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SELF-FOCUSING IN THE PERTURBED AND UNPERTURBED NONLINEAR SCHRÖDINGER EQUATION IN CRITICAL DIMENSION*

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Abstract. The formation of singularities of self-focusing solutions of the nonlinear Schrödinger equation (NLS) in critical dimension is characterized by a delicate balance between the focusing nonlinearity and diffraction (Laplacian), and is thus very sensitive to small perturbations. In this paper we introduce a systematic perturbation theory for analyzing the effect of additional small terms on self-focusing, in which the perturbed critical NLS is reduced to a simpler system of modulation equations that do not depend on the spatial variables transverse to the beam axis. The modulation equations can be further simplified, depending on whether the perturbed NLS is power conserving or not. We review previous applications of modulation theory and present several new ones that include dispersive saturating nonlinearities, self-focusing with Debye relaxation, the Davey–Stewartson equations, self-focusing in optical fiber arrays, and the effect of randomness. An important and somewhat surprising result is that various small defocusing perturbations lead to a generic form of the modulation equations, whose solutions have slowly decaying focusing-defocusing oscillations. In the special case of the unperturbed critical NLS, modulation theory leads to a new adiabatic law for the rate of blowup which is accurate from the early stages of self-focusing and remains valid up to the singularity point. This adiabatic law preserves the lens transformation property of critical NLS and it leads to an analytic formula for the location of the singularity as a function of the initial pulse power, radial distribution, and focusing angle. The asymptotic limit of this law agrees with the known loglog blowup behavior. However, the loglog behavior is reached only after huge amplifications of the initial amplitude, at which point the physical basis of NLS is in doubt. We also include in this paper a new condition for blowup of solutions in critical NLS and an improved version of the Dawes–Marburger formula for the blowup location of Gaussian pulses.

Key words. self-focusing, adiabatic, collapse, nonlinear Schrödinger equation, loglog law, Davey–Stewartson, Debye, fiber arrays, time-dispersion, nonparaxial

AMS subject classifications. 78A60, 35Q55

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1. Introduction.

The nonlinear Schrödinger equation in critical dimension (CNLS)

$$(1.1)\quad i\psi_z + \Delta_\perp \psi + |\psi|^2\psi = 0\ ,\quad \Delta_\perp = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\ ,\quad \psi(z=0,x,y) = \psi_0(x,y)$$

is the simplest model for the propagation of a laser beam in a medium with a Kerr nonlinearity. Here $\psi(z,x,y)$ is the electric field envelope, z is axial distance[1] in the direction of the wave propagation, and (x,y) are the coordinates in the transverse plane. In 1965, Kelley used (1.1) to show that for optical beams whose power is above a critical value, "self-focusing effect ... is not compensated for by diffraction" [32]. This result was a turning point in nonlinear optics, since until that time diffraction

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[1]Since the initial condition is given at $z = 0$ for all (x, y), the variable z plays the role of "time."

was believed to prevent singularity formation in optics, both linear and nonlinear, much as viscosity is believed to prevent singularity formation in fluid flow. Intensive experimental work followed, in which self-focusing and the existence of a critical power, above which beams may collapse, were observed. For a review of self-focusing experiments see [66].

Self-focusing in critical nonlinear Schrödinger equations (NLS) also attracted the attention of mathematicians, since it serves as a simple model of nonlinear dispersive wave propagation where a solution with smooth initial conditions can become singular in finite time (i.e., z). A lot has been accomplished in the last 30 years (e.g., [16, 61, 62, 69]), but the theory for CNLS self-focusing is far from complete. For example, sharp conditions for blowup or global existence in (1.1) are still unknown.

1.1. The loglog law. Considerable effort has been devoted to the study of the blowup rate near the singularity. Initially, self-focusing was analyzed by reducing CNLS to an ordinary differential equation for the beam width L by assuming that the solution maintains a modulated Gaussian profile. This approach was only partially successful. It predicted the existence of a critical power for self-focusing, but only up to a constant [7] and its prediction for the axial location of the singularity was quite inaccurate. There were also attempts to look for non-Gaussian self-similar solutions, but it gradually became clear that in critical transverse dimension $D = 2$ self-focusing solutions of (1.1) are only quasi self-similar and that the rate of focusing is determined by a delicate balance between the focusing nonlinearity and transverse diffraction. This delicate balance in critical self-focusing, which the Gaussian ansatz cannot capture, was at the heart of the difficulties in finding the blowup rate of CNLS. It is why it took so long for the structure and dynamics of the function ψ near the blowup point finally to be resolved by Fraiman [29] and, independently (and in a different manner), by Landman, Papanicolaou, C. Sulem, and P. Sulem [39] and Lemesurier, Papanicolaou, C. Sulem, and P. Sulem [43], who showed that as the beam approaches the singularity it follows the loglog law (3.23).

1.2. The adiabatic approach. Although with the loglog law the mathematical problem of finding the blowup rate was finally solved, it turned out that the loglog behavior is very hard to observe numerically, even in careful simulations where the solution was amplified by more than ten orders of magnitude (e.g., Figure 3.5). However, the validity of CNLS as a physical model for beam propagation breaks down much earlier, when the field intensity reaches the threshold for material breakdown. Even at subthreshold intensities, some small terms that are neglected in the derivation of (1.1) from Maxwell's equations (e.g., nonparaxial terms [22, 68], time-dispersion [27, 44, 59], etc.) may become important, because the delicate balance between the focusing nonlinearity and the defocusing Laplacian in critical dimension allows for even small terms to have a large effect on self-focusing and even to arrest it.

It is therefore clear that even though the loglog law is established, there is still a need for a description of CNLS self-focusing which is valid in the domain of physical interest and which can be extended to the analysis of the effect of small perturbations. In the last few years a new adiabatic approach was developed which achieves this. This approach is based on the main result of the analysis leading to the loglog law, which is the derivation of the reduced equation (3.13) for the slow rate of radiation losses of the focusing part of the solution. However, rather than solving this equation asymptotically, the adiabatic approach uses it only as a small correction to the adiabatic focusing-rate equation (3.5). This approach was first used by Malkin [47] and it led to an adiabatic law for the blowup rate of CNLS that is accurate in physically relevant regimes. An improved adiabatic law, which becomes accurate with even less amplification and also preserves the lens transformation property of CNLS, was later obtained by Fibich [24]. In this paper we give a detailed derivation of these laws and show that near the singularity, Fibich's adiabatic law reduces to Malkin's adiabatic law, whose asymptotic limit is in turn the loglog law. Thus, the three laws agree when the amplification is very large but have different domains of validity.

An immediate consequence of the adiabatic law is an analytic formula for the location of the singularity. A previous result of this type is that of Dawes and Marburger [20] and Marburger [48], derived by curve-fitting values obtained from numerical simulations with Gaussian initial conditions. We give here a new curve-fitted formula for Gaussian initial conditions which is more accurate than either the adiabatic formula or the one of Dawes and Marburger.

