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Monotone convergence of spreading processes on networks

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ABSTRACT

We analyze the Bass and SI models for the spreading of innovations and epidemics, respectively, on homogeneous complete networks, on one-dimensional networks, and on heterogeneous two-groups complete networks. We allow the network parameters to be time dependent, which is a prerequisite for the analysis of optimal promotional strategies on networks. Using a novel top-down analysis of the master equations, we present a simple proof for the monotone convergence of these models to their respective infinite-population limits. This leads to explicit expressions for the expected adoption or infection level in the Bass and SI models with time-dependent parameters on infinite homogeneous complete and circular networks, and on heterogeneous two-groups complete networks.

1. Introduction

Spreading processes on networks have attracted the attention of researchers in physics, mathematics, biology, computer science, social sciences, economics, and management science, as it concerns the spreading of "items" ranging from diseases and computer viruses to rumors, information, opinions, technologies, and innovations [1,18]. In this study, we focus on two prominent network models: The Bass model for the adoption of innovations [16,25] and the Susceptible-Infected (SI) model for the spread of epidemics [20]. These models were originally formulated as compartmental models, in which the population is divided into two compartments: adopters/infected and nonadopters/susceptible [3,4]. In recent years, research has shifted to studying these models on networks.

In the Bass and SI models on networks, the adoption/infection event by each node is stochastic. Since a direct analysis of stochastic particle models is hard, there has been a considerable research effort to rigorously derive a deterministic ODE for the macroscopic behavior of the expected adoption/infection level as a function of time. Niu [23] *derived* the ODE for the infinite-population limit of the Bass model on homogeneous complete networks. The approach in [23], however, does not extend to other types of networks. Fibich and Gibori obtained an explicit expression for the expected adoption level in the Bass model on infinite circles [7]. They did not prove rigorously, however, that this expression is the limit of the Bass model on circles with M nodes as $M \to \infty$. In [10], Fibich et al. rigorously derived the infinite-population limit of the Bass model on homogeneous complete networks, on heterogeneous complete networks with K groups, and on circular networks, and also the rate of convergence for these three cases. We are not aware of similar convergence results for the SI model on networks.

The common theme of the above studies has been to derive an explicit expression for the expected adoption/infection level as a function of time, and use it to analyze the effect of the network structure and parameters. We note, however, that another important application of the Bass and SI models has been to compute optimal strategies that influence the spreading process in a desired fashion. For example, one can use the Bass model to compute optimal promotional campaigns that maximize the profit [14,17]. So far, this optimal-control problem has only been analyzed in the context of the compartmental Bass model, which implicitly assume that the social network is a complete homogeneous network. Similarly, the compartmental SI model has been used to compute optimal government restrictions that minimize disease spread while keeping the economy healthy [2].

To apply the machinery of optimal-control theory to stochastic spreading on networks, one first needs to derive deterministic ODEs for the macroscopic dynamics. Moreover, if one wants to allow for time-dependent optimal strategies, the network parameters should be time-dependent. In this paper, *we present the first rigorous derivation of ODEs for the expected adoption/infection level in the Bass and* SI *models on infinite networks with time-varying parameters*. These ODEs, in turn, are used in a companion study [8] to compute and analyze optimal promotional strategies in the Bass model on infinite networks.

To the best of our knowledge, this paper presents the first derivation of the infinite-population limit of the Bass model on networks with time-varying parameters, and of the SI model on networks, with and without time-varying parameters. From a methodological point of view, this paper presents a unified treatment of the Bass and SI models on networks and introduces *a novel "top-down" analysis of the (bottom-up) master equations*, where one proves the monotonicity property at the "top" level of the master equation for the whole population, and then proves that this property remains valid as the number of nodes is reduced one at a time, until reaching the desired "bottom" level of the master equation for a single node.

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Once established, we can use the monotonicity property to obtain explicit lower and upper bounds for the expected adoption level on finite networks. Moreover, the monotonicity leads to pointwise convergence, which together with the dominated convergence theorem, proves the convergence of the expected adoption/infection level as the population becomes infinite. This convergence proof is much simpler than the one in [10] for networks with time-independent parameters that used space-time estimates. In addition, extending the proof in [10] to time-varying parameters is not as straightforward. As noted, this is a prerequisite for optimal-control applications on networks.

A frequently-used alternative approach to analyzing epidemiological models on networks and approximating them numerically is the mean-field approximation [20,24]. The advantage of the mean-field approach is that it yields relatively simple equations, whose solutions are therefore amenable to analysis and numerical computation. However, the accuracy of the mean-field approximation has been questioned [15,22]. Recent results [19] show that it only yields a good approximation of the underlying stochastic dynamics when the number of nodes is large and the harmonic mean of the degrees of the nodes is small, which essentially means the network is dense. A related approach is to close the master equations at the level of pairs [20]. This approach is exact on loopless networks [21,26]. In that case, however, the dynamics is governed by several ODEs, unlike our approach where it is governed by a single ODE. Moreover, in the case of network with cycles, this closure is not exact, and its accuracy is not clear. Finally, we note that the methodology and results of this study have the potential to be extended to other types of networks (Cartesian, random, …), to hypernetworks [12], and to other types of spreading models (SIS, SIR, Bass-SIR, etc.) [20,5,6].

To our knowledge, this is the first paper that highlights the role played by the monotonicity with respect to the network size. Although we only consider three network types, the monotonicity property seems to extend far more broadly. Indeed, in Appendix A we show that the expected adoption/infection level increases monotonically with M in d-dimensional Cartesian networks, d-regular networks, and Erdős-Rényi networks. How to exploit the monotonicity property on these and other networks, however, is a matter of future research.

2. Bass/SI model on networks

The Bass model describes the adoption of new products or innovations within a population. In this framework, all individuals start as non-adopters and can transition to becoming adopters due to two types of influences: external factors, such as exposure to mass media, and internal factors where individuals are influenced by their peers who have already adopted the product. The SI model is used to study the spreading of infectious diseases within a population. In this model, some individuals are initially infected (the "patient zero" cases), all subsequent infections occur through internal influences, whereby infected individuals transmit the disease to their susceptible peers, and infected individuals remain contagious indefinitely. In both models, once an individual becomes an adopter/infected, it remains so at all later times. In particular, she or he remain "contagious" forever. The difference between the SI model and the Bass model is the lack of external influences in the former, and the lack of initial adopters in the latter. It is convenient to unify these two models into a single model, the Bass/SI model on networks, as follows. Consider M individuals, denoted by $\mathcal{M} := \{1, \dots, M\}$. We denote by $X_j(t)$ the state of individual j at time t, so that

$$X_j(t) = \begin{cases} 1, & \text{if } j \text{ is adopter/infected at time } t, \\ 0, & \text{otherwise}, \end{cases} \qquad j \in \mathcal{M}.$$

The initial conditions at $t = 0$ are stochastic, so that

$$X_j(0) = X_j^0 \in \{0, 1\}, \qquad j \in \mathcal{M}, \tag{2.1a}$$

where

$$\mathbb{P}(X_j^0 = 1) = I_j^0, \quad \mathbb{P}(X_j^0 = 0) = 1 - I_j^0, \quad I_j^0 \in [0, 1], \qquad j \in \mathcal{M}, \tag{2.1b}$$

and

$$\text{the random variables } \{X_j^0\}_{j \in \mathcal{M}} \text{ are independent.} \tag{2.1c}$$

Deterministic initial conditions are a special case where $I_j^0 \in \{0, 1\}$.

So long that j is a nonadopter/susceptible, its adoption/infection rate at time t is

$$\lambda_j(t) = p_j(t) + \sum_{k \in \mathcal{M}} q_{k,j}(t) X_k(t), \qquad j \in \mathcal{M}. \tag{2.1d}$$

Here, $p_j(t)$ is the rate of external influences on j, and $q_{k,j}(t)$ is the rate of internal influences by k on j at time t, provided that k is already an adopter/infected. Once j becomes an adopter/infected, it remains so at all later times.[1] Hence, as $\Delta t \to 0$,

$$\mathbb{P}(X_j(t + \Delta t) = 1 \mid \mathbf{X}(t)) = \begin{cases} \lambda_j(t)\,\Delta t, & \text{if } X_j(t) = 0, \\ 1, & \text{if } X_j(t) = 1, \end{cases} \qquad j \in \mathcal{M}, \tag{2.1e}$$

where $\mathbf{X}(t) := \{X_j(t)\}_{j \in \mathcal{M}}$ is the state of the network at time t, and

$$\text{the random variables } \{X_j(t + \Delta t) \mid \mathbf{X}(t)\}_{j \in \mathcal{M}} \text{ are independent.} \tag{2.1f}$$

In the Bass model there are no adopters when the product is first introduced into the market, and so $I_j^0 \equiv 0$. In the SI model there are only internal influences for $t > 0$, and so $p_j(t) \equiv 0$. The quantity of most interest is the expected adoption (infection) level $f(t) := \frac{1}{M} \sum_{j=1}^{M} f_j(t)$, where $f_j := \mathbb{E}[X_j]$ in the adoption/infection probability of node j.

