\ h_{j,q} \in \mathbb{R}$ for $j = 1, \dots, M$ such that $\sum_{j=1}^{M} h_{j,p} = 0$ and $\sum_{j=1}^{M} h_{j,q} = 0$. Consider a vertex-transitive network with M nodes that is heterogeneous in p and mildly heterogeneous in q (see (9)), that is,*

$$p_j(\varepsilon) = p(1 + \varepsilon h_{j,p}), \qquad q_j(\varepsilon) = q(1 + \varepsilon h_{j,q}), \qquad j = 1, \dots, M.$$

Then, for $\varepsilon \ll 1$,

$$\begin{aligned} f^{\text{het}}(\varepsilon) &:= f^{\text{het}}(t; \{p_j(\varepsilon)\}, \{q_j(\varepsilon)\}) \\ &= f^{\text{hom}}(t; p, q) + \frac{\varepsilon^2}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} (p^2 h_{i,p} h_{j,p} a_{i,j} + 2pq h_{i,p} h_{j,q} b_{i,j} + q^2 h_{i,q} h_{j,q} c_{i,j}) + O(\varepsilon^3), \end{aligned} \tag{55}$$

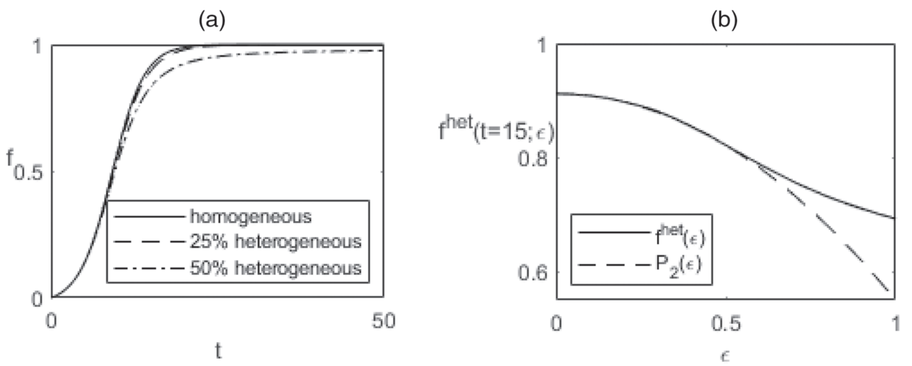
where

$$a_{i,j} := \frac{\partial^2 f^{\text{het}}}{\partial p_i \partial p_j}\Big|_{\varepsilon=0}, \qquad b_{i,j} := \frac{\partial^2 f^{\text{het}}}{\partial p_i \partial q_j}\Big|_{\varepsilon=0}, \qquad c_{i,j} := \frac{\partial^2 f^{\text{het}}}{\partial q_i \partial q_j}\Big|_{\varepsilon=0}.$$

Proof. The proof is similar to Fibich [8] and Fibich et al. [9], except that here we prove the smoothness of f^{het} using the novel master Equations (19). By (22), $f^{\text{het}}(t)$ is a linear combination of $\{[S^k](t)\}$. Because $\{[S^k](t)\}$ are solutions of the master Equations (19), which are linear constant-coefficient ODEs, they depend smoothly on $\{p_j\}$ and $\{q_{i,j}\}$, and so f^{het} depends smoothly on ε. Furthermore, because the network is vertex transitive, $f(t; \{p_j\}, \{q_j\})$ is weakly symmetric, that is, $f(t; \{(p, \dots, p) + \eta_1 \mathbf{e}_i\}, \{(q, \dots, q) + \eta_2 \mathbf{e}_i\})$ does not depend on $1 \le i \le M$ for any p, q, η_1, η_2, where $\mathbf{e}_i$ is the unit vector in the i th coordinate. Therefore, by the averaging principle (Fibich et al. [9]), $\frac{\partial f}{\partial \varepsilon} \equiv 0$, and so relation (55) holds. □

Lemma 16 shows that the effect of heterogeneity increases smoothly and monotonically with the variance in $\{p_j\}$ and $\{q_j\}$, at least for a weak heterogeneity. *The effects of the variances of* $\{p_j\}$ and $\{q_j\}$, *however, are* $O(\varepsilon^2)$ *small, that is, are much smaller than the effects of their means.* These conclusions also hold for any network structure for $t \ll 1$

Figure 11. (a) Fractional adoption in a complete homogeneous (solid) and mildly heterogeneous in q networks with $\varepsilon = 25\%$ (dashes) and $\varepsilon = 50\%$ (dash-dot). Plots show the average of 10,000 simulations with $M = 1{,}000$, $p = 0.01$, $q = 0.4$, and where $\{h_{j,q}\}$ are generated by a standard normal distribution. (b) $f^{\text{het}}(t = 15; \varepsilon)$ as a function of ε (solid). The fitted parabola is $P_2(\varepsilon) = 0.9122 - 0.3616\varepsilon^2$ (dashes).



(Appendix G) and were observed numerically in Fibich [8], Fibich et al. [9], Fibich and Gibori [10], and Goldenberg et al. [13].

For example, consider a complete network that is mildly heterogeneous in q. Then $f^{\text{het}} - f^{\text{hom}} < 0$ by Theorem 3, and so a higher variance leads to slower diffusion (Figure 11(a)). This slowdown effect, however, is quite small and barely noticeable for $\varepsilon = 25\%$. The $O(\varepsilon^2)$ effect of heterogeneity holds not only for very small values of ε but rather for $0 \le \varepsilon \le 50\%$ (Figure 11(b)). Therefore, although Lemma 16 is formally stated for small values of ε, in practice it holds for larger values of ε.

Acknowledgments
The authors thank Eilon Solan for help with the proof of Theorem 1 and Tomer Levin for useful discussions.

Appendix A. Proof of Lemma 2
Let $(S^{m_1}, \ldots, S^{m_n}, I^l)(t)$ denote the event that at time t, nodes $\{m_1, \ldots, m_n\}$ are nonadopters and node l_j is an adopter, where $1 \le n < M$, $m_i \in \{1, \ldots, M\}$, $m_i \ne m_j$ if $i \ne j$, and $l_j \notin \{m_1, \ldots, m_n\}$. Let $[S^{m_1}, \ldots, S^{m_n}, I^{l_j}](t)$ denote the probability that such an event occurs.

The configuration $(S^{m_1}, \ldots, S^{m_n})$ cannot be created. It is destroyed when

1. Any of n nonadopters adopts the product through external influence, which happens at rate p_{m_i} for the m_i^{th} nonadopter.
2. Any of the n nonadopters in $(S^{m_1}, \ldots, S^{m_n}, I^{l_j})$ adopts because of external influence by node l_j for all $n+1 \le j \le M$, which happens at rate q_{l_j,m_i}.

Therefore,

$$\frac{d}{dt}[S^{m_1}, \ldots, S^{m_n}](t) = -\left(\sum_{i=1}^{n} p_{m_i}\right)[S^{m_1}, \ldots, S^{m_n}](t) - \sum_{j=n+1}^{M}\left(\sum_{i=1}^{n} q_{l_j,m_i}\right)[S^{m_1}, \ldots, S^{m_n}, I^{l_j}](t).$$

By the total probability theorem,

$$[S^{m_1}, \ldots, S^{m_n}, I^{l_j}](t) = [S^{m_1}, \ldots, S^{m_n}](t) - [S^{m_1}, \ldots, S^{m_n}, S^{l_j}](t).$$

Combining these two relations gives (19a).