1.3. Perturbed critical self-focusing. Since the adiabatic approach is valid in regimes of physical interest, it may be used to analyze the effect of small perturbations on critical self-focusing. The main result of this paper is an extension of this approach to a general modulation theory for analyzing the effects of small perturbations on self-focusing. In this modulation theory the perturbed CNLS is averaged around a modulated Townes soliton ψ_R (see (4.6)) over the transverse variables, leading to a simpler system of reduced equations (Proposition 4.1). The analysis of the reduced system of modulation equations is further simplified by distinguishing between conservative perturbations (perturbations under which the total power (L^2 norm) is conserved) and nonconservative perturbations (Proposition 4.2). It is interesting to note that in the conservative case the modulation equations have a generic form (Proposition 4.3).

The adiabatic approach was used by Malkin to study the effect of a small defocusing fifth-power nonlinearity [47]. In [27], Fibich, Malkin, and Papanicolaou analyzed the effect of small normal time-dispersion, using for the first time the systematic approach presented in this paper. This approach was also used by Fibich to analyze the effect of beam nonparaxiality [25], and the unperturbed CNLS [24] by Fibich and Papanicolaou to analyze the combined effect of time-dispersion and nonparaxiality [28], and by Fibich and Levy to analyze self-focusing in the complex Ginzburg–Landau limit of CNLS [26].

1.4. Outline. The paper is organized as follows. In section 2 we review the analytic theory of existence and blowup for CNLS and use the lens transformation of critical NLS to derive a new condition for blowup and to relate solutions of CNLS with solutions of CNLS with an additional quadratic potential term. In section 3 we present the adiabatic approach for self-focusing in the unperturbed CNLS, derive and compare the three laws for self-focusing, derive the formula for the location of the blowup point, and present a new empirical formula for Gaussian initial conditions. In section 4 we develop the modulation theory for analyzing the effect of small perturbations of CNLS self-focusing, which is summarized by Propositions 4.1–4.4. In section 5 we review previous applications of this approach and apply it to several new situations (see Table 1.1).

2. Basic theory of self-focusing for CNLS. We begin with a review of the basic theory of self-focusing. More details can be found in [16, 61, 62, 69, 70]. We emphasize the importance of the lens transformation (2.16) in the analysis of critical NLS and use it to derive new results regarding blowup.

In order to understand the special character of blowup in the cubic Schrödinger equation in two dimensions, it is instructive to begin with the two-dimensional (2D)

Table 1.1
Perturbations of critical NLS which are analyzed in this paper using modulation theory.

Perturbed CNLS	**Application**	**Section**
$i\psi_z + \Delta_\perp\psi + \lvert\psi\rvert^2\psi + \epsilon\psi_{xxxx} + \frac{2}{5}\epsilon^2\psi_{xxxxxx} = 0$	fiber arrays	5.1
$i\psi_z + \Delta_\perp\psi + \lvert\psi\rvert^2\psi - \epsilon\lvert\psi\rvert^4\psi = 0$	quintic nonlinearity	5.2
$i\psi_z + \Delta_\perp\psi + \frac{1-\exp(-2\epsilon\lvert\psi\rvert^2)}{2\epsilon}\psi = 0$	saturating nonlinearity	5.3
$i\psi_z + \Delta_\perp\psi + \frac{\lvert\psi\rvert^2}{1+\epsilon\lvert\psi\rvert^2}\psi = 0$	saturating nonlinearity	5.3
$i\psi_z + \Delta_\perp\psi + \lvert\psi\rvert^2\psi - \epsilon\phi_x\psi = 0$, $\alpha\phi_{xx} + \phi_{yy} = -(\lvert\psi\rvert^2)_x$	Davey–Stewartson equation	5.4
$i\psi_z + \Delta_\perp\psi + \lvert\psi\rvert^2\psi + \epsilon\psi_{zz} = 0$	nonparaxiality	5.5
$i\psi_z + \Delta_\perp\psi + \lvert\psi\rvert^2\psi + \epsilon(x^2+y^2)h(z)\psi = 0,$ h random	randomness	5.6
$i\psi_z + \Delta_\perp\psi + \lvert\psi\rvert^2\psi + \epsilon_1(x^2+y^2)h(z)\psi - \epsilon_2\lvert\psi\rvert^4\psi = 0,$ h random	quintic nonlinearity + randomness	5.6
$i\psi_z + \Delta_\perp\psi + \lvert\psi\rvert^2\psi - \epsilon\psi_{tt} = 0$	time-dispersion	5.7.1
$i\psi_z + \Delta_\perp\psi + N\psi = 0$, $\epsilon N_t + N = \lvert\psi\rvert^2$	Debye relaxation	5.7.2
$i\psi_z + \Delta_\perp\psi + \lvert\psi\rvert^2\psi \ + \ \epsilon_1\psi_{zz}$ $+\epsilon_2\left[2i\frac{n_0c_g}{c}(\lvert\psi\rvert^2\psi)_t - \psi_{zt}\right] - \epsilon_3\psi_{tt} = 0$	time-dispersion + nonparaxiality	5.8

NLS with a general power nonlinearity

$$i\psi_z + \Delta_\perp\psi + \kappa|\psi|^{2\sigma}\psi = 0, \quad \kappa = \pm 1, \tag{2.1}$$

where κ positive/negative corresponds to the focusing/defocusing NLS, respectively.

Two important invariants of (2.1) are the power[2]

$$N = \frac{1}{2\pi} \int |\psi|^2 \, dx \, dy \equiv N(0) \tag{2.2}$$

and the Hamiltonian

$$H = \frac{1}{2\pi} \left(\int |\nabla_\perp \psi|^2 \, dx \, dy - \frac{\kappa}{\sigma + 1} \int |\psi|^{2\sigma+2} \, dx \, dy \right) \equiv H(0), \tag{2.3}$$

where

$$\nabla_\perp = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right).$$

We say that a solution exists at z if it has a finite H^1 norm

$$||\psi(z, \cdot)||_{H^1} < \infty \ , \quad ||\psi||_{H^1} = \left(\int |\psi|^2 \, dxdy + \int |\nabla_\perp \psi|^2 \, dxdy \right)^{1/2}$$

and that ψ blows up at $z = Z_c$ if it exists for $0 \le z < Z_c$ and

$$\lim_{z \to Z_c} ||\psi(z, \cdot)||_{H^1} = \infty,$$

which by (2.2) is equivalent to blowup of the gradient norm

$$\lim_{z \to Z_c} \int |\nabla_\perp \psi|^2 \, dxdy = \infty.$$

From the theory for local existence of solutions of (2.1), it is known that if $||\psi(z, \cdot)||_{H^1}$ is bounded, the solution exists for all z [30, 31]. As a result, when NLS is defocusing ($\kappa < 0$), conservation of the Hamiltonian implies that $\int |\nabla_\perp \psi|^2 \, dxdy$ is bounded and the solution exists globally. Since we are interested in singularity formation, from now on we restrict ourselves to the case of focusing NLS $\kappa = 1$.