3. Bass and SI models on complete networks

Consider a complete homogeneous network where everyone is connected to each other, all the nodes have the same initial condition, and all the nodes and all the edges have the same weights. Thus,[2]

$$I_j^0 \equiv I^0, \qquad p_j(t) \equiv p(t), \qquad q_{k,j}(t) \equiv \frac{q(t)}{M - 1} \mathbb{1}_{k \neq j}, \qquad k, j \in \mathcal{M}. \tag{3.1a}$$

Hence, the adoption rate of each of the nonadopting nodes is, see (2.1d),

$$\lambda^{\text{complete}}(t) := p(t) + \frac{q(t)}{M - 1} N(t), \qquad N := \sum_{k \in \mathcal{M}} X_k, \tag{3.1b}$$

where $N(t)$ is the number of adopters/infected in the network. Note that we allow the weights to be time-dependent, which is essential for the analysis of time-dependent promotional strategies on networks [8]. We assume that the parameters satisfy

$$0 \le I^0 < 1, \qquad q(t) > 0, \quad p(t) \ge 0, \qquad I^0 + p(t) > 0, \quad t > 0. \tag{3.1c}$$

Furthermore, we assume that $p(t)$ and $q(t)$ are piecewise continuous. We denote the expected adoption/infection level in the Bass/SI model on the complete network (3.1) by $f^{\text{complete}}(t; p(t), q(t), I^0, M)$. Specifically, in the case of the Bass model,

$$I^0 = 0, \qquad q(t) > 0, \quad p(t) > 0, \quad t > 0, \tag{3.2}$$

and the expected adoption level is $f_{\text{Bass}}^{\text{complete}}(\cdot) := f^{\text{complete}}(\cdot, I^0 = 0)$. In the case of the SI model,

$$0 < I^0 < 1, \qquad q(t) > 0, \quad p(t) \equiv 0, \quad t > 0, \tag{3.3}$$

and expected infection level is $f_{\text{SI}}^{\text{complete}}(\cdot) := f^{\text{complete}}(\cdot, p = 0)$.

[1] I.e., the only admissible transition is $X_j = 0 \to X_j = 1$.

[2] The internal influences are normalized, so that the maximal internal influence $\sum_{k \in \mathcal{M}} q_{k,j}(t)$ is independent of M.

3.1. Monotone convergence of f^{complete}

Consider the Bass/SI model (2.1), (3.1) on a complete network. As the network size M increases, each nonadopter can be influenced by more and more adopters, but the influence rate $q_{k,j} = \frac{q(t)}{M-1}$ of each adopter decays. Therefore, a priori, it is not clear whether f^{complete} should be monotonically decreasing or increasing in M. The following lemma settles this issue:

Lemma 3.1. *Let $t > 0$. Then $f^{\text{complete}}(t; p(t), q(t), I^0, M)$ is monotonically increasing in M.*

Proof. See Appendix C.2. □

Using the monotonicity in M, we can prove the convergence of f^{complete} as $M \to \infty$ and compute its limit:

Theorem 3.2. *Consider the Bass/SI model (2.1), (3.1) on a complete network. Then*

$$\lim_{M\to\infty} f^{\text{complete}}(t; p(t), q(t), I^0, M) = f^{\text{compart}}(t; p(t), q(t), I^0), \tag{3.4}$$

where f^{compart} is the solution of the equation

$$\frac{df}{dt} = (1-f)\big(p(t) + q(t)f\big), \qquad f(0) = I^0. \tag{3.5}$$

Proof. See Appendix C.3. □

From Lemma 3.1 and Theorem 3.2 we have

Corollary 3.3. *f^{complete} monotonically converges to f^{compart} as $M \to \infty$.*

Real networks are finite, yet are often approximated with compartmental models that correspond to $M \to \infty$. If we did not know that the convergence is monotone, we would not know that f^{compart} is below f^{complete}. Furthermore, we can use the monotonicity to obtain a lower bound for f^{complete}:

Corollary 3.4. *Consider the Bass/SI model (2.1), (3.1) on a complete network. Then*

$$\begin{aligned} 1-(1-I^0)e^{-\int_0^t p\, ds} &< f^{\text{complete}}(t; p(t), q(t), I^0, M) \\ &< f^{\text{compart}}(t; p(t), q(t), I^0), \qquad t>0, \quad M = 2,3,\dots \end{aligned} \tag{3.6}$$

Proof. This follows from Lemma 3.1, eq. (B.3), and Theorem 3.2. □

3.2. Time-independent parameters

When p and q are independent of time and $I^0 = 0$, we obtain from Theorem 3.2 the well-known compartmental limit

$$\lim_{M\to\infty} f^{\text{complete}}_{\text{Bass}}(t; p, q, M) = f^{\text{compart}}_{\text{Bass}}(t; p, q), \qquad f^{\text{compart}}_{\text{Bass}} := \frac{1-e^{-(p+q)t}}{1+\frac{q}{p}e^{-(p+q)t}}, \tag{3.7}$$

where $f^{\text{compart}}_{\text{Bass}}$ is the solution of the compartmental Bass model [3]

$$f'(t) = (1-f)(p+qf), \qquad f(0) = 0.$$

The monotone convergence of $f^{\text{complete}}_{\text{Bass}}$ to $f^{\text{compart}}_{\text{Bass}}$ is illustrated in Fig. 1.

Similarly, when $p = 0$ and q is independent of time, we obtain

$$\begin{aligned} &\lim_{M\to\infty} f^{\text{complete}}_{\text{SI}}(t; q, I^0, M) = f^{\text{compart}}_{\text{SI}}(t; q, I^0), \\ &f^{\text{compart}}_{\text{SI}} := \frac{1}{1+(\frac{1}{I^0}-1)e^{-qt}}, \end{aligned} \tag{3.8}$$

where $f^{\text{compart}}_{\text{SI}}$ is the solution of the compartmental SI model

$$f'(t) = q(1-f)f, \qquad f(0) = I^0.$$

The monotone convergence of $f^{\text{complete}}_{\text{SI}}$ to $f^{\text{compart}}_{\text{SI}}$ is illustrated in Fig. F.5 in Appendix F.

The limit (3.7) was proved in [10,23]. To the best of our knowledge, this is the first rigorous derivation of the limit (3.8). Furthermore, this is the first proof that $f^{\text{complete}}_{\text{Bass}}$ and $f^{\text{complete}}_{\text{SI}}$ converge *monotonically* to their respective limits.

4. Bass and SI models on circles

Consider now the Bass and SI models on the circle, where each node can only be influenced by its left and right neighbors. We can allow peer effects to be *anisotropic*, so that the influence rates of the left and right neighbors are q^{L} and q^{R}, respectively. Thus,

$$\begin{aligned} &I^0_j \equiv I^0, \quad p_j \equiv p(t), \\ &q_{k,j} \equiv q^{\text{L}}(t)\, \mathbb{1}_{(j-k) \bmod M = 1} + q^{\text{R}}(t)\, \mathbb{1}_{(j-k) \bmod M = -1}, \qquad k,j \in \mathcal{M}. \end{aligned} \tag{4.1a}$$

Hence, the adoption rate of j is

$$\lambda^{\text{circle}}_j(t) := p + q^{\text{L}}(t) X_{j-1}(t) + q^{\text{R}}(t) X_{j+1}(t), \tag{4.1b}$$

where $X_0 := X_M$ and $X_{M+1} := X_1$. The model parameters satisfy

$$\begin{aligned} &0 \le I^0 < 1, \qquad q^{\text{L}}(t), q^{\text{R}}(t) \ge 0, \quad q^{\text{L}}(t) + q^{\text{R}}(t) > 0, \quad p(t) \ge 0, \\ &I^0 + p(t) > 0, \quad t > 0. \end{aligned} \tag{4.1c}$$

Furthermore, we assume that $p(t)$, $q^{\text{L}}(t)$, and $q^{\text{R}}(t)$ are piecewise continuous. We denote the expected adoption/infection level in the Bass/SI model (2.1), (4.1) on the circle by $f^{\text{circle}}(t; p(t), q(t), I^0, M)$. Specifically, we denote the expected adoption level in the Bass model by $f^{\text{circle}}_{\text{Bass}}(\cdot) := f^{\text{circle}}(\cdot, I^0 = 0)$, and the expected infection level in the SI model by $f^{\text{circle}}_{\text{SI}}(\cdot) := f^{\text{circle}}(\cdot, p = 0)$.

4.1. Monotone convergence of f^{circle}

Consider the Bass/SI model (2.1), (4.1) on a circle. Both the results and proofs are very similar to the complete network.