The configuration $(S^1, S^2, \ldots, S^M)$ cannot be created. It is destroyed when any of the "S"s turns into an "I," which happens at the rate p_i for the i^{th} node, hence, giving (19b).

The initial conditions (19c) follow from (4).

Appendix B. Derivation of Equation (21) for $[S^k](t)$
Substituting (20) in (19a) with $n = M-1$ and $l_1 = \{1, \ldots, M\} \backslash \{m_1, \ldots, m_{M-1}\}$ gives

$$\frac{d}{dt}[S^{m_1}, \ldots, S^{m_{M-1}}](t) = -\left(\sum_{i=1}^{M-1}(p_{m_i} + q_{l_1,m_i})\right)[S^{m_1}, \ldots, S^{m_{M-1}}](t) + \left(\sum_{i=1}^{M-1} q_{l_1,m_i}\right)e^{-(\Sigma_{j=1}^{M} p_j)t}.$$

The solution of this ODE with the initial condition (19c) is

$$[S^{m_1}, \ldots, S^{m_{M-1}}](t) = c_0 e^{-(\Sigma_{j=1}^{M} p_j)t} + c_1 e^{-(\Sigma_{i=1}^{M-1}(p_{m_i} + q_{l_1,m_i}))t},$$

where $c_0 = \frac{\sum_{i=1}^{M-1} q_{l_1,m_i}}{\sum_{i=1}^{M-1} q_{l_1,m_i} - p_{l_1}}$ and $c_1 = -\frac{p_{l_1}}{\sum_{i=1}^{M-1} q_{l_1,m_i} - p_{l_1}}$. As this process is repeated for $n = M-2, M-3, \ldots, 1$, at each stage we solve an ODE of the form

$$y' + ay = \sum_j c_j e^{-b_j} t, \qquad a := \sum_{i=1}^{n} p_{m_i} + \sum_{j=n+1}^{M} \sum_{i=1}^{n} q_{l_j,m_i}.$$

The solution of this ODE is of the form $y = c_0 e^{-at} + \sum_j \tilde{c}_j e^{-b_j} t$. Thus, y is a linear combination of all the "old" right-hand-side exponents plus the new exponent e^{-at}. Therefore, for $n = 1$ and $m_1 = k$ we get (21).

Appendix C. Proof of Corollary 1

When p is homogeneous and q is mildly heterogeneous, the master Equation (19a) reads

$$\frac{d}{dt}[S^{m_1}, \ldots, S^{m_n}](t) = -\left(np + \frac{M-n}{M-1}\sum_{i=1}^{n} q_{m_i}\right)[S^{m_1}, \ldots, S^{m_n}](t) + \sum_{j=n+1}^{M}\left(\sum_{i=1}^{n}\frac{q_{m_i}}{M-1}\right)[S^{m_1}, \ldots, S^{m_n}, S^{l_j}](t), \tag{C.1}$$

Substituting $n = 1$ in (C.1) and differentiating gives

$$\frac{d^2}{dt^2}[S^{m_1}](t) = -(p + q_{m_1})[S^{m_1}](t) + \sum_{j=2}^{M}\frac{q_{m_1}}{M-1}[S^{m_1}, S^{m_j}](t). \tag{C.2}$$

Substituting $[S^{m_1}]'(0) = -p$ and $[S^{m_1,l}]'(0) = -2p$ (see (28)) gives

$$\frac{d^2}{dt^2}[S^i](0) = (p + q_i)(p) - (2p)q_i = p(p - q_i). \tag{C.3a}$$

Similarly,

$$\frac{d^2}{dt^2}[S^i, S^j](0) = \left(2p + \frac{M-2}{M-1}(q_i + q_j)\right)(2p) - (3p)(M-2)\left(\frac{q_i}{M-1} + \frac{q_j}{M-1}\right) = p\left(4p - \frac{M-2}{M-1}(q_i + q_j)\right). \tag{C.3b}$$

Differentiating (C.2) and using Equations (C3.a,b) gives

$$\frac{d^3}{dt^3}[S^i](0) = -(p + q_i)p(p - q_i) + \frac{q_i}{M-1}\sum_{j=1, j\neq i}^{M} p\left(4p - \frac{M-2}{M-1}(q_i + q_j)\right),$$

and so we get the desired result by (22).

Appendix D. Proof of Theorem 1

Inequalities (33) imply that for every i, there is a probability space (Ω_i', P_i') and two random variables $t_i'^A\omega_i')$ and $t_i'^B(\omega_i')$ that satisfy

$$\begin{aligned} &F_{t_i^A}(\tau) = F_{t_i'^A}(\tau), \qquad 1 \le i \le M, \qquad \tau \ge 0,\\ &F_{t_i^B}(\tau) = F_{t_i'^B}(\tau), \qquad 1 \le i \le M, \qquad \tau \ge 0, \end{aligned} \tag{D.1a}$$

and also the pointwise dominance condition

$$t_i'^A(\omega_i') \ge t_i'^B(\omega_i'), \qquad \forall i,\ \forall \omega_i' \in \Omega_i', \qquad t_i'^A(\omega_i') > t_i'^B(\omega_i') \qquad \forall \omega_i' \in \Omega_i'. \tag{D.1b}$$

For example, let $0 \le \omega_i' \le 1$, let P_i' be the Lebesgue measure of $\Omega_i' = [0, 1]$, and let $t_i'^A = F_{t_i^A}^{-1}$. Then

$$F_{t_i'^A}(t) = \text{Prob}\left(t_i'^A(\omega_i') \le t\right) = \mu\left(0 \le \omega_i' \le (t_i'^A)^{-1}(t)\right) = \mu\left(0 \le \omega_i' \le F_{t_i^A}(t)\right) = F_{t_i^A}(t),$$

which gives (D.1a). In addition, because $F_{t_i^A}(t) \le F_{t_i^B}(t)$, then $t_i'^A = F_{t_i^A}^{-1} \ge F_{t_i^B}^{-1} = t_i'^B$, and because $F_{t_j^A}(t) < F_{t_j^B}(t)$, then $t_j'^A > t_j'^B$. Therefore, we have (D.1b). By (D.1b), $T_m'^A(\omega') \ge T_m'^B(\omega')$ for all m and ω' and, furthermore, $T_k'^A(\omega') > T_k'^B(\omega')$ for $j \le k \le M$. Hence, for $0 < t < \infty$,

$$\begin{cases} \text{Prob}(T_m'^A \le t) \le \text{Prob}(T_m'^B \le t), & 1 \le k \le j-1,\\ \text{Prob}(T_k'^A \le t) < \text{Prob}(T_k'^B \le t), & j \le k \le M. \end{cases} \tag{D.2}$$

Because $\text{Prob}(N_A(t) \ge m) = \sum_{k=m}^{M} \text{Prob}(N_A(t) = k)$, then

$$\sum_{m=1}^{M} \text{Prob}(N_A(t) \ge m) = \sum_{k=1}^{M} k \cdot \text{Prob}(N_A(t) = k) = E[N_A(t)]. \tag{D.3}$$