Because of the minus sign in the Hamiltonian of the focusing NLS, its conservation does not prohibit $\int |\nabla_\perp \psi|^2 \, dxdy$ from growing to infinity. To see that this can indeed happen, we note that solutions of (2.1) satisfy the *variance identity* [75]

$$V_{zz} \equiv 8H - \frac{8(\sigma - 1)}{2\pi(\sigma + 1)} \int |\psi|^{2\sigma+2} dxdy, \quad V(\psi) = \frac{1}{2\pi} \int (x^2 + y^2) |\psi|^2 \, dxdy. \tag{2.4}$$

From the variance identity (2.4), the invariance of the Hamiltonian (2.3), and the uncertainty principle

$$N^2(\psi) \le V(\psi) \int |\nabla_\perp \psi|^2 \, dxdy,$$

it follows that for $\sigma \ge 1$ the condition

$$H(0) < 0 \tag{2.5}$$

is sufficient for blowup in a finite z.

[2]In the nonlinear optics context, the L^2 norm corresponds to the power of the laser beam.

In the supercritical case $\sigma > 1$, sharper conditions for blowup can be obtained [36, 73] and singularity formation is characterized by dominance of self-focusing over wave diffraction, resulting in a finite z blowup which is stable under small perturbations. Conversely, for $\sigma < 1$, the subcritical case, there is no finite z blowup and the solution exists globally [69], as in the case of solitons in the cubic NLS in one transverse dimension. In the physically important case of critical self-focusing $\sigma = 1$ which we study here, wave diffraction and self-focusing are nearly balanced and blowup is extremely sensitive to perturbations and to changes in the initial condition (a physical argument that explains the role of criticality in the balance between nonlinearity and diffraction is given at the beginning of section 4). A necessary condition for blowup in critical NLS (1.1) is that

$$N(0) \geq N_c \ , \tag{2.6}$$

where $N_c \cong 1.862$ is the critical power for self-focusing. More precisely, there is no blowup when $N < N_c$ but for any $\epsilon \geq 0$, there exist solutions with $N = N_c + \epsilon$ for which there is finite z blowup [76].

The proof of these results makes use of the Gagliardo–Nirenberg inequality

$$||f||_{2\sigma+2}^{2\sigma+2} \leq C_\sigma ||\nabla f||_2^{2\sigma} ||f||_2^2 \ , \quad 0 < \sigma, \tag{2.7}$$

which holds for any function on 2D space that has square integrable derivatives. Specifically, by combining

$$||\psi||_2^2 = H + \frac{1}{\sigma+1} ||\psi||_{2\sigma+2}^{2\sigma+2} \ , \tag{2.8}$$

the invariance of the Hamiltonian and the Gagliardo–Nirenberg inequality, we obtain an estimate for the L^2 norm of the gradient of solutions of NLS:

$$||\psi||_2^2 \leq H + \frac{C_\sigma}{\sigma+1} ||\nabla\psi||_2^{2\sigma} ||\psi||_2^2 \ . \tag{2.9}$$

When $\sigma < 1$ this gives a bound for the L^2 norm of the gradient of the solution, since the L^2 norm of the solution and H are constants. When $\sigma = 1$, the critical case, the optimal constant C_σ can be used to get a bound for the L^2 norm of the gradient of the solution provided that $N(0) < N_c$. The optimal constant in the Gagliardo–Nirenberg inequality is obtained when $f = R$ (the waveguide solution of the next section), in which case (2.7) becomes an equality [76].

We note that if in (2.1) the transverse Laplacian is in D dimensions, the subcritical, critical, and supercritical cases correspond to the product σD being less than, equal to, or greater than 2, respectively. For this reason, the case $D = 2$ for NLS with cubic nonlinearity is called "critical dimension."

2.1. Waveguide solutions and the Townes soliton. From now on we restrict our analysis to the critical case $\sigma = 1$ and D=2. Critical NLS (1.1) has radially symmetric waveguide solutions

$$\psi(z, r) = \exp(iz)R(r), \quad r = \sqrt{x^2 + y^2}, \tag{2.10}$$

where R satisfies the nonlinear boundary value problem

$$\Delta_\perp R - R + R^3 = 0, \quad R'(0) = 0, \quad \lim_{r\to\infty} R(r) = 0 \ , \quad \Delta_\perp = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right). \tag{2.11}$$

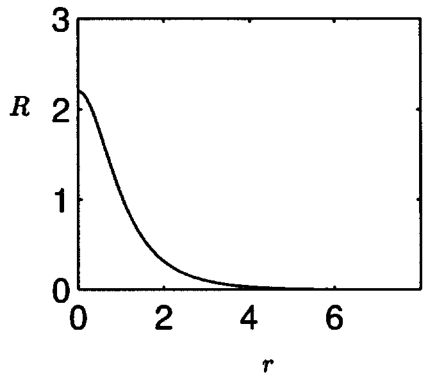

FIG. 2.1. *The Townes soliton $R(r)$.*

This ordinary differential equation has an enumerable set of solutions (see [61] and its references). Of most interest is the solution with the lowest power (ground state), often called the Townes soliton. The Townes soliton is positive and monotonically decreasing (Figure 2.1). In addition, it has exactly the critical power for blowup [76]

$$\int_0^\infty R^2 \, r dr = N_c \tag{2.12}$$

and its Hamiltonian is equal to zero,

$$H(R) = 0. \tag{2.13}$$

Therefore, the waveguide solution (2.10), being a borderline case for blowup, is unstable.

Some additional relations, which will be used later are (Lemma A.1)

$$\int_0^\infty \left(\frac{dR}{dr}\right)^2 r dr = N_c \ , \quad \int_0^\infty R^4 \, r dr = 2N_c \ . \tag{2.14}$$

The asymptotic behavior of R is given by

$$R(r) \sim A_R r^{-1/2} \exp(-r), \qquad 1 \ll r, \tag{2.15}$$

where

$$A_R = \left(\frac{\pi}{2}\right)^{1/2} \int_0^\infty R^3(r') I_0(r') \, r' dr' \cong 3.52$$

and I_0 is the modified Bessel function.

The Townes soliton plays an important role in CNLS theory and can be used to construct exact and approximate blowup solutions, as will be seen in the following sections. Although Gaussians look roughly like the Townes soliton (Figure 2.1), in the critical case there is no Gaussian that can satisfy the two conditions (2.12) and (2.13) simultaneously. Therefore, Gaussians cannot capture the delicate balance between diffraction and nonlinear focusing in critical self-focusing, which is the reason why CNLS analysis that is based on representing the solution by a modulated Gaussian is unreliable.