Lemma 4.1. *Let $t > 0$. Then $f^{\text{circle}}(t; p, q, I^0, M)$ is monotonically increasing in M.*

Proof. See Appendix D. □

Using the monotonicity in M, we can prove the convergence of f^{circle} as $M \to \infty$ and compute its limit:

Theorem 4.2. *Consider the Bass/SI model (2.1), (4.1) on the circle. Then*

$$\lim_{M\to\infty} f^{\text{circle}}(t; p(t), q(t), I^0, M) = f^{\text{1D}}(t; p(t), q(t), I^0), \tag{4.2a}$$

where f^{1D}, the expected level of adoption/infection for the Bass/SI model on the one-dimensional lattice with nearest-neighbor interactions, is the solution of

$$\frac{df}{dt} = (1-f)\Big(p(t) + q(t)\big(1-(1-I^0)e^{-\int_0^t p(\tau)}\big)\Big), \qquad f(0) = I^0. \tag{4.2b}$$

Proof. See Appendix D. □

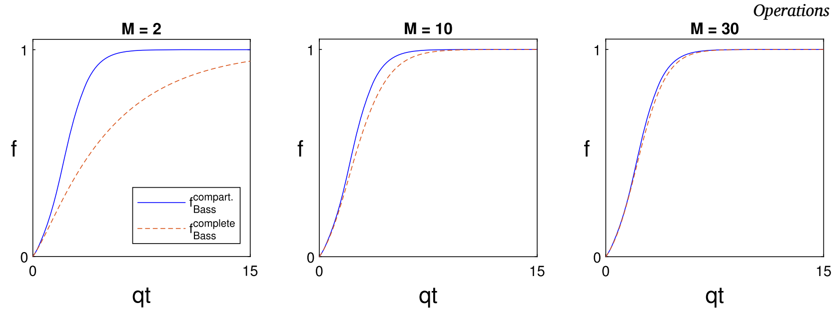

Fig. 1. Monotone convergence of $f_{\text{Bass}}^{\text{complete}}$ (dashes) to $f_{\text{Bass}}^{\text{compart}}$ (solid). Here $\frac{q}{p} = 10$, $I^0 = 0$, and $M = 2, 10, 30, 200$.

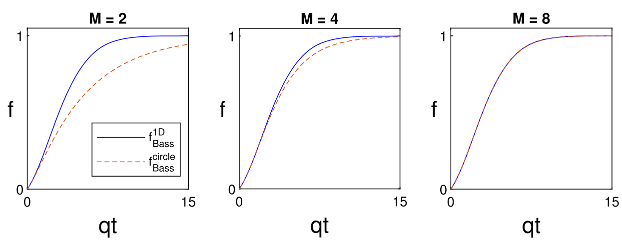

Fig. 2. Monotone convergence of $f_{\text{Bass}}^{\text{circle}}$ (dashes) to $f_{\text{Bass}}^{\text{1D}}$ (solid). Here $\frac{q}{p} = 10$, $I^0 = 0$, and $M = 2, 4, 8, 16$.

From Lemma 4.1 and Theorem 4.2 we have

Corollary 4.3. *f^{circle} monotonically converges to f^{1D} as $M \to \infty$.*

Proof. This follows from Lemma 4.1 and Theorem 4.2. □

We can also use the monotonicity to obtain lower and upper bounds for f^{circle}:

Corollary 4.4. *Consider the Bass/SI model (2.1), (4.1) on the circle. Then*

$$1 - (1 - I^0)e^{-\int_0^t p\,ds} < f^{\text{circle}}(t; p(t), q(t), I^0, M) < f^{\text{1D}}(t; p(t), q(t), I^0),$$
$$t > 0, \quad M = 2, 3, \ldots, \tag{4.3}$$

where f^{1D} is the solution of (4.2b).

Proof. This follows from Lemma 4.1, eq. (B.3), and Theorem 4.2. □

4.2. Time-independent parameters

When p and q are independent of time, we obtain from Theorem 4.2 the explicit limits

$$\lim_{M \to \infty} f_{\text{Bass}}^{\text{circle}}(t; p, q, M) = f_{\text{Bass}}^{\text{1D}}(t; p, q), \qquad f_{\text{Bass}}^{\text{1D}} := 1 - e^{-(p+q)t + q\frac{1-e^{-pt}}{p}},$$

and

$$\lim_{M \to \infty} f_{\text{SI}}^{\text{circle}}(t; q, I^0, M) = f_{\text{SI}}^{\text{1D}}(t; q, I^0), \qquad f_{\text{SI}}^{\text{1D}} := 1 - (1 - I^0)e^{-qI^0 t}.$$

The former limit was first derived in [7] and rigorously justified in [10], the latter limit is new. The monotone convergence of $f_{\text{Bass}}^{\text{circle}}$ to $f_{\text{Bass}}^{\text{1D}}$ is illustrated in Fig. 2, and of $f_{\text{SI}}^{\text{circle}}$ to $f_{\text{SI}}^{\text{1D}}$ in Fig. F.6.

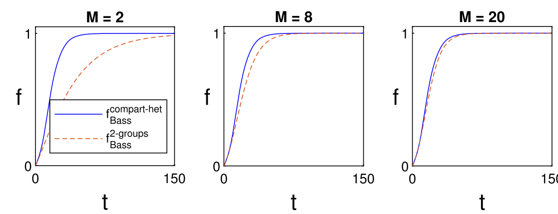

Fig. 3. Monotone convergence of $f_{\text{Bass}}^{\text{2-groups}}$ (dashes) to $f_{\text{Bass}}^{\text{compart-het}}$ (solid). Here $\frac{q_1}{p_1} = \frac{q_2}{p_2} = 10$, $p_2 = 2p_1$, $I^0 = 0$, and $M = 2, 8, 20, 80$.

5. Heterogeneous complete networks with two groups

Consider now the Bass/SI model on a heterogeneous complete network with two groups, each of size M. The parameters for node k_n in group k are[3]

$$I_{k_n}^0 \equiv I_k^0, \quad p_{k_n}(t) \equiv p_k(t), \quad q_{m,k_n}(t) \equiv \frac{q_k(t)}{2M},$$
$$m \in \mathcal{M}, \quad n = 1, \ldots, M, \qquad k = 1, 2,$$

where $\mathcal{M} := \{1, \ldots, 2M\}$. Hence, the adoption rate of a node in group k is

$$\lambda_k(t) = p_k(t) + \frac{q_k(t)}{2M} \sum_{i=1}^{2M} X_i(t), \qquad k = 1, 2. \tag{5.1a}$$

The model parameters satisfy

$$0 \le I_k^0 < 1, \quad q_k(t) > 0, \quad p_k(t) \ge 0, \quad I_k^0 + p_k(t) > 0, \quad t > 0, \qquad k = 1, 2. \tag{5.1b}$$

Furthermore, we assume that $p_k(t)$ and $q_k(t)$ are piecewise continuous. We denote the expected adoption/infection level in the Bass/SI model (2.1), (5.1) on a complete network with two groups by $f^{\text{2-groups}}(t; p_1(t), p_2(t), q_1(t), q_2(t), I_1^0, I_2^0, 2M)$. Specifically, we denote the expected adoption level in the Bass model by $f_{\text{Bass}}^{\text{2-groups}}(\cdot) := f^{\text{2-groups}}(\cdot, I_1^0 = I_2^0 = 0)$, and the expected infection level in the SI model by $f_{\text{SI}}^{\text{2-groups}}(\cdot) := f^{\text{2-groups}}(\cdot, p_1 = p_2 = 0)$.

5.1. Monotone convergence of $f^{\text{2-groups}}$

We can show the monotone convergence of the Bass/SI model (2.1), (5.1) on a complete network with two groups. In this case, both the results and proofs are very similar to the homogeneous complete network.

Lemma 5.1. *Let $t > 0$. Then $f^{\text{2-groups}}(t; p_1(t), p_2(t), q_1(t), q_2(t), I_1^0, I_2^0, 2M)$ is monotonically increasing in M.*

[3] There is no normalization for which the maximal internal influence $\sum_{m \in \mathcal{M}} q_{m,k_n}(t)$ is independent of M.

Proof. See Appendix E. □

Using the monotonicity in M, we can prove the convergence of $f^{2-\text{groups}}$ as $M \to \infty$ and compute its limit:

Theorem 5.2. *Consider the Bass/SI model (2.1), (5.1) on a complete network with two groups. Then*

$$\lim_{M\to\infty} f^{2-\text{groups}}(t; p_1(t), p_2(t), q_1(t), q_2(t), I_1^0, I_2^0, 2M) = f^{\text{compart-het}}(t), \tag{5.2a}$$

where $f^{\text{compart-het}} := f_1 + f_2$, and f_1, f_2 are the solutions of

$$\frac{df_k}{dt} = \left(\frac{1}{2} - f_k\right)\Big(p_k(t) + q_k(t)\left(f_1 + f_2\right)\Big), \qquad f_k(0) = \frac{I_k^0}{2}, \qquad k = 1, 2. \tag{5.2b}$$

Here $0 \le f_k \le \frac{1}{2}$ denotes the fraction of adopters from group i in the population.