Therefore,

$$\mathbf{E}_{\omega'}[N'_A(t)] = \sum_{m=1}^{M} \mathrm{Prob}(N'_A(t) \geq m) = \sum_{m=1}^{M} \mathrm{Prob}(T'^A_m \leq t) < \sum_{m=1}^{M} \mathrm{Prob}(T'^B_m \leq t) = \mathbf{E}_{\omega'}[N'_B(t)], \tag{D.4}$$

where the inequality follows from (D.2). Because the (t_i^A)'s are independent, then so are the (t'^A_i)'s. Therefore, because t_i^A and t'^A_i are identically distributed (see (D.1a)), then

$$\mathrm{Prob}(T^A_m \leq t) = \mathrm{Prob}(T'^A_m \leq t) \quad \forall m,\ \forall t.$$

By (32), this equality can be rewritten as

$$\mathrm{Prob}(N_A(t) \geq m) = \mathrm{Prob}(N'_A(t) \geq m), \qquad \forall m,\ \forall t. \tag{D.5}$$

Therefore, by (D.5) and (D.3),

$$\mathbf{E}_{\omega}[N_A(t)] = \mathbf{E}_{\omega'}[N'_A(t)], \qquad t \geq 0.$$

Similarly,

$$\mathbf{E}_{\omega}[N_B(t)] = \mathbf{E}_{\omega'}[N'_B(t)], \qquad t \geq 0.$$

The result follows from (D.4) and the last two relations.

Appendix E. Proof of Lemma 4

We first recall an auxiliary lemma:

Lemma E.1 (Fibich et al. [12, lemma E.1.]). *Let $\sigma(t)$ be the solution of*

$$\frac{d}{dt}\sigma(t) + K\sigma(t) = b(t), \qquad t > 0, \qquad \sigma(0) = 0,$$

where K is a constant, and $b(t) > 0$ for $t > 0$. Then $\sigma(t) > 0$ for $t > 0$.

To prove Lemma 4, let $\delta := p_1 - p = -(p_2 - p) > 0$. By (22),

$$f^{\mathrm{hom}}(t) - f^{\mathrm{het}}(t) = \frac{([S^1]^{\mathrm{het}}(t) + [S^2]^{\mathrm{het}}(t)) - ([S^1]^{\mathrm{hom}}(t) + [S^2]^{\mathrm{hom}}(t))}{2}. \tag{E.1}$$

Because for both networks $p_1 + p_2 = 2p$, then by (20),

$$[S^1, S^2]^{\mathrm{het}}(t) = [S^1, S^2]^{\mathrm{hom}}(t) = [S_2](t), \qquad [S_2](t) := 1 - e^{-2pt}.$$

By (23a),

$$\frac{d}{dt}[S^i]^{\mathrm{het}}(t) = -(p_i + q)[S^i]^{\mathrm{het}}(t) + q[S_2](t), \tag{E.2}$$

and so

$$\frac{d}{dt}([S^1]^{\mathrm{het}} + [S^2]^{\mathrm{het}})(t) = -(p + \delta + q)[S^1]^{\mathrm{het}}(t) + q[S_2](t) - (p - \delta + q)[S^2]^{\mathrm{het}}(t) + q[S_2](t), \tag{E.3}$$

and

$$\frac{d}{dt}([S^1]^{\mathrm{hom}} + [S^2]^{\mathrm{hom}})(t) = 2\frac{d}{dt}[S^1]^{\mathrm{hom}}(t) = 2(-(p + q)[S^1]^{\mathrm{hom}}(t) + q[S_2](t)). \tag{E.4}$$

Let $y(t) := ([S^1]^{\mathrm{het}}(t) + [S^2]^{\mathrm{het}}(t)) - ([S^1]^{\mathrm{hom}}(t) + [S^2]^{\mathrm{hom}}(t)) = 2(f^{\mathrm{hom}}(t) - f^{\mathrm{het}}(t))$, see (E.1). We need to prove that $y(t) > 0$ for $0 < t < \infty$. By (E.3) and (E.4),

$$\frac{d}{dt}y(t) = -(p + q)y(t) - \delta[S^1]^{\mathrm{het}}(t) + \delta[S^2]^{\mathrm{het}}(t), \qquad y(0) = 0.$$

Therefore, by Lemma E.1, it suffices to show that

$$-\delta([S^1]^{\mathrm{het}}(t) - [S^2]^{\mathrm{het}}(t)) > 0. \tag{E.5}$$

By (E.2)

$$\left(\frac{d}{dt}[S^1]^{\mathrm{het}} - \frac{d}{dt}[S^2]^{\mathrm{het}}\right)(t) + (p + q)([S^1]^{\mathrm{het}} - [S^2]^{\mathrm{het}})(t) = -\delta([S^1]^{\mathrm{het}} + [S^2]^{\mathrm{het}})(t).$$

Applying Lemma E.1 to this ODE gives that $[S^1]^{\mathrm{het}}(t) < [S^2]^{\mathrm{het}}(t)$ for $t > 0$, hence (E.5) holds.

Appendix F. CDF Dominance Condition

We begin with two auxiliary lemmas.

Lemma F.1. *Assume that the time t_j at which j adopts is exponentially distributed with parameter λ_j, where $j = 1 : J$. Then*

$$\text{Prob}\begin{pmatrix} i \text{ adopts before} \\ \text{all others} \end{pmatrix} = \frac{\lambda_i}{\sum_{j=1}^J \lambda_j}. \tag{F.1}$$

Proof.

$$\begin{aligned} \text{Prob}\begin{pmatrix} i \text{ adopts before} \\ \text{all others} \end{pmatrix} &= \int_0^\infty f(i \text{ adopts at } t) \prod_{j=1, j\neq i}^J \left(\int_t^\infty f(j \text{ adopts at } \tau_j)\, d\tau_j \right) dt \\ &= \int_0^\infty \lambda_i e^{-\lambda_i t} \prod_{j=1, j\neq i}^J \left(\int_t^\infty \lambda_j e^{-\lambda_j \tau_j}\, d\tau_j \right) dt = \int_0^\infty \lambda_i e^{-\lambda_i t} \prod_{j=1, j\neq i}^J e^{-\lambda_j t}\, dt \\ &= \int_0^\infty \lambda_i e^{-(\sum_{j=1}^J \lambda_j)t}\, dt = \frac{\lambda_i}{\sum_{j=1}^J \lambda_j}. \quad \square \end{aligned}$$

Lemma F.2. *Let $a = \frac{1}{n}\sum_{i=1}^n a_i$ and $w_i = g(a_i)$, where g is monotonically increasing and $\sum_{i=1}^n w_i = 1$. Then $\sum_{i=1}^n w_i a_i \ge a$.*

Proof. Without loss of generality, $a_1 \le a_2 \le \cdots \le a_n$. Therefore, from the monotonicity of g, $w_1 \le w_2 \le \ldots \le w_n$. Hence, one can apply the *Chebyshev sum inequality*

$$\frac{1}{n}\sum_{i=1}^n a_i w_i \ge \left(\frac{1}{n}\sum_{i=1}^n a_i\right)\left(\frac{1}{n}\sum_{i=1}^n w_i\right) = \frac{a}{n}. \quad \square \tag{F.2}$$

We now prove the CDF dominance condition:

Proof of Lemma 5. The time t_1^{hom} until the first adoption in the homogeneous network is exponentially distributed with parameter Mp. Therefore, the corresponding CDF is

$$F_1^{\text{hom}}(t) := \text{Prob}(t_1^{\text{hom}} \le t) = 1 - e^{-Mpt}.$$

Similarly, the time t_1^{het} until the first adoption in the heterogeneous network is exponentially distributed with parameter $\sum_{i=1}^M p_i$. Therefore, the corresponding CDF is

$$F_1^{\text{het}}(t) := \text{Prob}(t_1^{\text{het}} \le t) = 1 - e^{-\left(\sum_{i=1}^M p_i\right)t}.$$

Hence, by definition of p,

$$F_1^{\text{hom}}(t) = F_1^{\text{het}}(t).$$

In the homogeneous case, the time t_2^{hom} between the first and second adoptions is exponentially distributed with parameter $(M-1)\left(p + \frac{q}{M-1}\right) = (M-1)p + q$. Therefore, the corresponding CDF is

$$F_2^{\text{hom}}(t) := \text{Prob}(t_2^{\text{hom}} \le t) = 1 - e^{-((M-1)p+q)t}.$$

In the heterogeneous case, let w_k denote the probability that the first adopter was k. In that case, the time between the first and second adoptions is exponentially distributed with parameter $\sum_{i=1, i\neq k}^M \left(p_i + \frac{q_i}{M-1}\right) = Mp - p_k + \frac{Mq - q_k}{M-1}$, and so the corresponding conditional CDF is

$$F_{2,k}(t) := \text{Prob}(t_2^{\text{hom}} \le t | \text{1st adopter was } k) = 1 - e^{-\left(Mp - p_k + \frac{Mq-q_k}{M-1}\right)t}.$$

Therefore, the overall CDF for t_2 is

$$\begin{aligned} F_2^{\text{het}}(t) &= \sum_{k=1}^M \text{Prob}(t_2^{\text{hom}} \le t | \text{1st adopter was } k) \cdot \text{Prob}(\text{1st adopter was } k) \\ &= \sum_{k=1}^M w_k F_{2,k}(t) = \sum_{k=1}^M w_k \left(1 - e^{-\left((M-1)p + \frac{Mq-q_k}{M-1}\right)t}\right) < 1 - e^{-\sum_{k=1}^M w_k \left((Mp - p_k + \frac{Mq-q_k}{M-1}\right)t}, \end{aligned}$$

where the last inequality follows from the strict concavity ($G_{\lambda\lambda} < 0$) of the function $G = 1 - e^{-\lambda t}$ when $0 < t < \infty$.

By (F.1),

$$w_k = \frac{p_k}{\sum_{j=1}^M p_i} = \frac{p_k}{Mp}. \tag{F.3}$$

Therefore, w_k is monotonically increasing in p_k; so by Lemma F.2,

$$\sum_{k=1}^{M} w_k p_k \geq \frac{1}{M}\sum_{k=1}^{M} p_k = p. \tag{F.4}$$

Hence, because $G = 1 - e^{-\lambda t}$ is monotonically increasing in λ,

$$F_2^{\text{het}}(t) < 1 - e^{-((M-1)p+q)t} = F_2^{\text{hom}}(t), \qquad 0 < t < \infty.$$

In the homogeneous case, the time t_3 between the second and third adoptions is exponentially distributed with parameter $(M-2)\left(p + \frac{2q}{M-1}\right)$. Therefore, the corresponding CDF is given by

$$F_3^{\text{hom}}(t) := \text{Prob}(t_3^{\text{hom}} \leq t) = 1 - e^{-\left((M-2)p + 2q\frac{M-2}{M-1}\right)t}. \tag{F.5}$$

In the heterogeneous case, let $w_{k,m}$ denote the probability that the first and second adopters were k and m, respectively. In that case, the time between the second and third adoptions is exponentially distributed with parameter $\sum_{i=1, i\neq k,m}^{M}\left(p_i + 2\frac{q_i}{M-1}\right) = Mp - p_k - p_m + 2\frac{Mq - q_k - q_m}{M-1}$, and so the corresponding conditional CDF is $F_{3,k,m}(t) = 1 - e^{-\left(Mp - p_k - p_m + 2\frac{Mq-q_k-q_m}{M-1}\right)t}$. Therefore, the overall CDF for t_3^{het} is

$$\begin{aligned} F_3^{\text{het}}(t) &= \sum_{\substack{k,m=1\\ m\neq k}}^{M} w_{k,m} F_{3,k,m}(t) = \sum_{\substack{k,m=1\\ m\neq k}}^{M} w_{k,m}\left(1 - e^{-\left(Mp - p_k - p_m + 2\frac{Mq-q_k-q_m}{M-1}\right)t}\right) \\ &< 1 - e^{-\sum_{\substack{k,m=1\\ m\neq k}}^{M} w_{k,m}\left(Mp - p_k - p_m + 2\frac{Mq-q_k-q_m}{M-1}\right)t}, \end{aligned} \tag{F.6}$$

where the last inequality follows from the strict concavity of $G = 1 - e^{-\lambda t}$ when $0 < t < \infty$. By (F.1),

$$w_{k,m} = w_k \cdot \text{Prob(2nd adopter is m} \mid \text{1st adopter was } k) = \frac{p_k}{Mp}\left(\frac{p_m + \frac{q_m}{M-1}}{Mp - p_k + \frac{Mq - q_k}{M-1}}\right).$$

In addition,

$$\begin{aligned} \sum_{\substack{k,m=1\\ m\neq k}}^{M} w_{k,m}\left(p_k + \frac{2q_k}{M-1}\right) &= \sum_{k=1}^{M} w_k\left(p_k + \frac{2q_k}{M-1}\right)\underbrace{\sum_{m=1,\, m\neq k}^{M} \text{Prob(2nd adopter is m} \mid \text{1st adopter was } k)}_{=1} \\ &= \left(p + 2\frac{q}{M-1}\right), \end{aligned}$$

and

$$\begin{aligned} \sum_{\substack{k,m=1\\ m\neq k}}^{M} w_{k,m}\left(p_m + \frac{2q_m}{M-1}\right) &= \sum_{k=1}^{M} \frac{p_k}{Mp}\sum_{m=1,\, m\neq k}^{M} \frac{p_m + \frac{q_m}{M-1}}{Mp - p_k + \frac{Mq-q_k}{M-1}}\left(p_m + \frac{2q_m}{M-1}\right) \\ &\geq \sum_{k=1}^{M}\frac{1}{M}\sum_{m=1,\, m\neq k}^{M} \frac{p_m + \frac{q_m}{M-1}}{(M-1)p + \frac{Mq-q}{M-1}}\left(p_m + \frac{2q_m}{M-1}\right) \\ &\geq \sum_{k=1}^{M}\frac{1}{M}\sum_{m=1,\, m\neq k}^{M} \frac{p + \frac{q}{M-1}}{(M-1)p + \frac{Mq-q}{M-1}}\left(p + \frac{2q}{M-1}\right) = \left(p + \frac{2q}{M-1}\right), \end{aligned}$$

where the two inequalities follow from Lemma F.2. Therefore,

$$\sum_{\substack{k,m=1\\ m\neq k}}^{M} w_{k,m}\left(p_k + p_m + 2\frac{q_k + q_m}{M-1}\right) \geq 2p + 4\frac{q}{M-1}.$$