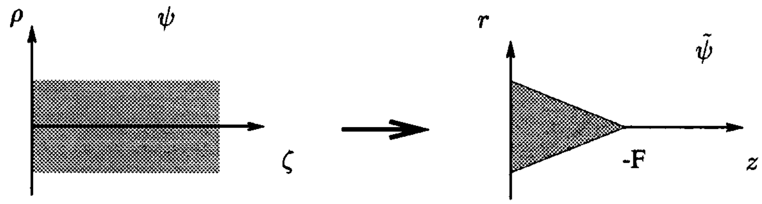

FIG. 2.2. *The lens transformation* (2.16) *with* $L(z)$ *given by* (2.17) *and* $F < 0$ *maps the values of* ψ *in the shaded semi-infinite strip into the corresponding values of* $\tilde{\psi}$ *in the shaded triangle.*

2.2. The lens transformation. An important tool in the analysis of critical NLS is the lens transformation. Let ψ and $\tilde{\psi}$ be related through

(2.16)
$$\tilde{\psi}(z,x,y) = \frac{1}{L(z)}\psi(\zeta,\xi,\eta)\exp\left(i\frac{L_z}{L}\frac{r^2}{4}\right), \quad \xi = \frac{x}{L}\ , \quad \eta = \frac{y}{L}\ , \quad \zeta = \int_0^z \frac{1}{L^2(z')}\,dz'.$$

Then, as noted by Talanov [71], if L depends linearly on z,

$$L = 1 + \frac{z}{F}\ , \quad F \text{ constant,} \tag{2.17}$$

and if ψ is a solution of (1.1) with initial condition ψ_0, then $\tilde{\psi}$ is also an exact solution of (1.1) with the initial condition

$$\tilde{\psi}_0(x,y) = \psi_0(x,y)\exp\left(i\frac{r^2}{4F}\right). \tag{2.18}$$

The addition of a quadratic phase term to the initial condition corresponds to adding at $z = 0+$ a thin lens whose focal point is at $(z = -F, 0, 0)$. Since z and ζ are related by

$$\frac{1}{z} + \frac{1}{F} = \frac{1}{\zeta} \tag{2.19}$$

and in addition

$$\rho := \sqrt{\xi^2 + \eta^2} = \frac{r}{L}\ ,$$

the lens transformation (2.16) shows that the effect of the lens in the diffractive case (linear, or with cubic nonlinearity) is to map the solution exactly as in ray optics (Figure 2.2). It is interesting to note that the lens transformation is valid in the linear case in all dimensions but the only nonlinearity for which the transformation will remain valid is the critical one.

The lens transformation can also be used to analyze CNLS with an additional quadratic potential term

$$i\tilde{\psi}_z + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\tilde{\psi} + |\tilde{\psi}|^2\tilde{\psi} + \tilde{\gamma}(z)(x^2 + y^2)\tilde{\psi} = 0\ . \tag{2.20}$$

In the linear case this is the basic equation of Gaussian optics, which in the non-isotropic case can be solved by the ABCD law (e.g., [80]). Let $\tilde{\psi}(z,x,y)$ be a solution of (2.20) with $\tilde{\gamma}(z)$ given, and define $\psi(\zeta,\xi,\eta)$ by the lens transformation (2.16) with a general $L(z)$, which is not necessarily linear in z as in (2.17). Then $\psi(\zeta,\xi,\eta)$ also satisfies CNLS with a quadratic potential term

$$i\psi_\zeta + \left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\psi + |\psi|^2\psi + \gamma(\zeta)(\xi^2+\eta^2)\psi = 0, \tag{2.21}$$

where

$$\gamma(\zeta) = \left(\frac{-L^3L_{zz}}{4} + L^4\tilde{\gamma}(z(\zeta))\right), \quad z(\zeta) = \int_0^\zeta L^2(\zeta')\,d\zeta'.$$

Therefore, the family of solutions of (2.20) with a general $\tilde{\gamma}(z)$ is closed under the lens transformation with a general $L(z)$. If $L(z)$ is chosen so that it satisfies the ordinary differential equation

$$L_{zz} = 4\tilde{\gamma}(z)L$$

and ψ is a solution of CNLS (1.1), then $\tilde{\psi}(z,x,y)$ satisfies the nonlinear Gaussian optics equation (2.20). Thus, (2.20) can always be reduced to (1.1).

2.3. Applications of the lens transformation. By applying the lens transformation to the CNLS waveguide solution (2.10) with $L = Z_c - z$, we get that

$$\psi_{ex}(z,r) = \frac{1}{Z_c - z}R\left(\frac{r}{Z_c - z}\right)\exp\left(i\frac{1 - r^2/4}{Z_c - z}\right) \tag{2.22}$$

is an exact solution of CNLS which blows up at Z_c:

$$\lim_{z\to Z_c} ||\psi_{ex}||_{H^1} = \infty.$$

This solution has a linear blowup rate, $L = (Z_c - z)$, and the power concentration property

$$|\psi_{ex}(z,r)|^2 \to N_c\delta(r), \qquad \text{as } z \to Z_c.$$

However, it is unstable [61, 76] since $N(\psi_{ex}) = N_c$, and it has not been seen in numerical experiments.

We can also use ψ_{ex} to construct exact blowup solutions of

$$i\psi_z + \Delta_\perp\psi + |\psi|^2\psi + \alpha(x^2+y^2)\psi = 0, \quad \alpha \text{ constant}, \tag{2.23}$$

by defining

$$\tilde{\psi}_{ex}(z,r) = \frac{1}{L(z)(Z_c - \zeta(z))}R\left(\frac{r}{L(z)(Z_c-\zeta(z))}\right)\exp\left(i\frac{1-r^2/4L^2(z)}{Z_c-\zeta(z)}\right)\exp\left(i\frac{L_z}{L}\frac{r^2}{4}\right),$$
$$\zeta = \int_0^z \frac{1}{L^2(z')}\,dz', \quad L = L_0\mathrm{Re}[\exp(\pm 2\sqrt{\alpha}z)] .$$

Note that with a proper choice of L_0 and the sign in L, this construction of exact blowup solutions for (2.23) works for both positive and negative values of α.

The variance identity in the critical case has the form

$$V_{zz} = 8H(0). \tag{2.24}$$

Therefore, in addition to (2.5), it can be used to derive conditions for blowup when $H(0) \geq 0$ which involve $V(0)$ and $V_z(0)$ [76]. However, these conditions are not sharp, since they are based on the vanishing of variance and typically blowup occurs well before the vanishing point of the variance [41]. Thus, the problem of finding sharp conditions for global existence or blowup in CNLS is still open. The following new result rules out many potential candidates.

PROPOSITION 2.1. *Let ψ be a solution of* (1.1) *such that $V(\psi_0) < \infty$. Any condition which involves only the absolute value of the initial condition $|\psi_0|$ cannot be sufficient for blowup.*

COROLLARY 2.2. *There is no critical threshold N_{TH} such that*

$$N(\psi_0) > N_{TH}$$

is a sufficient condition for blowup.