Proof. See Appendix E. □

From Lemma 5.1 and Theorem 5.2 we have

Corollary 5.3. *$f^{2-\text{groups}}$ monotonically converges to $f^{\text{compart-het}}$ as $M \to \infty$.*

Proof. This follows from Lemma 5.1 and Theorem 5.2. □

The monotone convergence of $f_{\text{Bass}}^{2-\text{groups}}$ to $f_{\text{Bass}}^{\text{compart-het}}$ is illustrated in Fig. 3, and of $f_{\text{SI}}^{2-\text{groups}}$ to $f_{\text{SI}}^{\text{compart-het}}$ in Fig. F.7.

CRediT authorship contribution statement

Gadi Fibich: Writing – review & editing, Methodology, Conceptualization, Writing – original draft, Formal analysis. **Amit Golan:** Writing – original draft, Formal analysis, Writing – review & editing, Methodology, Conceptualization. **Steven Schochet:** Writing – review & editing, Methodology, Conceptualization, Writing – original draft, Formal analysis.

Appendix A. Monotone convergence on various sparse networks

This work proves the monotonicity of the adoption/infection level in M for three types of networks. The monotonicity property, however, is much more widespread. For example, in [11], it was shown analytically and numerically that the adoption level in the Bass model on bounded one-dimensional and multi-dimensional Cartesian domains is monotonically increasing with M. In Fig. A.4, we show numerically the monotone convergence of $f(t; M)$ on a two-dimensional toroidal domain where each node is connected to its four nearest neighbors, and in sparse d-regular and Erdős-Rényi networks [13].

Appendix B. Master equations

The key tool in the analysis of the Bass/SI model (2.1) are the master equations. Let $\emptyset \ne \Omega \subset \mathcal{M}$ be a nontrivial subset of the nodes, let $\Omega^c := \mathcal{M} \setminus \Omega$, and let

$$S_\Omega(t) := \{X_m(t) = 0, \ \forall m \in \Omega\}, \qquad [S_\Omega](t) := \mathbb{P}(S_\Omega(t)), \tag{B.1}$$

denote the event that all nodes in Ω are nonadopters at time t, and the probability of this event, respectively. To simplify the notations, we introduce the notation

$$S_{\Omega_1,\Omega_2} := S_{\Omega_1 \cup \Omega_2}, \qquad \Omega_1, \Omega_2 \subset \mathcal{M}.$$

Thus, for example, $S_{\Omega,k} := S_{\Omega\cup\{k\}}$. We also denote the sum of the external influences on the nodes in Ω and the sum of the internal influences by node k on the nodes in Ω by

$$p_\Omega(t) := \sum_{m\in\Omega} p_m(t), \qquad q_{k,\Omega}(t) := \sum_{m\in\Omega} q_{k,m}(t),$$

respectively.

Theorem B.1 *([9]). The master equations for the Bass/SI model (2.1) are*

$$\frac{d[S_\Omega]}{dt} = -\left(p_\Omega(t) + \sum_{k\in\Omega^c} q_{k,\Omega}(t)\right)[S_\Omega] + \sum_{k\in\Omega^c} q_{k,\Omega}(t)\,[S_{\Omega,k}], \tag{B.2a}$$

subject to the initial conditions

$$[S_\Omega](0) = [S_\Omega^0], \qquad [S_\Omega^0] := \prod_{m\in\Omega}(1 - I_m^0), \tag{B.2b}$$

for all $\emptyset \ne \Omega \subset \mathcal{M}$.

For example, the solution of (B.2) for $M = 1$ is

$$[S](t; p(t), I^0, M = 1) = (1 - I^0)e^{-\int_0^t p}, \qquad t \ge 0. \tag{B.3}$$

For $M > 1$, if we can solve the $2^M - 1$ equations (B.2), then we have $f(t)$ from $f(t) = \frac{1}{M}\sum_{j=1}^M f_j(t)$, and $f_j = 1 - [S_j]$. As noted, we do not compute the $\{[S_j]\}$ using the simpler mean-field approach, since it may lead to approximate results whose accuracy is not always clear, especially on sparse networks.

Appendix C. Proof of Lemma 3.1 and Theorem 3.2

C.1. Reduced master equations

As noted, there are $2^M - 1$ master equations for $\{[S_\Omega]\}_{\Omega\subset\mathcal{M}}$, see Theorem B.1. Because of the symmetry of the complete network (3.1), however, $[S_\Omega]$ only depends on the number of nodes in Ω, and not on the identity of the nodes in Ω. Therefore, we can denote by

$$[S^n] := [S_\Omega \mid |\Omega| = n] \tag{C.1}$$

the probability that for any given subset of n nodes, all its nodes are nonadopters at time t. This substitution replaces the $2^M - 1$ master

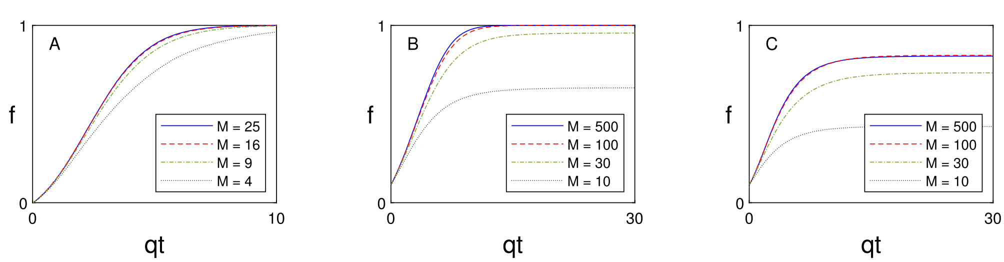

Fig. A.4. Monotone convergence of $f(t; M)$. A) Bass model on a $\sqrt{M} \times \sqrt{M}$ toroidal domain. B) SI model on a 3-regular network. C) SI model on a sparse Erdős-Rényi network.

equations (B.2) for $\{[S_\Omega]\}_{\Omega\subset\mathcal{M}}$ with a reduced system of M equations for $\{[S^n]\}_{n=1}^{M}$:

Lemma C.1. *The reduced master equations for the Bass/SI model (2.1), (3.1) on a complete network are*

$$\frac{d[S^n]}{dt} = -n\left(p(t)+q(t)\frac{M-n}{M-1}\right)[S^n] + nq(t)\frac{M-n}{M-1}[S^{n+1}],\qquad n=1,\dots,M-1, \tag{C.2a}$$

$$\frac{d[S^M]}{dt} = -Mp(t)[S^M], \tag{C.2b}$$

subject to the initial conditions

$$[S^n](0) = (1-I^0)^n, \qquad n\in\mathcal{M}. \tag{C.2c}$$

Proof. This follows from the substitution of (3.1) and (C.1) in the master equations (B.2). □

C.2. Monotonicity of $\{[S^n]\}$ in M

Let us recall the following auxiliary result:

Lemma C.2. *Let $\alpha(t):\mathbb{R}\to\mathbb{R}$ be piecewise continuous, and let $y(t)$ satisfy the differential inequality*

$$\frac{dy}{dt}+\alpha(t)y>0,\quad t>0,\qquad y(0)=0.$$

Then $y(t)>0$ for $t>0$.

Proof. Multiplying the differential inequality by the integrating factor $e^{\int_0^t \alpha(s)\,ds}$ gives

$$\frac{d}{dt}\left(e^{\int_0^t \alpha(s)\,ds}y\right)>0.$$

Integrating between zero and t and using the initial condition gives the result. □

We are now ready to prove Lemma 3.1:

Proof of Lemma 3.1. Let

$$y_n(t) := [S^n](t;M) - [S^n](t;M+1), \qquad n=1,\dots,M, \tag{C.3}$$

where $[S^n](t;M)$ is the solution of the master equations (C.2). Then

$$\frac{dy_M}{dt} + Mpy_M = z_M(t), \qquad y_M(0)=0, \tag{C.4a}$$

where

$$z_M := q\Big([S^M](t;M+1) - [S^{M+1}](t;M+1)\Big). \tag{C.4b}$$

Note that $[S^M](t;M+1)-[S^{M+1}](t;M+1)=[IS^M](t;M+1)>0$, where $[IS^M](t,M+1)$ denotes the probability that exactly one of $M+1$ nodes is an adopter/infected. Hence $z_M(t)>0$ for $t>0$. Therefore, applying Lemma C.2 to (C.4) shows that

$$y_M(t)>0, \qquad t>0. \tag{C.5}$$

We can rewrite the master equations (C.2) for $n=1,\dots,M-1$ as

$$\frac{d[S^n]}{dt}(t;M) = -n\left(p+q_M^n\right)[S^n](t;M) + nq_M^n[S^{n+1}](t;M),$$
$$[S^n](0;M)=1,\quad q_M^n := q\frac{M-n}{M-1}.$$