Hence, because $G = 1 - e^{-\lambda t}$ is monotonically increasing in λ, then using (F.5) and (F.6),

$$F_3^{\text{het}}(t) < 1 - e^{-\left((M-2)p + 2q\frac{M-2}{M-1}\right)t} = F_3^{\text{hom}}(t), \qquad 0 < t < \infty.$$

Proceeding similarly, one can show that (36) also holds for $k = 4, \ldots, M$. $\square$

Proof of Lemma 6. We need to prove that for any $m \in \{2, \ldots, M\}$, $t_m \in \{t_m^{\text{het}}, t_m^{\text{hom}}\}$, and $\{\tau_1, \ldots, \tau_m\} \in (\mathbb{R}^+)^m$,

$$\text{Prob}(t_m \leq \tau_m) = \text{Prob}(t_m \leq \tau_m | t_1 \leq \tau_1, \ldots, t_{m-1} \leq \tau_{m-1}).$$

In the homogeneous case, the random variable $t_m^{\rm hom}$ and the conditional random variable $t_m^{\rm hom}|(t_1^{\rm hom} \le \tau_1, \ldots, t_{m-1}^{\rm hom} \le \tau_{m-1})$ are both exponentially distributed with parameter $(M-(m-1))\Big(p+(m-1)\frac{q)}{M-1}\Big)$. Hence,

$$\mathrm{Prob}(t_m^{\rm hom} \le \tau_m) = 1 - e^{-\left((M-(m-1))p+(m-1)q\frac{M-(m-1)}{M-1}\right)t} = \mathrm{Prob}(t_m^{\rm hom} \le \tau_m | t_1^{\rm hom} \le \tau_1, \ldots, t_{m-1}^{\rm hom} \le \tau_{m-1}).$$

In the heterogeneous case, to simplify the calculations, we present the proof for $m = 2$, that is,

$$\mathrm{Prob}(t_2^{\rm het} \le t) = \mathrm{Prob}(t_2^{\rm het} \le t | t_1^{\rm het} \le \tau_1). \tag{F.7}$$

The proof for the other cases is identical. Recall that

$$\mathrm{Prob}(t_2^{\rm het} \le t) = \sum_{k=1}^{M} \mathrm{Prob}(t_2^{\rm het} \le t | \text{1st adopter was } k) \cdot \mathrm{Prob}(\text{1st adopter was } k).$$

Therefore,

$$\mathrm{Prob}(t_2^{\rm het} \le t | t_1^{\rm het} \le \tau_1) = \sum_{k=1}^{M} \mathrm{Prob}(t_2^{\rm het} \le t | \text{1st adopter was } k,\ t_1^{\rm het} \le \tau_1) \cdot \mathrm{Prob}(\text{1st adopter was } k | t_1^{\rm het} \le \tau_1).$$

The conditional random variables $t_2^{\rm het}|(\text{1st adopter was } k)$ and $t_2^{\rm het}|(\text{1st adopter was } k,\ t_1^{\rm het} \le \tau_1)$ are both exponentially distributed with parameter $(Mp - p_k + q)$. Therefore,

$$\mathrm{Prob}(t_2^{\rm het} \le t | \text{1st adopter was } k) = \mathrm{Prob}(t_2^{\rm het} \le t | \text{1st adopter was } k,\ t_1^{\rm het} \le \tau_1)$$

In addition, by (F.3), Prob(1st adopter was k) $= w_k = \frac{p_k}{\sum_{j=1}^M p_i}$ is independent of $t_1^{\rm het}$, and so

$$\mathrm{Prob}(\text{1st adopter was } k) = \mathrm{Prob}(\text{1st adopter was } k | t_1^{\rm het} \le \tau_1).$$

Therefore, we proved (F.7). □

Appendix G. Small-Time Analysis

Many results in this paper can be easily proven for $t \ll 1$ using the explicit expressions for $f'(0), f''(0)$, and $f'''(0)$ in Section 3.2:

- Theorem 2:

By (26), when the network is heterogeneous in p and homogeneous in q, then

$$(f^{\rm het})'(0) = \frac{1}{M}\sum_{i=1}^{M} p_i, \qquad (f^{\rm het})''(0) = \frac{1}{M}\sum_{i=1}^{M} p_i(q - p_i) = \frac{q}{M}\sum_{i=1}^{M} p_i - \frac{1}{M}\sum_{i=1}^{M} p_i^2,$$

By (31), $(f^{\rm hom})'(0) = p$ and $(f^{\rm hom})''(0) = qp - p^2$. Therefore, under the conditions of Theorem 2[6],

$$(f^{\rm het})'(0) = (f^{\rm hom})'(0), \qquad (f^{\rm hom})''(0) - (f^{\rm het})''(0) = \mathrm{Var}\{p_1, \ldots, p_m\} > 0.$$

Hence, $f^{\rm het}(t) < f^{\rm hom}(t)$ for $0 < t \ll 1$.

- Theorem 3:

Under the conditions of Theorem 3, by (29) and (31), $f'(0) = p$ and $f''(0) = p(q - p)$, both for the heterogeneous and homogeneous networks. Furthermore, by (30) and (31), $(f^{\rm het})'''(0) = p^3 + pq\Big(\frac{M(M-2)}{(M-1)^2}q - 4p\Big) - \frac{(2M-3)}{M(M-1)^2}p\sum_{i=1}^M q_i^2$ and $(f^{\rm hom})'''(0) = p^3 + pq\Big(\frac{M(M-2)}{(M-1)^2}q - 4p\Big) - \frac{(2M-3)}{(M-1)^2}pq^2$. Therefore, $(f^{\rm hom})'''(0) - (f^{\rm het})'''(0) = p\frac{(2M-3)}{(M-1)^2}\mathrm{Var}\{q_1, \ldots, q_M\} > 0$. Hence, $f^{\rm het}(t) < f^{\rm hom}(t)$ for $0 < t \ll 1$.

- Theorem 4:

Under the conditions of Theorem 4, by (26) and (31), $(f^{\rm het})'(0) = \frac{1}{M}\sum_{i=1}^M p_i = p = (f^{\rm hom})'(0)$. By the Chebyshev sum inequality (see (F.2)), because $\{p_i\}$ and $\{q_i\}$ are positively correlated, then $\sum_{i=1}^M p_i q_i \ge \frac{1}{M}\sum_{j=1}^M q_j \sum_{i=1}^M p_i$; so by (27),

$$(f^{\rm het})''(0) \le \frac{1}{M(M-1)}\left[\sum_{j=1}^{M} q_j \sum_{i=1}^{M} p_i - \frac{1}{M}\sum_{j=1}^{M} q_j \sum_{i=1}^{M} p_i\right] - \frac{1}{M}\sum_{i=1}^{M} p_i^2 = pq - \frac{1}{M}\sum_{i=1}^{M} p_i^2.$$

Therefore, because $(f^{\rm hom})''(0) = pq - p^2$ (see (31)), then $(f^{\rm hom})''(0) - (f^{\rm het})''(0) \ge \mathrm{Var}\{p_1, \ldots, p_m\} > 0$. Hence, $f^{\rm het}(t) < f^{\rm hom}(t)$ for $0 < t \ll 1$.

- Lemma 7:

By (26) and (31), $(f^{\rm het})'(0) = (f^{\rm hom})'(0) = p$. By (26) and (31), $(f^{\rm het})''(0) = 2p(q - p)$, and $(f^{\rm hom})''(0) = p(q - p)$. Therefore, the difference $(f^{\rm het})''(0) - (f^{\rm hom})''(0) = p(q - p)$ is positive if $q > p$ and negative if $q < p$, which implies Lemma 7 for $0 < t \ll 1$.