Proof of Proposition 2.1. Assume that there is such a condition. Let ψ be a solution of CNLS with initial condition ψ_0 that satisfies this condition and $V(\psi_0) < \infty$. Then there exists $0 < Z_c < \infty$ such that

$$\lim_{z \to Z_c} \int |\nabla_\perp \psi|^2 \, dxdy = \infty.$$

Let $\tilde{\psi}$ be the solution of (1.1) corresponding to the initial condition (2.18) with

$$0 < F < Z_c. \tag{2.25}$$

Then $\tilde{\psi}$ is given by (2.16). Since $|\tilde{\psi}_0| = |\psi_0|$, there exists $0 < Z_c^* < \infty$ such that

$$\lim_{z \to Z_c^*} \int |\nabla_\perp \tilde{\psi}|^2 \, dxdy = \infty.$$

To see that this leads to a contradiction we first note that

$$\int |\nabla_\perp \tilde{\psi}(z,x,y)|^2 \, dxdy \leq \frac{2}{L^2} \int |\nabla_\perp \psi(\zeta(z),x,y)|^2 \, dxdy + 2L^2 V(\psi(\zeta(z))). \tag{2.26}$$

Since $F > 0$, at any finite value of z the value of L and of L^{-1} are finite. In addition, by (2.19),

$$0 \leq \zeta(z) < F < Z_c \qquad \text{for} \quad 0 \leq z, \tag{2.27}$$

$V(\psi(\zeta(z)))$ is well defined and finite (2.24). Therefore, the right-hand side of (2.26) can become infinite only at $\tilde{Z}_c$ such that $0 \leq \tilde{Z}_c \leq Z_c^*$ and

$$\lim_{z \to \tilde{Z}_c} \int |\nabla_\perp \psi(\zeta(z),x,y)|^2 \, dxdy = \infty.$$

Clearly,

$$\frac{1}{\tilde{Z}_c} + \frac{1}{F} = \frac{1}{Z_c} \tag{2.28}$$

and $\zeta(\tilde{Z}_c) = Z_c$, leading to a contradiction with (2.27).

The proof shows that any initial condition, however large its L^2 norm may be, will not result in blowup if defocused at $z = 0$ by a sufficiently strong defocusing lens that maps the blowup point Z_c "beyond infinity" (Figure 2.3). The converse, of course, is not true. If the initial power is below critical, no focusing lens can cause the solution to blow up.

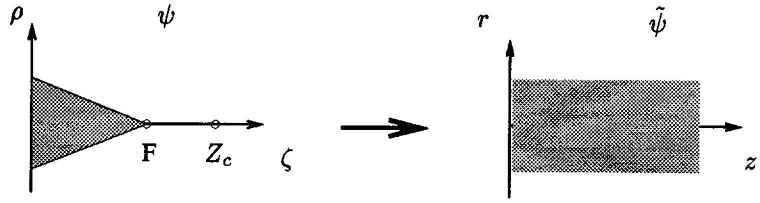

FIG. 2.3. *Sketch of proof of Proposition* 2.1*: If ψ, a CNLS solution which blows up at Z_c, is defocused so that the point F is mapped to infinity, the defocused solution $\tilde{\psi}$ will not blowup.*

2.4. Theoretical results on the nature of blowup. There is substantial numerical evidence that near the blowup point the ground state R serves as an attractor for the radial profile of the solution[3]

$$|\psi| \sim \frac{1}{L(z)} R\left(\frac{r}{L}\right) \qquad \text{as } z \to Z_c \tag{2.29}$$

and we will be using this assumption in the asymptotic analysis of the next section. Partial support for (2.29) can be found in the following result, due to Weinstein [78], which in the radially symmetric case is as follows.

THEOREM 2.3. *Let ψ be a radially symmetric solution of* (1.1) *such that $\psi_0 \in H^1$ and $\lim_{z\to Z_c} \int |\nabla_\perp \psi|^2 = \infty$. Then, for any sequence $z_k \to Z_c$ there is a subsequence z_{k_j} such that*

$$\frac{1}{L(z)} \psi\left(\frac{r}{L(z)}, z\right) \exp(i\gamma(z)) \to \Psi(r) \not\equiv 0$$

in L^p for $2 < p < \infty$, where $\gamma(z) \in [0\ , 2\pi)$. Furthermore,

$$\int |\Psi|^2 \geq N_c\ .$$

Note that in order to make (2.29) rigorous one has to show that $\Psi \equiv R(r)$.

Relation (2.29) implies that blowup solutions of critical NLS have a unique local power concentration property

$$|\psi|^2 \sim N_c \delta(r) \qquad \text{as } z \to Z_c,$$

namely, the amount of power which goes into the singularity is always equal to the critical power for self-focusing, independent of the initial condition. Based on simulations and asymptotic arguments it is also widely believed that the rate of blowup is slightly faster than a square root,

$$(Z_c - z)^{1/2+\epsilon} \ll L(z) \ll (Z_c - z)^{1/2} \qquad \text{as } z \to Z_c \ \ \forall \epsilon > 0.$$

Partial support for this can be found in the concentration theorems of Merle and Tsutsumi [53] and Tsutsumi [72] which in the radial case is as follows.

THEOREM 2.4. *Let ψ be a radially symmetric solution of* (1.1) *that blows up at a finite Z_c.*

[3]For clarity, we are considering here the radially symmetric case with a single blowup point located at the origin. For the possibility of multiple singularity points, see, e.g., [51, 55].

1. *If $a(z)$ is a decreasing function from $[0,\ Z_c)$ to R^+ such that $\lim_{z\to Z_c} a(z) = 0$ and $\lim_{z\to Z_c}(Z_c - z)^{1/2}/a(z) = 0$, then*

$$\liminf_{z\to Z_c} \int_{r<a(z)} |\psi(z)|^2 \geq N_c \ .$$

2. *For any $\epsilon > 0$, there exists a $K > 0$ such that*

$$\liminf_{z\to Z_c} \int_{r<K(Z_c-z)^{1/2}} |\psi(z)|^2 \geq (1-\epsilon) N_c \ .$$

Note that the two theorems (2.3, 2.4) give only an upper bound on the amount of power that goes into the singularity ($\geq N_c$). Strictly speaking, we cannot hope to prove that the power that goes into the singularity is exactly N_c, because there are exact blowup solutions which do not satisfy it. For example, if in the focusing waveguide solution (2.22), R is any of the nonground state solutions of (2.11), the power going into the singularity is greater than N_c.

Similarly, the concentration theorem suggests an upper bound on the blowup rate $L \ll (Z_c - z)^{1/2-\epsilon}$. In fact, it has been proved [17, 52] that for z near Z_c, $\int |\nabla\psi|^2 \geq c(Z_c - z)^{-1/2}$ which implies

$$L \leq C(Z_c - z)^{1/2} \ .$$

However, we cannot hope to prove that for all blowup solutions $L \gg (Z_c - z)^{1/2+\epsilon}$ because the blowup rate of the focusing waveguide (2.22) is $L = Z_c - z$. Of course, these exact blowup solutions are unstable but their mere existence helps to explain the difficulty in making a completely rigorous theory for CNLS self-focusing.

The concentration Theorem 2.4 illustrates the fact that blowup in critical NLS is a local phenomenon, which is why global quantities, such as N, H, and the variance, cannot capture the sharp conditions for blowup. For example, if ψ_0 is composed of K well-separated pulses, each of which would not blow up by itself, e.g.,

$$\psi_0 = \sum_{k=1}^{K} 0.8R(\sqrt{x^2 + (y-100k)^2}) \ ,$$

then ψ will not blow up, although $N(\psi_0) > N_c$. Similarly, if

$$\psi_0 = 1.1R(\sqrt{x^2 + y^2}) + 0.8R(\sqrt{x^2 + (y-100)^2}) \ ,$$

ψ would have a finite variance when its $1.1R$ component blows up, due to its $0.8R$ component.

3. Self-focusing in the unperturbed CNLS—an adiabatic approach. In this section we describe the local structure and dynamics of self-focusing near the blowup point. Unlike the previous section, most of the results presented in this section have not been made rigorous at present (see section 2.4).