Similarly,

$$\frac{d[S^n]}{dt}(t;M+1) = -n\left(p+q_{M+1}^n\right)[S^n](t;M+1) + nq_{M+1}^n[S^{n+1}](t;M+1),\quad [S^n](0;M+1)=1.$$

Taking the difference of these two equations gives

$$\frac{dy_n}{dt} + n\left(p+q_{M+1}^n\right)y_n = nq_{M+1}^n y_{n+1} + z_n(t),\qquad y_n(0)=0,\qquad n=1,\dots,M-1, \tag{C.6a}$$

where

$$z_n = n(q_M^n - q_{M+1}^n)\big(-[S^n](t;M)+[S^{n+1}](t;M)\big). \tag{C.6b}$$

Since

$$q_M^n - q_{M+1}^n = q\frac{M-n}{M-1} - q\frac{M+1-n}{M} = q\frac{1-n}{(M-1)M}\le 0,$$

and

$$-[S^n]+[S^{n+1}] = -[IS^n]<0,\qquad t>0,$$

we have that $z_n\ge 0$ for $t>0$. Therefore, applying Lemma C.2 to (C.6) shows that

$$y_{n+1}(t)>0,\quad t>0\quad\Rightarrow\quad y_n(t)>0,\quad t>0,\qquad n=1,\dots,M-1. \tag{C.7}$$

From relations (C.5) and (C.7) we get by reverse induction on n that

$$y_n(t)>0,\qquad n\in\mathcal{M}, \tag{C.8}$$

i.e., that $\{[S^n](t;M)\}_{n\in\mathcal{M}}$ are monotonically decreasing in M. In particular,

$$0<y_1(t) = [S](t;M)-[S](t;M+1) = f^{\text{complete}}(t;M+1) - f^{\text{complete}}(t;M). \quad □$$

C.3. Convergence of f^{complete}

Next, we utilize the monotonicity of $\{[S^n]\}$ in M to prove the convergence of f^{complete}:

Proof of Theorem 3.2. Consider the master equations (C.2). If we formally fix n and let $M\to\infty$, we get the ODE

$$\frac{d[S_\infty^n]}{dt} = -n(p+q)[S_\infty^n] + nq[S_\infty^{n+1}],\qquad [S_\infty^n](0)=(1-I_0)^n,\qquad n\in\mathbb{N}. \tag{C.9}$$

This does not immediately imply that $\lim_{M\to\infty}[S^n]=[S_\infty^n]$. Indeed, this limit does not follow from the standard theorems for continuous dependence of solutions of ODEs on parameters, because the number of ODEs in (C.2) increases with M, and becomes infinite in the limit, and also because of the presence of the unbounded factor n on the right-hand sides of (C.2) and (C.9). In Lemma C.3 below, however, we will rigorously prove that

$$\lim_{M\to\infty}[S^n](t;M) = [S_\infty^n](t),\qquad n\in\mathbb{N}. \tag{C.10}$$

Therefore, we can proceed to solve the infinite system (C.9). The ansatz

$$[S_\infty^n] = [S_\infty]^n,\qquad n\in\mathbb{N} \tag{C.11}$$

transforms the system (C.9) into

$$n[S_\infty]^{n-1}\frac{d[S_\infty]}{dt} = -n(p+q)[S_\infty]^n + nq[S_\infty]^{n+1},\qquad [S_\infty](0)=1-I^0.$$

Dividing by $n[S_\infty]^{n-1}$, we find that the infinite system reduces to the single ODE

$$\frac{d}{dt}[S_\infty] = -(p+q)[S_\infty] + q[S_\infty]^2,\qquad [S_\infty](0)=1-I^0. \tag{C.12}$$

Let $f^{\text{compart}} = 1-[S_\infty]$. Then f^{compart} satisfies (3.5). The limit (3.4) follows from (C.10) and (C.11) with $n=1$. □

In the proof of Theorem 3.2 we used the following convergence result:

Lemma C.3. *For any $n \in \mathbb{N}$ and $t \geq 0$, the solution $[S^n](t; M)$ of equations (C.2) converges monotonically as $M \to \infty$ to the solution $[S^n_\infty](t)$ of (C.9).*

Proof. Let $n \in \mathbb{N}$ and let $M \geq n$. Taking the integral of ODE (C.2a) for $[S^n]$ from zero to t and using the initial condition (C.2c) gives

$$[S^n](t;M) - 1 = -n \int_0^t \left(p(s) + q(s)\frac{M-n}{M-1} \right) [S^n](s;M)\, ds + n\frac{M-n}{M-1} \int_0^t q(s)[S^{n+1}](s;M)\, ds. \tag{C.13}$$

Let us consider the limit of (C.13) as $M \to \infty$. Since $[S^n](t;M)$ is monotonically decreasing in M, see (C.3) and (C.8), and since $[S^n] \geq 0$ as a probability, this implies that $[S^n](t;M)$ converges pointwise as $M \to \infty$ to some limit $[S^n_\infty](t)$. Therefore, as $M \to \infty$, the left-hand side of (C.13) converges to $[S^n_\infty] - 1$. In addition, since $[S^n]$ is a probability, $0 \leq [S^n](t;M) \leq 1$, and so by the dominated convergence theorem, the integrals of $[S^n]$ and $[S^{n+1}]$ on the right-hand side of (C.13) converge to the integrals of the limits. Since the coefficients also converge, the limit of (C.13) as $M \to \infty$ is

$$[S^n_\infty](t) - 1 = -n \int_0^t (p(s) + q(s))\, [S^n_\infty](s)\, ds + n \int_0^t q(s)[S^{n+1}_\infty](s)\, ds. \tag{C.14}$$

Since $[S^n_\infty](t)$ is the pointwise limit of a sequence of measurable functions, it is also measurable. Therefore, it follows from (C.14) that it is continuous, hence differentiable. Differentiating (C.14), we conclude that $[S^n_\infty]$ satisfies the limit ODE (C.9). □

Appendix D. Proof of Lemma 4.1 and Theorem 4.2

Let

$$S^n := S_{j+1,\dots,j+n}, \qquad n \in \mathcal{M},$$

denote the event that the n adjacent nodes $\{j+1,\dots,j+n\}$ are nonadopters at time t, and let $[S^n]$ denote the probability of this event. Note that the probabilities $\{[S^n]\}$ are independent of j, because of translation invariance. Then we have

Lemma D.1 *([7]). The reduced master equations for the Bass/SI model (2.1), (4.1) on the circle are*

$$\frac{d[S^n]}{dt} = -\big(np(t) + q(t)\big)[S^n] + q(t)[S^{n+1}], \qquad n = 1,\dots,M-1, \tag{D.1a}$$

$$\frac{d[S^M]}{dt} = -Mp(t)[S^M], \tag{D.1b}$$

where $q(t) = q^{\mathrm{R}}(t) + q^{\mathrm{L}}(t)$, subject to the initial conditions

$$[S^n](0) = (1 - I^0)^n, \qquad n = 1,\dots,M. \tag{D.1c}$$

D.1. Monotonicity of $\{[S^n]\}$ in M

Proof of Lemma 4.1. Let

$$y_n(t) := [S^n](t;M) - [S^n](t;M+1), \qquad n \in \mathcal{M}.$$

By the master equations (D.1),

$$\frac{dy_M}{dt} + Mpy_M = qz_M(t), \qquad y_M(0) = 0,$$

where

$$z_M := [S^M](t;M+1) - [S^{M+1}](t;M+1) = [IS^M](t;M+1),$$

where $[IS^M](t;M+1)$ is the probability that the nodes $\{1,\dots,M\}$ are nonadopters and the remaining node is an adopter. Since $z_M(t) > 0$ for $t > 0$, we have from Lemma C.2 that

$$y_M(t) > 0, \qquad t > 0. \tag{D.2a}$$

Similarly, by (D.1),

$$\frac{dy_n}{dt} + (np+q)y_n = qy_{n+1}(t), \qquad y_n(0) = 0, \qquad n = 1,\dots,M-1.$$

Therefore, by Lemma C.2,

$$y_{n+1}(t) > 0, \quad t > 0 \qquad \Rightarrow \qquad y_n(t) > 0, \qquad t > 0. \tag{D.2b}$$

From relations (D.2), we get by reverse induction on n that

$$y_n(t) > 0, \qquad n \in \mathcal{M}, \tag{D.3}$$

i.e., that $\{[S^n](t;M)\}_{n\in\mathcal{M}}$ are monotonically decreasing in M. In particular,

$$0 < y_1(t) = [S](t;M) - [S](t;M+1) = f^{\mathrm{circle}}(t;M+1) - f^{\mathrm{circle}}(t;M). \qquad \square$$

The result and proof of Lemma 4.1 are similar to those in Lemma 3.1 for a complete network. Note, however, that while the addition of nodes is accompanied by a reduction of the weight of the edges in a complete network, this is not the case on the circle.