• Level of heterogeneity (Section 9):
By Lemma 3 and Corollary 1, $f'(0)$ only depends on the mean of p, whereas $f''(0)$ depends also on the variance of p. Similarly, $f''(0)$ only depends on the mean of q, whereas $f'''(0)$ depends also on the variance of q. Hence, the effects of the variances are much smaller than those of the parameters themselves.

Appendix H. Proof of Lemma 7

First, assume that $p \neq q$. By Equation (24),

$$f^{\text{hom}}(t) = 1 - \left(\frac{p}{p-q}\right)e^{-(p+q)t} + \left(\frac{q}{p-q}\right)e^{-2pt}. \tag{H.1}$$

We cannot use (24) to obtain $f^{\text{het}}(t)$ because $b_1 = \frac{q_{2,1}}{p_2 - q_{2,1}} = \frac{0}{0}$. Therefore, we solve (23) directly. By (20), $[S^1, S^2](t) = e^{-2pt}$. By (23a) and (19c),

$$\begin{cases} \dfrac{d}{dt}[S^1](t) = -2p[S^1](t), & [S^1](0) = 1, \\ \dfrac{d}{dt}[S^2](t) = -2q[S^2](t) + 2qe^{-2pt}, & [S^2](0) = 1. \end{cases}$$

Therefore,

$$[S^1](t) = e^{-2pt}, \qquad [S^2](t) = \frac{q}{q-p}e^{-2pt} - \frac{p}{q-p}e^{-2qt},$$

and so, by (22),

$$f^{\text{het}}(t) = 1 - \frac{2q-p}{2(q-p)}e^{-2pt} + \frac{p}{2(q-p)}e^{-2qt}. \tag{H.2}$$

Hence,

$$g(t) := f^{\text{het}}(t) - f^{\text{hom}}(t) = \frac{p}{q-p}\left(\frac{1}{2}(e^{-2pt} + e^{-2qt}) - e^{-(p+q)t}\right). \tag{H.3}$$

For any strictly convex function ($f'' > 0$), we have that

$$f\left(\frac{x_1 + x_2}{2}\right) < \frac{1}{2}(f(x_1) + f(x_2)). \tag{H.4}$$

Because $f = e^{at}$ is convex, Equations (H.3) and (H.4) give that $g(t) > 0$ if $q > p$ and $g(t) < 0$ if $q < p$. By continuity, $g(t) \equiv 0$ if $p = q$.

Appendix I. Proof of Lemma 11

By (19a), for $1 \leq n \leq M-1$,

$$\begin{aligned} \frac{d}{dt}[S_k^j](t) &= \frac{d}{dt}[S^{j-k+1}, S^{j-k+2}, \ldots, S^j](t) \\ &= -\left(\left(\sum_{i=j-k+1}^{j} p_i\right) + \sum_{\substack{l=1 \\ l \notin \{j-k+1, j-k+2, \ldots, j\}}}^{M} \left(\sum_{i=j-k+1}^{j} q_{l,i}\right)\right)[S^{j-k+1}, S^{j-k+2}, \ldots, S^j](t) \\ &\quad + \sum_{\substack{l=1 \\ l \notin \{j-k+1, j-k+2, \ldots, j\}}}^{M} \left(\sum_{i=j-k+1}^{j} q_{l,i}\right)[S^{j-k+1}, S^{j-k+2}, \ldots, S^j, S^l](t). \end{aligned}$$

Because the network is a one-sided circle, $q_{l,i} \neq 0$ only if $(i-l) \bmod M = 1$. The only nonzero $q_{l,i}$ in the sum is $q_{j-k,j-k+1}$ because $i \in \{j-k+1, j-k+2, \ldots, j\}$ and $l \notin \{j-k+1, j-k+2, \ldots, j\}$. Equations (42b) and (42c) follow directly from (19b) and (19c), respectively.

Appendix J. Proof of Theorem 5

We can rewrite (42) as a system of M linear constant-coefficient ODEs

$$[\dot{S}^j] = A^j[S^j], \tag{J.1a}$$

where

$$[\boldsymbol{S}^j] = \begin{pmatrix} [S_1^j](t) \\ [S_2^j](t) \\ \vdots \\ (t) \end{pmatrix}, \qquad [\dot{\boldsymbol{S}^j}] = \begin{pmatrix} [\dot{S_1^j}]\,(t) \\ [\dot{S_2^j}]\,(t) \\ \vdots \\ [\dot{S_M^j}]\,(t) \end{pmatrix}.$$

and A^j is the bidiagonal matrix whose two nonzero diagonals are

$$\begin{pmatrix} a_{1,1} \\ a_{2,2} \\ \vdots \\ a_{k,k} \\ \vdots \\ a_{M,M} \end{pmatrix} = \begin{pmatrix} -p_j - q_j \\ -p_{j-1} - p_j - q_{j-1} \\ \vdots \\ \left(-\sum_{i=j-k+1}^{j} p_i\right) - q_{j-k+1} \\ \vdots \\ -\sum_{i=1}^{M} p_i \end{pmatrix}, \qquad \begin{pmatrix} a_{1,2} \\ a_{2,3} \\ \vdots \\ a_{k-1,k} \\ \vdots \\ a_{M-1,M} \end{pmatrix} = \begin{pmatrix} q_j \\ q_{j-1} \\ \vdots \\ q_{j-k+1} \\ \vdots \\ q_{j-M+2} \end{pmatrix},$$

together with the initial condition

$$[\boldsymbol{S}^j]_{|t=0} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{J.1b}$$

Lemma J.1. *The solution of* (J.1a,b) *is*

$$[\boldsymbol{S}^j] = \sum_{k=1}^{M} c_k^j \boldsymbol{v}_k^j e^{\lambda_k^j t}, \tag{J.2}$$

where λ_k^j *is given by* (44a), c_k^j *is given by* (44c), *and*

$$v_k^j(n) = \begin{cases} \prod_{m=j-k+2}^{j-n+1} \left(\frac{-q_m}{(\sum_{i=j-k+1}^{m-1} p_i) + q_{j-k+1} - q_m} \right), & \text{if } n \le k-1, k = 1, \ldots, M-1, \\ \prod_{m=j-M+2}^{j-n+1} \left(\frac{-q_m}{(\sum_{i=j-M+1}^{m-1} p_i) - q_m} \right), & \text{if } n \le M-1, k = M, \\ 1, & \text{if } n = k, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The eigenvalues of A^j are its diagonal elements $\{\lambda_k^j\}_{k=1}^M$. Hence, we can solve for their corresponding eigenvectors $\{v_k^j\}_{k=1}^M$. The coefficients $\{c_k^j\}_{k=1}^M$ are determined from (J.2) and (J.1b), which gives $\sum_{k=1}^M c_k^j \boldsymbol{v}_k^j = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. □

Because $\text{Prob}(X_j(t) = 0) = [S_1^j](t)$, the expected fraction of adopters is given by (43).