3.1. Derivation of reduced equations—modulation theory. Self-focusing in critical dimension has the unique property that the amount of power which goes into the singularity is always equal to the critical power for blowup N_c. For this

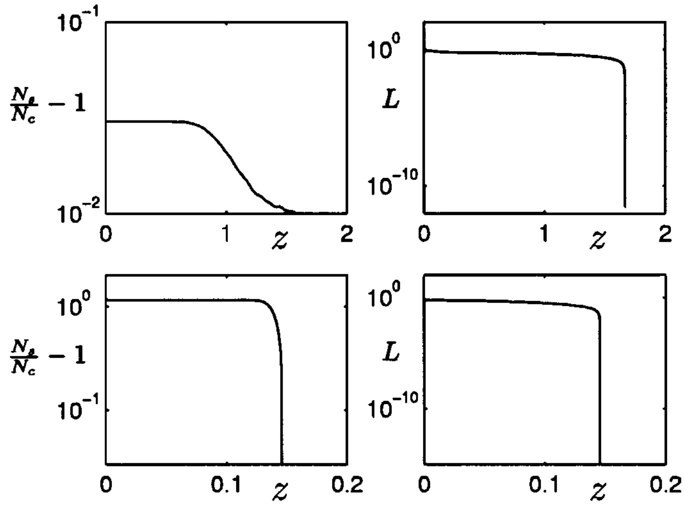

FIG. 3.1. *Most self-focusing ($L \searrow 0$) and power loss ($N_S \searrow N_c$) occur near the singularity. Initial conditions are $\psi_0 = 2.77\exp(-r^2)$ with near critical power (top), and $\psi_0 = 4\exp(-r^2)$ (bottom).*

to happen as the total beam power is conserved (2.2), the beam separates into two components as it propagates,[4]

$$\psi = \psi_s + \psi_{back},$$

where ψ_s is the high intensity inner core of the beam which self-focuses toward its center axis and ψ_{back} is the low intensity outer part which propagates forward following the usual linear propagation mode, i.e., it diffracts and slowly diverges. This "reorganization" stage takes place almost until the singularity in terms of the axial distance z (Figure 3.1) and is characterized by relatively slow focusing and fast power transfer from ψ_s to ψ_{back} (*nonadiabatic* self-focusing). Close enough to the singularity, ψ_s has only small excess power above the critical one and it approaches the radially symmetric[5] asymptotic profile (see Figure 3.2)

$$\psi_s(r,z) = \frac{1}{L(z)} V(\zeta,\rho) \exp\left[i\zeta + i\frac{L_z}{L}\frac{r^2}{4}\right], \quad \arg V(\zeta,0) = 0, \tag{3.1}$$

where $L(z)$ is a yet undetermined function that is used to rescale ψ_s and the independent variables

$$\rho = \frac{r}{L}, \quad \frac{d\zeta}{dz} = \frac{1}{L^2}. \tag{3.2}$$

Note that (3.1) can be viewed as a generalized lens transformation (2.16), in which nonlinear self-focusing is replaced by a continuum of thin lenses with a variable focal length.

[4]If $N > 2N_c$ the beam may split into several self-focusing filaments. In this case our discussion is applicable to each filament.

[5]The convergence of nonisotropic initial conditions toward a radially symmetric profile around the singularity was observed numerically in [40].

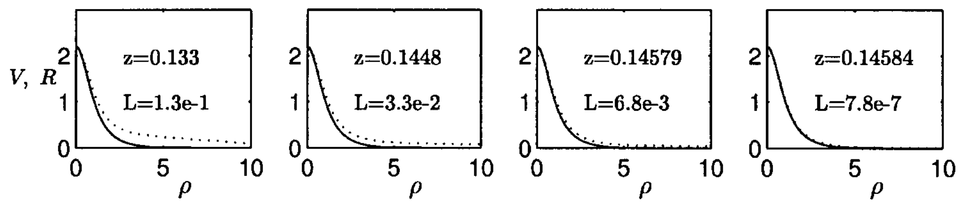

FIG. 3.2. *Convergence of the radial profile* $V(\rho) = |\psi_s(r/L)|/L$ *(dots) to the Townes soliton* $R(\rho)$ *(solid) for the initial condition* $\psi_0 = 4\exp(-r^2)$.

Because L is a measure of the radial width of ψ_s, we can use it to give a more precise definition of ψ_s and ψ_{back}. A possible definition is[6]

$$\psi = \begin{cases} \psi_s & 0 \le r \le \rho_c L(t) \\ \psi_{back} & \rho_c L(t) \le r \end{cases} \qquad \text{with } 1 \ll \rho_c \text{ constant.} \tag{3.3}$$

The resulting equation for V is

$$iV_\zeta + \Delta_\perp V - V + |V|^2 V + \frac{1}{4}\beta\rho^2 V = 0 \tag{3.4}$$

with

$$\beta(z) = -L^3 L_{zz}. \tag{3.5}$$

As the beam is focusing, $\beta \searrow 0$. In addition, we shall see that when $0 < \beta \ll 1$ its rate of change is exponentially small compared with that of the focusing (L). Therefore, if we expand V in an asymptotic series

$$V \sim V_0 + V_1 + \dots, \tag{3.6}$$

the leading order solution of (3.4) is quasisteady, i.e., $V_0 = V_0(\rho; \beta(\zeta))$. This suggests that the equation for V_0 is

$$\Delta_\perp V_0 - V_0 + |V_0|^2 V_0 + \frac{1}{4}\beta\rho^2 V_0 = 0, \quad V'(0) = 0, \quad V(\infty) = 0. \tag{3.7}$$

However, if V_0 satisfies this real equation then $V_0 \sim \rho^{-1}\cos(\sqrt{\beta}\rho^2/4)$ for $\rho \gg \beta^{-1/2}$. Since $\sqrt{\beta} \sim -LL_z$ (J.1),

$$\psi_s \sim \frac{1}{\rho}\exp\left(i\frac{\sqrt{\beta}}{2}\rho^2\right)$$

and it is not possible to match ψ_s with ψ_{back} which has no such fast oscillations.

The difficulty in resolving the asymptotics of ψ_s for large ρ was the main reason it took so long to determine the blowup rate of CNLS. Eventually, it was shown that for the leading order quasi-steady solution V_0 to have the correct behavior for large ρ, one has to add to (3.7) a term which is exponentially small in β [39, 43]:

$$\Delta_\perp V_0 - V_0 + |V_0|^2 V_0 + \frac{1}{4}\beta\rho^2 V_0 - i\frac{M}{2N_c}\nu(\beta)V_0 = 0\ , \tag{3.8}$$

[6]Recall that $|\psi_s| \sim R(\rho)/L$ has an exponential decay (2.15).

where

$$\nu(\beta) \sim \frac{2A_R^2}{M} e^{-\pi/\sqrt{\beta}} \ , \quad M = \frac{1}{4}\int_0^\infty r^3 R^2(r)\, dr \cong 0.55 \ .$$

The original asymptotics beyond all orders derivation of (3.8) is based on an analysis of (3.4) in the supercritical case $d > 2$. By defining $\Delta = \partial_{\rho\rho} + (d-1)/\rho\partial_\rho$ and allowing d to vary continuously, it is shown that for every $d > 2$ there is a positive limit $\lim_{\zeta\to\infty} \beta(\zeta) = \beta_*(d) > 0$. Taking the limit of $\beta_*(d)$ as $d \searrow 2$ leads to the $\nu(\beta)$ term. Parts of this derivation were later made rigorous in [34]. A clear presentation of this derivation is given in [70].