We can motivate the result of Lemma 4.1 as follows. Any node $j \in \mathcal{M}$ adopts either externally or internally. In the latter case, the adoption of j can be traced back to an *adoption path* that starts from another node i that adopted externally, and progresses through a series of internal adoptions that ultimately reach node j. The probability of j to adopt either externally or internally due to some adoption path of length $\leq M-1$, is the same on circles with M and with $M+1$ nodes. On the circle with $M+1$ nodes, however, j can also adopt due to adoption paths of length M. Therefore, its overall adoption probability on the larger circle is higher.

D.2. Convergence of f^{circle}

Proof of Theorem 4.2. The proof is similar to that for complete networks (Theorem 3.2). Our starting point are the master equations (D.1). If we formally fix n and let $M \to \infty$ in (D.1), we get the limiting system

$$\frac{d[S^n_\infty]}{dt} = -(np+q)[S^n_\infty] + q[S^{n+1}_\infty], \qquad [S^n_\infty](0) = (1-I^0)^n, \qquad n \in \mathbb{N}. \tag{D.4}$$

This does not immediately imply that $\lim_{M\to\infty}[S^n] = [S^n_\infty]$. Indeed, this limit does not follow from the standard theorems on continuous dependence of solutions of ODEs on parameters, because the number of ODEs in (D.1) increases with M, and becomes infinite in the limit, and because of the presence of the unbounded factor n on the right-hand sides of (D.1) and (D.4). In Lemma D.2 below, however, we will rigorously prove that for any $n \in \mathbb{N}$,

$$\lim_{M\to\infty} [S^n](t;M) = [S^n_\infty](t). \tag{D.5}$$

Therefore, we can proceed to solve the infinite system (D.4). To do that, we note that the ansatz

$$[S^n_\infty] = \left((1-I^0)e^{-\int_0^t p(\tau)} \right)^{n-1} [S^{1\mathrm{D}}] \tag{D.6}$$

reduces the infinite system (D.4) to the single ODE

$$\frac{d[S^{1D}]}{dt} = -\left(p + q(1-(1-I^0)e^{-\int_0^t p(\tau)})\right)[S^{1D}], \qquad [S^{1D}](0) = 1 - I^0. \tag{D.7}$$

Since $f^{\text{circle}} = 1 - [S^1]$ and $f^{1D} = 1 - [S^{1D}]$, the result follows from relation (D.5) with $n = 1$. The monotonicity in M follows from Lemma 4.1. □

The proof of Theorem 4.2 makes use of

Lemma D.2. *For any $n \in \mathbb{N}$ and $t \geq 0$, the solution $[S^n](t; M)$ of the master equations (D.1) converges monotonically as $M \to \infty$ to the solution $[S^n_\infty](t)$ of equation (D.4).*

Proof. The proof is nearly identical to that of Lemma C.3. Let $n \in \mathbb{N}$ and $M \geq n$. Integrating the ODE (D.1a) for $[S^n](t; M)$ from zero to t and using the initial condition (D.1c) gives

$$[S^n](t; M) - (1 - I^0)^n = -\int_0^t (np(s) + q(s))\,[S^n](s; M)\,ds + \int_0^t q(s)[S^{n+1}](s; M)\,ds. \tag{D.8}$$

Let us consider the limit of (D.8) as $M \to \infty$. Since $[S^n](t; M)$ is monotonically decreasing in M, see (D.3), and since $[S^n] \geq 0$ as a probability, this implies that $[S^n](t; M)$ converges pointwise as $M \to \infty$ to some limit $[S^n_\infty](t)$. Therefore, as $M \to \infty$, the left-hand side of (D.8) converges to $[S^n_\infty] - (1 - I^0)^n$. In addition, since $[S^n]$ is a probability, $0 \leq [S^n](t; M) \leq 1$, and so, by the dominated convergence theorem, the integrals of $[S^n]$ and $[S^{n+1}]$ on the right-hand side of (D.8) converge to the integrals of the limits. Hence, the limit of (D.8) as $M \to \infty$ is

$$[S^n_\infty](t) - (1 - I^0)^n = -\int_0^t (np(s) + q(s))\,[S^n_\infty](s)\,ds + \int_0^t q(s)[S^{n+1}_\infty](s)\,ds. \tag{D.9}$$

Since a pointwise limit of a sequence of measurable functions is also measurable, $[S^n_\infty](t)$ is measurable. Hence, it follows from (D.9) that it is continuous, hence differentiable. Differentiating (D.9), we conclude that $[S^n_\infty]$ satisfies the limit ODE (D.4). □

Appendix E. Proof of Lemma 5.1 and Theorem 5.2

Let

$$A_M := \{0, \dots, M\}^2 \setminus (0, 0),$$

and let $[S^{k_1,k_2}](t; 2M)$ denote the probability that k_1 nodes in group 1 and k_2 nodes in group 2 are non-adopters at time t. Then

$$f^{2-\text{groups}} = f_1 + f_2 = 1 - \frac{1}{2}\left([S^{1,0}] + [S^{0,1}]\right).$$

Lemma E.1 *([10]). The reduced master equations for the Bass/SI model (2.1), (5.1) on a complete network with two groups are*

$$\frac{d[S^{k_1,k_2}]}{dt} = -\left(k_1 p_1(t) + k_2 p_2(t) + \frac{2M - k_1 - k_2}{2M}\left(k_1 q_1(t) + k_2 q_2(t)\right)\right)[S^{k_1,k_2}] + \left(\frac{M - k_1}{2M}[S^{k_1+1,k_2}] + \frac{M - k_2}{2M}[S^{k_1,k_2+1}]\right)\left(k_1 q_1(t) + k_2 q_2(t)\right), \quad (k_1, k_2) \in A_M, \tag{E.1a}$$

subject to the initial conditions

$$[S^{k_1,k_2}](0) = (1 - I_1^0)^{k_1}(1 - I_2^0)^{k_2}, \qquad (k_1, k_2) \in A_M. \tag{E.1b}$$

E.1. Monotonicity of $[S^{k_1,k_2}](t; 2M)$

Proof of Lemma 5.1. We proceed by reverse induction on $n = k_1 + k_2$. Let

$$y_{k_1,k_2}(t) := [S^{k_1,k_2}](t; 2M) - [S^{k_1,k_2}](t; 2(M+1)), \qquad (k_1, k_2) \in A_M, \tag{E.2}$$

where $[S^{k_1,k_2}](t; 2M)$ is the solution of the master equations (E.1). We begin with the induction base $n = 2M$. Then

$$\frac{dy_{M,M}}{dt} + M\left(p_1(t) + p_2(t)\right) y_{M,M} = z_{M,M}(t), \qquad y_{M,M}(0) = 0, \tag{E.3a}$$

where

$$z_{M,M} := \frac{M(q_1 + q_2)}{2(M+1)}\Big(\left([S^{M,M}](t; 2(M+1)) - [S^{M+1,M}](t; 2(M+1))\right) + \left([S^{M,M}](t; 2(M+1)) - [S^{M,M+1}](t; 2(M+1))\right)\Big). \tag{E.3b}$$

Since

$$[S^{M,M}](t; 2(M+1)) - [S^{M+1,M}](t; 2(M+1)) = [I^{1,0}S^{M,M}](t; 2(M+1)) > 0,$$

and

$$[S^{M,M}](t; 2(M+1)) - [S^{M,M+1}](t; 2(M+1)) = [I^{0,1}S^{M,M}](t; 2(M+1)) > 0,$$

where $[I^{j_1,j_2}S^{k_1,k_2}]$ denotes the probability that there are j_m adopters and k_m nonadopters in group m for $m = 1, 2$, then $z_{M,M}(t) > 0$ for $t > 0$. Therefore, applying Lemma C.2 to (E.3) shows that

$$y_{M,M}(t) > 0, \qquad t > 0. \tag{E.4}$$

Let $n \in \{1, \dots, 2M - 1\}$. Then

$$\frac{d[S^{k_1,k_2}]}{dt}(t; 2M) = -\left(k_1 p_1 + k_2 p_2 + \frac{2M - k_1 - k_2}{2M}\left(k_1 q_1 + k_2 q_2\right)\right)[S^{k_1,k_2}](t; 2M) + \left(\frac{M - k_1}{2M}[S^{k_1+1,k_2}](t; 2M) + \frac{M - k_2}{2M}[S^{k_1,k_2+1}](t; 2M)\right) \times \left(k_1 q_1 + k_2 q_2\right),$$