Appendix K. Proof of Lemma 12

By (19a), for $q \le n \le M-1$,

$$\frac{d}{dt}[S_{m,n}^j](t) = \frac{d}{dt}[S^{j-m}, \ldots, S^j, \ldots, S^{j+n}](t) =$$
$$-\left(\left(\sum_{i=j-m}^{j+n} p_i \right) + \sum_{\substack{l=1 \\ l \notin \{j-m, \ldots, j, \ldots, j+n\}}}^{M} \left(\sum_{i=1}^{n} q_{l,i} \right) \right) [S^{j-m}, \ldots, S^j, \ldots, S^{j+n}](t)$$
$$+ \sum_{\substack{l=1 \\ l \notin \{j-m, \ldots, j, \ldots, j+n\}}}^{M} \left(\sum_{i=j-m}^{j+n} q_{l,i} \right) [S^{j-m}, \ldots, S^j, \ldots, S^{j+n}, S^l](t).$$

Because this network is a two-sided circle, $q_{l,i} \neq 0$ only if $|i-l| \bmod M = 1$. Because $i \in \{j-m, \dots, j, \dots, j+n\}$ and $l \notin \{j-m, \dots, j, \dots, j+n\}$, the only non-zero $q_{l,i}$ are $q_{j-m-1,j-m}$ and $q_{j+n+1,j+n}$. Equations (46b) and (46c) follow directly from (19b) and (19c), respectively.

Appendix L. $[S^j_{0,0}]$ in the Two-Sided Case

Equation (46) can be written as the system of $\frac{M(M-1)}{2}+1$ linear constant-coefficient ODEs

$$[\dot{S^j}] = A^j[S^j], \tag{L.1a}$$

where

$$[S^j] = \begin{pmatrix} [S^j_{0,0}](t) \\ [S^j_{0,1}](t) \\ [S^j_{1,0}](t) \\ [S^j_{0,2}](t) \\ [S^j_{1,1}](t) \\ [S^j_{2,0}](t) \\ [S^j_{0,3}](t) \\ \vdots \\ [S^j_{M-2,0}](t) \\ [S^j_M](t) \end{pmatrix}, \quad [\dot{S^j}] = \begin{pmatrix} [\dot{S}^j_{0,0}](t) \\ [\dot{S}^j_{0,1}](t) \\ [\dot{S}^j_{1,0}](t) \\ [\dot{S}^j_{0,2}](t) \\ [\dot{S}^j_{1,1}](t) \\ [\dot{S}^j_{2,0}](t) \\ [\dot{S}^j_{0,3}](t) \\ \vdots \\ [\dot{S}^j_{M-2,0}](t) \\ [\dot{S}^j_M](t) \end{pmatrix}, A^j = \begin{pmatrix} a_1 & q_{j+1,j} & q_{j-1,j} & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & a_2 & 0 & q_{j+2,j+1} & q_{j-1,j} & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & a_3 & 0 & q_{j+1,j} & q_{j-2,j-1} & 0 & \ddots & \ddots \\ \vdots & \ddots & 0 & \ddots & \dots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & \ddots & a_4 & \dots & q_{j+b+1,j+b} & q_{j-a-1,j-a} & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & a_5 \\ 0 & \ddots & \ddots & \dots & \dots & \dots & 0 & 0 & -\sum_{i=1}^{M} p_i \end{pmatrix}$$

and

$$a_1 = -p_j - q_{j-1,j} - q_{j+1,j}, \quad a_2 = -p_j - p_{j+1} - q_{j-1,j} - q_{j+2,j+1}, \quad a_3 = -p_{j-1} - p_j - q_{j-2,j-1} - q_{j+1,j},$$
$$a_4 = \left(-\sum_{i=j-a}^{j+b} p_i\right) - q_{j-a-1,j-a} - q_{j+b+1,j+b}, \quad a_5 = q_{j-M+1,j-M+2} + q_{j+1,j},$$

together with the initial conditions

$$[S^j]_{|t=0} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{L.1b}$$

The eigenvalues of A^j are its diagonal elements

$$\lambda^j_{a,b} = \left(-\sum_{i=j-a}^{j+b} p_i\right) - q_{j-a-1,j-a} - q_{j+b+1,j+b}, \qquad a,b = 0,1,\dots,M-2, \qquad a+b \le M-2,$$

and $\lambda_M = -\sum_{i=1}^{M} p_i$.

We have not yet found a way to explicitly solve for the eigenvectors of A^j for a general M. Under the assumption that all eigenvalues are unique, for $M=2$, the two eigenvectors are

$$v^j_{0,0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v^j_2 = \begin{pmatrix} -\dfrac{q_{j-1,j}}{p_{j-1} - q_{j-1,j}} \\ 1 \end{pmatrix}.$$

When $M=3$, the four eigenvectors are

$$v^j_{0,0}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad v^j_{0,1}=\begin{pmatrix}-\dfrac{q_{j+1,j}}{p_{j+1}+q_{j+2,j+1}-q_{j+1,j}}\\1\\0\\0\end{pmatrix},\quad v^j_{1,0}=\begin{pmatrix}-\dfrac{q_{j-1,j}}{p_{j-1}+q_{j-2,j-1}-q_{j-1,j}}\\0\\1\\0\end{pmatrix},$$

$$v^j_3=\begin{pmatrix}\dfrac{q_{j+1,j}\left(\dfrac{q_{j-1,j}+q_{j+2,j+1}}{p_{j-1}-q_{j-1,j}-q_{j+2,j+1}}\right)+q_{j-1,j}\left(\dfrac{q_{j-2,j-1}+q_{j+1,j}}{p_{j+1}-q_{j-2,j-1}-q_{j+1,j}}\right)}{p_{j+1}+p_{j-1}-q_{j-1,j}-q_{j+1,j}}\\-\dfrac{q_{j-1,j}+q_{j+2,j+1}}{p_{j-1}-q_{j-1,j}-q_{j+2,j+1}}\\-\dfrac{q_{j-2,j-1}+q_{j+1,j}}{p_{j+1}-q_{j-2,j-1}-q_{j+1,j}}\\1\end{pmatrix}.$$

After solving for the eigenvectors of A^j, we get that

$$[S^j]=\sum_{\substack{a,b\in\{0,\ldots,M-2\}\\a+b\le M-1}}c^j_{a,b}\boldsymbol{v}^j_{a,b}e^{\lambda^j_{a,b}t},\tag{L.2}$$

where $(c_{a,b},\boldsymbol{v}_{a,b},\lambda_{a,b})=(c_M,\boldsymbol{v}_M,\lambda_M)$ when $a+b=M-1$, and the coefficients $c^j_{a,b}$ are determined by (L.2) and (L.1b):

$$\sum_{\substack{a,b\in\{0,\ldots,M-2\}\\a+b\le M-1}}c^j_{a,b}\boldsymbol{v}^j_{a,b}=\begin{pmatrix}1\\\vdots\\1\end{pmatrix}.$$

Appendix M. Proof of Equation (50)