Once it is known that V_0 satisfies (3.8), we can proceed with regular perturbations and expand V_0 in an asymptotic series in β:

$$V_0(\rho) \sim R(\rho) + \beta g(\rho) + O(\beta^2) \ , \quad g = \left.\frac{\partial V_0}{\partial \beta}\right|_{\beta=0} \ , \quad 0 < \beta \ll 1 \ . \tag{3.9}$$

The corresponding equations for R and g are (2.11) and

$$\Delta_\perp g + 3R^2 g - g = -\frac{1}{4}\rho^2 R \ , \quad g'(0) = 0 \ , \quad g(\infty) = 0 \ . \tag{3.10}$$

The leading order equation for V_1 follows from (3.4), (3.6), (3.8), and (3.9):

$$\Delta_\perp V_1 - V_1 + 2R^2 V_1 + R^2 V_1^* = -i\beta_\zeta g - i\frac{M}{2N_c}\nu(\beta) R \ . \tag{3.11}$$

The equation for the real part of V_1 is solvable, while the solvability condition for the imaginary part of V_1 is that R is perpendicular to the right-hand side of (3.11) (Lemma F.1):

$$\int_0^\infty R\left[g\beta_\zeta + \frac{M}{2N_c}\nu(\beta) R\right] \rho d\rho = 0 \ .$$

Using (2.12) and

$$\int_0^\infty Rg\, \rho d\rho = \frac{M}{2} \tag{3.12}$$

(Lemma B.1), the solvability condition leads to the important relation

$$\beta_\zeta \sim -\nu(\beta). \tag{3.13}$$

With this relation, the goal of reducing CNLS self-focusing to a system of equations which do not depend on the transverse variables ((3.2), (3.5), and (3.13)) is achieved.

In the original derivation of (3.13) in [39, 43], this relation was written as

$$a_\zeta \sim -\frac{1}{a}\exp(-\pi/a) \ , \quad a = -LL_z = -\frac{L}{L_\zeta} \ . \tag{3.14}$$

To see that this equation agrees with (3.13) we note that $\beta = a^2 + a_\zeta$ and that $a_\zeta \ll a^2$ (3.14), so that $\beta \sim a^2$. Nevertheless, when we later extend this approach to analyze perturbed CNLS it is better to use (3.13), because with the approximation $\beta \sim a^2$

we add a constraint that $\beta > 0$, while in many cases of perturbed CNLS β becomes negative.

We remark that we use the terminology *modulation theory* to emphasize that it is based on perturbations of the (focusing part of the) solution around a modulated Townes soliton:

$$\psi_s \sim \psi_R := \frac{1}{L} R(r/L) \exp\left[i\zeta + i\frac{L_z}{L}\frac{r^2}{4}\right].$$

The delicate balance between the nonlinearity and diffraction in critical self-focusing is reflected in the above analysis by the fact that self-focusing dynamics are determined from the $O(\beta)$ *deviation* of ψ_s from ψ_R.

3.2. Adiabatic self-focusing. Malkin suggested a different way to derive (3.13) [46, 47]. Expansion of V_0 in an asymptotic series in β shows that β is proportional to the excess power above critical of the focusing part of the beam (Lemma B.2):

$$N_s - N_c \sim \beta M, \quad |\beta| \ll 1, \tag{3.15}$$

where

$$N_s := N(\psi_s)$$

is the power of the focusing part of the beam. Note that relation (3.15), as well as adiabatic theory in general, have $O(\beta)$ accuracy, since they are based on the expansion (3.9).

When β is small, the problem of finding the rate of power radiation of ψ_s can be formulated by analogy with the probability of penetration through a potential barrier and it can be solved using the WKB method (Appendix C):

$$\frac{d}{d\zeta} N_s \sim -M\nu(\beta). \tag{3.16}$$

If we combine (3.15)–(3.16), we again get (3.13).[7] Thus, the small term $\nu(\beta)$ is the rate of power radiation of ψ_s. In particular, near the focal point β is small and self-focusing is essentially *adiabatic*, that is, the beam collapses much faster than the excess power $N_s - N_c$ goes to zero.

The rate of change of the Hamiltonian of ψ_s is given by (Appendix C)

$$\frac{d}{d\zeta} H_s \sim -\frac{M}{L^2}\nu(\beta), \quad H_s := H(\psi_s). \tag{3.17}$$

From (3.15), (3.16), and (3.17) we see that as the solution approaches the blowup point,

$$\lim_{z \to Z_c} N_s = N_c \ , \quad \lim_{z \to Z_c} H_s = -\infty \ .$$

The rate at which H_s goes to infinity is given by (Appendix D):

$$H_s \sim -\frac{M}{2}\frac{\nu(\beta)}{\sqrt{\beta}}\frac{1}{L^2}.$$

These characteristics of adiabatic self-focusing can be seen in Figure 3.3.

[7] In [21] $\nu(\beta)$ was calculated using a nonlinear eigenvalue formulation. Recently, Pelinovsky suggested a derivation of the relation (3.13) from a multiple-scales argument [58].

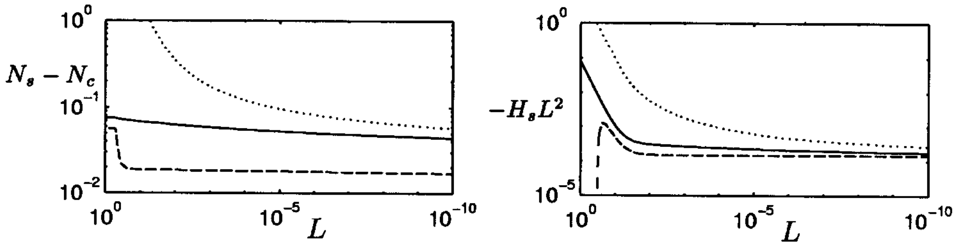

FIG. 3.3. *After an initial non-adiabatic stage with relatively fast power radiation and slow self-focusing, self-focusing become adiabatic: Changes in N_s are very slow and $H_sL^2 \sim$ constant $= o(\beta)$. Initial conditions are $\psi_0 = 1.02R(r)$ (solid line), $\psi_0 = 2.77\exp(-r^2)$ (dashed line) and $\psi_0 = 4\exp(-r^2)$ (dots).*

3.3. The loglog law. Equation (3.13) cannot be solved analytically. In order to solve it asymptotically, we rewrite it as

$$\lambda_\zeta = \frac{c_\nu}{2\pi^2}\lambda^3 \exp(-\lambda), \quad \lambda = \frac{\pi}{\sqrt{\beta}}\ , \quad c_\nu = \frac{2A_R^2}{M}\ . \tag{3.18}$$