subject to

$$[S^{k_1,k_2}](t; 2M)(0) = (1 - I_1^0)^{k_1}(1 - I_2^0)^{k_2}.$$

Similarly,

$$\frac{d[S^{k_1,k_2}]}{dt}(t; 2(M+1)) = -\left(k_1 p_1 + k_2 p_2 + \frac{2M + 2 - k_1 - k_2}{2M + 2}\left(k_1 q_1 + k_2 q_2\right)\right) \times [S^{k_1,k_2}](t; 2(M+1)) + \left(\frac{M + 1 - k_1}{2M + 2}[S^{k_1+1,k_2}](t; 2(M+1)) + \frac{M + 1 - k_2}{2M + 2}[S^{k_1,k_2+1}](t; 2(M+1))\right)\left(k_1 q_1 + k_2 q_2\right),$$

subject to

$$[S^{k_1,k_2}](t; 2(M+1))(0) = (1 - I_1^0)^{k_1}(1 - I_2^0)^{k_2}.$$

Taking the difference of these two equations gives

$$\frac{dy_{k_1,k_2}}{dt}+\left(k_1p_1+k_2p_2+\frac{2M+2-k_1-k_2}{2M+2}\left(k_1q_1+k_2q_2\right)\right)y_{k_1,k_2}$$
$$=\left(k_1q_1+k_2q_2\right)\left(\frac{M+1-k_1}{2M+2}y_{k_1+1,k_2}+\frac{M+1-k_2}{2M+2}y_{k_1,k_2+1}\right)$$
$$+z_{k_1,k_2}(t),\quad y_{k_1,k_2}(0)=0, \tag{E.5a}$$

where

$$z_{k_1,k_2}=\left(k_1q_1+k_2q_2\right)\left(\frac{-k_1}{M(2M+2)}\left(-[S^{k_1,k_2}](t;2M)+[S^{k_1+1,k_2}](t;2M)\right)+\frac{-k_2}{M(2M+2)}\left(-[S^{k_1,k_2}](t;2M)+[S^{k_1,k_2+1}](t;2M)\right)\right). \tag{E.5b}$$

Since

$$\frac{-k_1}{M(2M+2)}\le 0,\qquad \frac{-k_2}{M(2M+2)}\le 0,$$

and

$$-[S^{k_1,k_2}]+[S^{k_1+1,k_2}]=-[I^{1,0}S^{k_1,k_2}]<0,$$
$$-[S^{k_1,k_2}]+[S^{k_1,k_2+1}]=-[I^{0,1}S^{k_1,k_2}]<0\qquad t>0,$$

we have that $z_{k_1,k_2}\ge 0$ for $t>0$.

Therefore, applying Lemma C.2 to (E.5) shows that for any $(k_1,k_2)\in A_M\setminus(M,M)$,

$$\left\{y_{k_1+1,k_2}(t)\quad\text{and}\quad y_{k_1,k_2+1}(t)>0,\quad t>0\right\}\quad\Rightarrow\quad y_{k_1,k_2}(t)>0,\quad t>0. \tag{E.6}$$

From relations (E.4) and (E.6) we get by reverse induction on n that

$$y_{k_1,k_2}(t)>0,\qquad (k_1,k_2)\in A_M, \tag{E.7}$$

i.e., that $\{[S^{k_1,k_2}](t;2M)\}$ are monotonically decreasing in M. ☐

E.2. Convergence of $f^{2-\text{groups}}$

Proof of Theorem 5.2. Let $M\to\infty$. Then the master equations (E.1a) converge to

$$\frac{d}{dt}[S_\infty^{k_1,k_2}]=-\left(k_1\left(p_1+q_1\right)+k_2\left(p_2+q_2\right)\right)[S_\infty^{k_1,k_2}]+\frac{1}{2}\left(k_1q_1+k_2q_2\right)\left([S_\infty^{k_1+1,k_2}]+[S_\infty^{k_1,k_2+1}]\right), \tag{E.8a}$$

with the initial condition

$$[S_\infty^{k_1,k_2}](0)=(1-I_1^0)^{k_1}(1-I_2^0)^{k_2},\qquad (k_1,k_2)\in A_M. \tag{E.8b}$$

We will prove in Lemma E.2 below that $\lim_{M\to\infty}[S^{k_1,k_2}]=[S_\infty^{k_1,k_2}]$. Substituting the ansatz

$$[S_\infty^{k_1,k_2}]=[S_\infty^{1,0}]^{k_1}[S_\infty^{0,1}]^{k_2},$$

in (E.8) gives

$$k_1[S_\infty^{0,1}]\frac{d}{dt}[S_\infty^{1,0}]+k_2[S_\infty^{1,0}]\frac{d}{dt}[S_\infty^{0,1}]$$
$$=-\left(k_1\left(p_1+q_1\right)+k_2\left(p_2+q_2\right)\right)[S_\infty^{1,0}][S_\infty^{0,1}]+\frac{1}{2}\left(k_1q_1+k_2q_2\right)\left([S_\infty^{1,0}]^2[S_\infty^{0,1}]+[S_\infty^{1,0}][S_\infty^{0,1}]^2\right). \tag{E.9}$$

Let $[S_\infty^{1,0}]$ and $[S_\infty^{0,1}]$ be the solutions of

$$\frac{d}{dt}[S_\infty^{1,0}]=-\left(p_1+q_1\right)[S^{1,0}]+\frac{q_1}{2}\left([S_\infty^{1,0}]^2+[S_\infty^{1,0}][S_\infty^{0,1}]\right),\quad [S_\infty^{1,0}](0)=1-I_1^0,$$
$$\frac{d}{dt}[S_\infty^{0,1}]=-\left(p_2+q_2\right)[S_\infty^{0,1}]+\frac{q_2}{2}\left([S_\infty^{0,1}]^2+[S_\infty^{1,0}][S_\infty^{0,1}]\right),\quad [S_\infty^{0,1}](0)=1-I_2^0. \tag{E.10}$$

Then $[S_\infty^{1,0}]$ and $[S_\infty^{0,1}]$ satisfy (E.9). Substituting $f_1:=\frac{1}{2}\left(1-[S_\infty^{1,0}]\right)$ and $f_2:=\frac{1}{2}\left(1-[S_\infty^{0,1}]\right)$ in (E.10) gives

$$\frac{df_1}{dt}=\left(\frac{1}{2}-f_1\right)\left(p_1+q_1\left(f_1+f_2\right)\right),\qquad f_1(0)=\frac{I_1^0}{2},$$
$$\frac{df_2}{dt}=\left(\frac{1}{2}-f_2\right)\left(p_2+q_2\left(f_1+f_2\right)\right),\qquad f_2(0)=\frac{I_2^0}{2}. \tag{E.11}$$

Since $f^{\text{compart-het}}:=f_1+f_2$, the result follows. ☐

Lemma E.2. *For any $t\ge 0,(k_1,k_2)\in A_M$, the solution $[S^{k_1,k_2}](t;2M)$ of the master equations (E.1a) converges monotonically as $M\to\infty$ to the solution $[S_\infty^{k_1,k_2}](t)$ of equation (5.2b)*

Proof. The proof is nearly identical to that of Lemma C.3. Integrating the ODE (E.1a) for $[S^{k_1,k_2}](t;2M)$ from zero to t gives

$$[S^{k_1,k_2}](t;2M)-(1-I_1^0)^{k_1}(1-I_2^0)^{k_2}=$$
$$-\int_0^t\left(k_1p_1(s)+k_2p_2(s)+\frac{2M-k_1-k_2}{2M}\left(k_1q_1(s)+k_2q_2(s)\right)\right)\times[S^{k_1,k_2}](s;2M)\,ds$$
$$+\int_0^t\left(k_1q_1(s)+k_2q_2(s)\right)\left(\frac{M-k_1}{2M}[S^{k_1+1,k_2}](s;2M)\right)ds$$
$$+\int_0^t\left(k_1q_1(s)+k_2q_2(s)\right)\left(\frac{M-k_2}{2M}[S^{k_1,k_2+1}](s;2M)\right)ds. \tag{E.12}$$

Let us consider the limit of (E.12) as $a\to\infty$. Since $[S^{k_1,k_2}](t;2M)$ is monotonically decreasing in M, see (E.7), and since $[S^{k_1,k_2}]\ge 0$ as a probability, this implies that $[S^{k_1,k_2}](t;2M)$ converges pointwise as $M\to\infty$ to some limit $[S_\infty^{k_1,k_2}](t)$. Therefore, as $M\to\infty$, the left-hand side of (E.12) converges to $[S_\infty^{k_1,k_2}]-1$. In addition, since $[S^{k_1,k_2}]$ is a probability, $0\le[S^{k_1,k_2}](t;2M)\le 1$, and so, by the dominated convergence theorem, the integrals of $[S^{k_1,k_2}],[S^{k_1+1,k_2}]$ and $[S^{k_1,k_2+1}]$ on the right-hand side of (E.12) converge to the integrals of the limits. Hence, the limit of (E.12) as $M\to\infty$ is

$$[S_\infty^{k_1,k_2}](t;M)-(1-I_1^0)^{k_1}(1-I_2^0)^{k_2}=$$
$$-\int_0^t\left(k_1p_1(s)+k_2p_2(s)+\left(k_1q_1(s)+k_2q_2(s)\right)\right)[S_\infty^{k_1,k_2}](s)\,ds$$
$$+\frac{1}{2}\int_0^t\left(k_1q_1(s)+k_2q_2(s)\right)[S_\infty^{k_1+1,k_2}](s)\,ds$$
$$+\frac{1}{2}\int_0^t\left(k_1q_1(s)+k_2q_2(s)\right)[S_\infty^{k_1,k_2+1}](s)\,ds. \tag{E.13}$$

Since a pointwise limit of a sequence of measurable functions is also measurable, $[S_\infty^{k_1,k_2}](t)$ is measurable. Hence, it follows from (E.13) that it is continuous, hence differentiable. Differentiating (E.13), we conclude that $[S_\infty^{k_1,k_2}]$ satisfies the limit ODE (5.2b). ☐

Appendix F. Monotone convergence of f_{SI}

In Figs. F.5–F.7, we illustrate numerically the monotone convergence of f_{SI} on complete, circular, and 2-groups networks, respectively.