The equations for $f'_D(0)$ and $f''_D(0)$ follow from (29) by letting $M\to\infty$. In order to derive $f'''_D(0)$, we first note that because each node is connected to its $2D$ neighbors, then by (19a) and (49),

$$\frac{d}{dt}[S^{\mathbf{i}}](t)=-\left(p+\sum_{j=1}^{2D}\frac{q}{2D}\right)[S^{\mathbf{i}}](t)+\sum_{j=1}^{D}\frac{q}{2D}[S^{\mathbf{i}},S^{\mathbf{i}+\mathbf{e}_j}](t)+\sum_{j=1}^{D}\frac{q}{2D}[S^{\mathbf{i}},S^{\mathbf{i}-\mathbf{e}_j}](t),$$

where $\mathbf{e}_j(i)$ is the unit vector in the j th coordinate. By translational invariance, $[S^{\mathbf{i}}]$ and $[S^{\mathbf{i}},S^{\mathbf{i}\pm\mathbf{e}_j}]$ are independent of $\mathbf{i}$, $\mathbf{e}_j$, and $\pm$, and so

$$\frac{d}{dt}[S^{\mathbf{i}}](t)=-(p+q)[S^{\mathbf{i}}](t)+q[S_2](t).\tag{M.1}$$

Let $(S_3^-)(t)$ denote the event that any configuration in which three adjacent nonadopters are colinear appears at time t, for example, $(S^{\mathbf{i}},S^{\mathbf{i}+\mathbf{e}_j},S^{\mathbf{i}+2\mathbf{e}_j})(t)$, and let $(S_3^L)(t)$ denote the event that any configuration in which three adjacent nonadopters form an L-shape appears at time t, for example, $(S^{\mathbf{i}},S^{\mathbf{i}+\mathbf{e}_j},S^{\mathbf{i}+\mathbf{e}_j+\mathbf{e}_k})(t)$, where $j\ne k$. Let $[S_3^-](t)$ and $[S_3^L](t)$ denote the probabilities of these events, respectively. By translational invariance, $[S_3^-](t)$ is the same for any such configuration and similarly for $[S_3^L](t)$. Therefore, by (19a) and (49),

$$\frac{d}{dt}[S_2](t)=-\left(2p+2\left(\frac{q}{2D}\right)+(4D-4)\left(\frac{q}{2D}\right)\right)[S_2](t)+2\left(\frac{q}{2D}\right)[S_3^-](t)+(4D-4)\left(\frac{q}{2D}\right)[S_3^L](t),\tag{M.2}$$

because each node in the configuration (S_2) is potentially influenced by $2D-1$ nodes, $2D-2$ of which form an L-shape when added to (S_2), and one of which forms a line. Differentiating (M.1) and using (19c) and (28) gives

$$\frac{d^2}{dt^2}[S](0)=-(p+q)[S]'(0)+q[S_2]'(0)=p(p-q).\tag{M.3a}$$

Furthermore, by (M.2),

$$\frac{d^2}{dt^2}[S_2](0)=\left(2p+(4D-2)\left(\frac{q}{2D}\right)\right)(2p)-(4D-2)\left(\frac{q}{2D}\right)(3p)=p\left(4p-\frac{2D-1}{D}q\right).\tag{M.3b}$$

Therefore, differentiating (M.1) twice and substituting in the right hand sides of Equations (M.3a,b) gives

$$\frac{d^3}{dt^3}[S_1](0) = -\Big(p + 2D\Big(\frac{q}{2D}\Big)\Big)p(p-q) + (2D)\Big(\frac{q}{2D}\Big)p\Big(4p - \frac{2D-1}{D}q\Big) = -p\Big(p^2 - 4pq + \frac{D-1}{D}q^2\Big),$$

and so we get the desired result.

Appendix N. Proof of Lemma 15

For any finite t, nodes whose distance from the interface between p_1 and p_2 is $\gg qt$ only "see" a homogeneous environment. Therefore, as $M \to \infty$, the interaction between nodes with different p_i values becomes negligible, and so

$$f_A^{\mathrm{het}}(t) \sim \frac{f_{\mathrm{line}}^{\text{1-sided}}\Big(t; p_1, q, \frac{M}{2}\Big) + f_{\mathrm{line}}^{\text{1-sided}}\Big(t; p_2, q, \frac{M}{2}\Big)}{2}.$$

Because $\lim_{M\to\infty} f_{\mathrm{line}}^{\text{1-sided}}(t;p,q,M) = \lim_{M\to\infty} f_{\mathrm{circle}}^{\text{1-sided}}(t;p,q,M)$ (see Fibich et al. [12]) and $\lim_{M\to\infty} f_{\mathrm{circle}}^{\text{1-sided}}(t;p,q,M) = f_{\mathrm{1D}}(t;p,q)$ (see (18)), result (53) for f_A follows.

In circle B, as $M \to \infty$, by translation symmetry, $[S_k^i](t) \equiv [S_k^{i+2}](t)$ for any i and k. Therefore,

$$f_B^{\mathrm{het}}(t) = 1 - \frac{[S_1^1](t) + [S_1^2](t)}{2}.$$

Let $p := \frac{p_1+p_2}{2}$. By (42a),

$$[\dot{S_k^1}](t) = \begin{cases} -((k-1)p + p_1 + q)[S_k^1](t) + q[S_{k+1}^1](t), & k\,\text{odd}, \\ -(kp+q)[S_k^1](t) + q[S_{k+1}^1](t), & k\,\text{even}, \end{cases}$$

where $[S_k^1](0) = 1$ for all k. The substitution

$$[S_k^1](t) = \begin{cases} e^{-(k-1)pt}T_1(t), & k\,\text{odd}, \\ e^{-kpt}R_1(t), & k\,\text{even}, \end{cases}$$

reduces this infinite system of ODEs to the two coupled ODEs

$$\dot{R}_1(t) = -qR_1(t) + qT_1(t), \qquad R_1(0) = 1,$$

$$\dot{T}_1(t) = -(p_1+q)T_1(t) + qe^{-2pt}R_1(t), \qquad T_1(0) = 1.$$

Furthermore, the substitutions $R_1(t) = e^{-qt}U_1(t)$ and $T_1(t) = e^{-(p_1+q)t}V_1(t)$ yield system (54b) for $U_1(t)$ and $V_1(t)$, and so

$$[S_1^1](t) = T_1(t) = e^{-(p_1+q)t}V_1(t) = \frac{1}{q}e^{-qt}\,\dot{U}_1(t).$$

Repeating this procedure for the infinite system $\{[S_k^2](t)\}_k$ yields $[S_1^2](t) = \frac{1}{q}e^{-qt}\,\dot{U}_2(t)$, where $U_2(t)$ is the solution of

$$\dot{U}_2(t) = qe^{-p_2 t}V_2(t), \quad \dot{V}_2(t) = qe^{-p_1 t}U_2(t) \qquad U_2(0) = V_2(0) = 1.$$

Comparing this system with (54b) shows that $V_1(t) \equiv U_2(t)$, and so $[S_1^2](t) = \frac{1}{q}e^{-qt}\,\dot{V}_1(t)$. Hence,

$$f_B(t) = 1 - \frac{1}{2}([S_1^1](t) + [S_1^2](t)) = 1 - \frac{1}{2q}e^{-qt}(\dot{V}_1(t) + \dot{U}_1(t)).$$

Endnotes

[1] This is the case, for example, for homogeneous Cartesian networks, see Equation (49) in Section 7.

[2] One motivation for asking this question is given in Section 4.1.

[3] Here we use the conventions of Remark 1.

[4] Note that $(S_M^j)(t) = (S_M)(t)$ is independent of j.

[5] Here we also use the conventions of Remark 1.

[6] Recall that $\frac{1}{M}\sum_{i=1}^M x_i^2 - \mathbf{E}[X]^2 = \frac{1}{M}\sum_{i=1}^M (x_i - \mathbf{E}[X])^2 = Var(X)$.

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