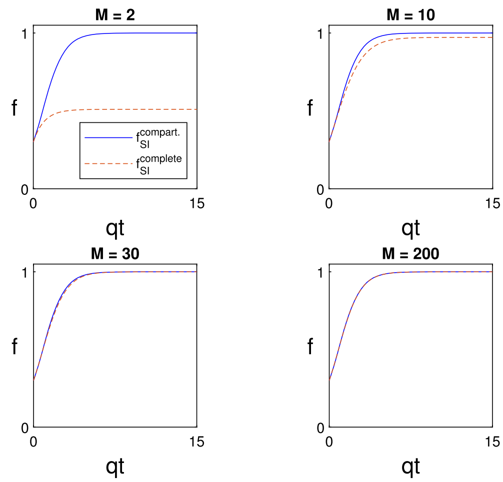

Fig. F.5. Monotone convergence of $f_{\mathrm{SI}}^{\mathrm{complete}}$ (dashes) to $f_{\mathrm{SI}}^{\mathrm{compart}}$ (solid). Here $p = 0$, $q = 0.1$, $I^0 = 0.3$, and $M = 2, 10, 30, 200$.

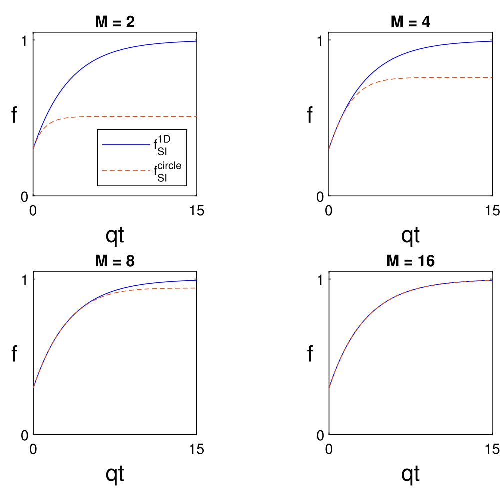

Fig. F.6. Monotone convergence of $f_{\mathrm{SI}}^{\mathrm{circle}}$ (dashes) to $f_{\mathrm{SI}}^{\mathrm{1D}}$ (solid). Here $p = 0$, $q = 0.1$, $I^0 = 0.3$, and $M = 2, 4, 8, 16$.

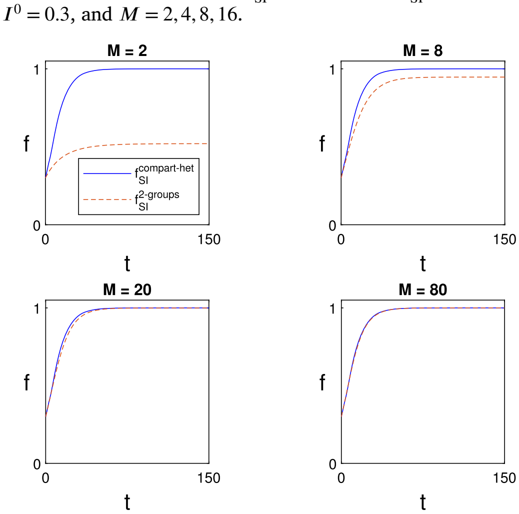

Fig. F.7. Monotone convergence of $f_{\mathrm{SI}}^{2-\mathrm{groups}}$ (dashes) to $f_{\mathrm{SI}}^{\mathrm{compart-het}}$ (solid). Here $p_1 = p_2 = 0$, $q_1 = 0.1$, $q_2 = 0.2$, $I_1^0 = 0.2$, $I_2^0 = 0.4$, and $M = 2, 8, 20, 80$.

References

[1] R.M. Anderson, R.M. May, Infectious Diseases of Humans, Oxford University Press, Oxford, 1992.
[2] R. Balderrama, J. Peressutti, J.P. Pinasco, F. Vazquez, C.S. de la Vega, Optimal control for a SIR epidemic model with limited quarantine, Sci. Rep. 12 (2022) 12583.
[3] F.M. Bass, A new product growth model for consumer durables, Manag. Sci. 15 (1969) 215–1227.
[4] F. Brauer, Mathematical epidemiology: past, present, and future, Infect. Dis. Model. 2 (2) (2017) 113–127.
[5] G. Fibich, Bass-SIR model for diffusion of new products in social networks, Phys. Rev. E 94 (2016) 032305.
[6] G. Fibich, Diffusion of new products with recovering consumers, SIAM J. Appl. Math. 77 (2017) 230–1247.
[7] G. Fibich, R. Gibori, Aggregate diffusion dynamics in agent-based models with a spatial structure, Oper. Res. 58 (2010) 1450–1468.
[8] G. Fibich, A. Golan, Optimal promotional strategies for spreading of new products on networks, arXiv:2504.06967.
[9] G. Fibich, A. Golan, Diffusion of new products with heterogeneous consumers, Math. Oper. Res. 48 (2023) 257–287.
[10] G. Fibich, A. Golan, S. Schochet, Compartmental limit of discrete Bass models on networks, Discrete Contin. Dyn. Syst., Ser. B 28 (2023) 3052–3078.
[11] G. Fibich, T. Levin, K. Gillingham, Boundary effects in the diffusion of new products on Cartesian networks, Oper. Res. 73 (2024) 2026–2044.
[12] G. Fibich, J.G. Restrepo, G. Rothmann, Explicit solutions of the Bass and SI models on hypernetworks, Phys. Rev. E 110 (2024) 054306.
[13] G. Fibich, Y. Warman, Explicit solutions of the SI and Bass models on sparse Erdős-Rényi and regular networks, arXiv:2411.12076.
[14] G.E. Fruchter, A. Prasad, C. Van den Bulte, Too popular, too fast: optimal advertising and entry timing in markets with peer influence, Manag. Sci. 68 (2022) 4725–4741.
[15] O. Givan, N. Schwartz, A. Cygelberg, L. Stone, Predicting epidemic thresholds on complex networks: limitations of mean-field approaches, J. Theor. Biol. 288 (2011) 21–28.
[16] J. Goldenberg, B. Libai, E. Muller, Using complex systems analysis to advance marketing theory development, Acad. Mark. Sci. Rev. 9 (2001) 1–19.
[17] D. Horsky, L.S. Simon, Advertising and the diffusion of new products, Mark. Sci. 2 (1983) 1–17.
[18] M.O. Jackson, Social and Economic Networks, Princeton University Press, Princeton and Oxford, 2008.
[19] D. Keliger, I. Horváth, Accuracy criterion for mean field approximations of Markov processes on hypergraphs, Phys. A 609 (2023) 128370.
[20] I.Z. Kiss, J.C. Miller, P.L. Simon, Mathematics of Epidemics on Networks, Springer, 2017.
[21] I.Z. Kiss, C.G. Morris, F. Sélley, P.L. Simon, R.R. Wilkinson, Exact deterministic representation of Markovian SIR epidemics on networks with and without loops, J. Math. Biol. 70 (2015) 437–464.
[22] B. Marcolongo, F. Peruani, G.J. Sibona, Assessing the forecasting power of mean-field approaches for disease spreading using active systems, Phys. A 648 (2024) 129916.
[23] S.C. Niu, A stochastic formulation of the Bass model of new product diffusion, Math. Prob. Eng. 8 (2002) 249–263.
[24] P.E. Paré, C.L. Beck, T. Başar, Modeling, estimation, and analysis of epidemics over networks: an overview, Annu. Rev. Control 50 (2020) 345–360.
[25] W. Rand, R.T. Rust, Agent-based modeling in marketing: guidelines for rigor, Int. J. Res. Mark. 28 (2011) 181–193.
[26] K.J. Sharkey, I.Z. Kiss, R.R. Wilkinson, P.L. Simon, Exact equations for SIR epidemics on tree graphs, Bull. Math. Biol. 77 (2015) 614–645.