Integration by parts of $\zeta = \int_{\lambda_0}^{\lambda} (\bar\lambda_\zeta)^{-1}\, d\bar\lambda$ shows that

$$\zeta \sim \frac{2\pi^2}{c_\nu}\frac{\exp\lambda}{\lambda^3}\ , \quad \lambda - \lambda(0) \gg 1$$

and

$$\lambda \sim \log\zeta\ , \quad \beta \sim \frac{\pi^2}{\log^2\zeta}, \tag{3.19}$$

where now c_ν has disappeared. Using (J.1), we can rewrite (3.19) as

$$A_{\zeta\zeta} - \frac{\pi^2}{\log^2\zeta}A \sim 0, \quad A = \frac{1}{L}\ .$$

The leading order solution for this equation is

$$A \sim A_0 \exp\left(\frac{\pi\zeta}{\log\zeta}\right).$$

Therefore,

$$\log\zeta \sim \log\log A \tag{3.20}$$

and

$$Z_c - z \sim \int_\zeta^\infty \frac{1}{A^2(\bar\zeta)}\, d\bar\zeta \sim \frac{\log\zeta}{2\pi A^2}\ . \tag{3.21}$$

Combining (3.20) and (3.21) gives

$$\log\zeta \sim \log\log\frac{1}{Z_c - z}\ , \tag{3.22}$$

which together with (3.21) results in the loglog law

$$L \sim \left(\frac{2\pi(Z_c - z)}{\ln\ln 1/(Z_c - z)} \right)^{\frac{1}{2}} . \tag{3.23}$$

Although mathematically correct, it turns out that the loglog law becomes applicable only for huge and nonphysical amplifications. This is because (3.19) becomes the leading order solution of (3.18) only at huge focusing factors. To see why this is true, we note that (3.19) holds when $\lambda - \lambda_0 \gg 1$. However, from (3.18) and $\lambda - \lambda_0 < \zeta\lambda_\zeta(0)$ we see that a necessary, but clearly not sufficient, condition for the loglog law to hold is that

$$\zeta \gg \frac{\beta_0^{3/2}}{\nu(\beta_0)} . \tag{3.24}$$

3.4. Adiabatic analysis of modulation equations. In order to derive an asymptotic law for critical self-focusing which is valid in the domain of physical interest, we note that (3.5) and (3.13) which govern self-focusing evolve on very different length scales:

$$L_{zz} = -\frac{\beta}{L^3} \qquad \text{small scale,} \tag{3.25}$$

$$\beta_z = -\frac{\nu(\beta)}{L^2} \qquad \text{large scale.} \tag{3.26}$$

The loglog law is derived by solving (3.26) to leading order and then using (3.25). However, since the length scale for power changes in (3.26) is exponentially long compared with the one for changes in the focusing rate in (3.25), we should do just the opposite: First integrate (3.25) while ignoring the slow changes in β (*strictly adiabatic* self-focusing) and only then use (3.26) in order to get the next order correction [24]. Therefore, strictly adiabatic self-focusing is given by

$$L_{zz} = -\frac{\beta}{L^3}, \quad \beta \equiv \beta_0 := \beta(0). \tag{3.27}$$

If we multiply (3.27) by $2L_z$ and integrate, we get

$$L_z^2 = \frac{\beta}{L^2} + C_0, \quad C_0 := C(0), \quad C(z) := L_z^2 - \frac{\beta}{L^2} = (L^2)_{zz}. \tag{3.28}$$

Multiplying (3.28) by L^2 gives

$$(L^2)_z = \pm 2 \left(\beta + C_0 L^2\right)^{1/2},$$

where the plus/minus sign corresponds to the cases of initial defocusing/focusing at $z = 0$, respectively. Integrating one more time and using the initial condition $L(0) = L_0$ gives the corrected version of adiabatic law of Fibich, first obtained in [24]:

$$L^2(z) \sim L_0^2 + (L^2)_z(0)\, z + C_0 z^2 . \tag{3.29}$$

Note that (3.29) implies that

$$(L^2)_{zz} \equiv (L^2)_{zz}(0) . \tag{3.30}$$

If we set $L(Z_c) = 0$ in (3.29) we get a quadratic equation for the blowup point Z_c whose smaller positive solution is [24]

$$Z_c \sim \begin{cases} \dfrac{L_0^2}{\sqrt{\beta} - L_0 L_z(0)} & L_0 L_z(0) \leq \beta, \\ \\ \text{no blowup} & L_0 L_z(0) > \beta \ . \end{cases} \tag{3.31}$$

It is instructive to compare (3.31) with the necessary and sufficient conditions for blowup (i.e., conditions (2.5)–(2.6)). Equation (3.31) shows that the condition $\beta > 0$ (i.e., power above critical) is necessary for blowup. This condition is also sufficient when $L_z(0) \leq 0$. However, if the beam is initially defocusing, the necessary and sufficient condition for blowup is $\beta \geq L_0 L_z(0)$.

The expression (3.31) for Z_c inherits the lens transformation property (2.28). To see this, let us consider the case of a collimated beam (ψ_0 real). Since in this case $L_z(0) = 0$, the strict adiabatic law for ψ_0 real is

$$L \sim L_0 \sqrt{1 - \frac{z^2}{Z_c^2}} \ , \quad Z_c = \frac{L_0^2}{\sqrt{\beta_0}} \ . \tag{3.32}$$

If we add a lens with focal length F at $z = 0$ the initial condition becomes (2.18). Since this change does not affect the beam radius and power at $z = 0+$, $\tilde{L}_0 = L_0$ and $\tilde{\beta}_0 \sim \beta_0$ (the tildes denote the corresponding parameters for $\tilde{\psi}$). However,

$$\tilde{L}_z(0) = L_0/F.$$

Therefore, from (3.31) we see that the blowup point for $\tilde{\psi}$ is at

$$\tilde{Z}_c = \frac{L_0^2}{\sqrt{\beta_0} - L_0^2/F} \ ,$$

which is related to Z_c by

$$\frac{1}{\tilde{Z}_c} = \frac{1}{Z_c} - \frac{1}{F} \ ,$$

showing that the adiabatic law (3.29) preserves the lens transformation property of CNLS.

3.5. Nonadiabatic effects. Self-focusing, as given by (3.27) or by (3.32), is strictly adiabatic, i.e., radiation losses are completely neglected. Therefore, if we are interested in maintaining the $O(\beta)$ accuracy of the adiabatic law up to the blowup point, the slow scale changes in β and $C(z)$ must be included. This can be done by solving the fast equation (3.25) coupled with the slow equation (3.26), as in Figure 3.4.

It may seem that we can get a more accurate asymptotic law than the strictly adiabatic one if we replace (3.27) with its Euler approximation,

$$L_{zz} = -\frac{\beta}{L^3} \ , \quad \beta = \beta(0) - \zeta\nu(\beta(0)) \ , \quad \frac{d\zeta}{dz} = \frac{1}{L^2} \ . \tag{3.33}$$

However, this approximation is better than (3.27) only during the initial stage of self-focusing and eventually becomes worse than the strict adiabatic approximation (Figure 3.4(B)).

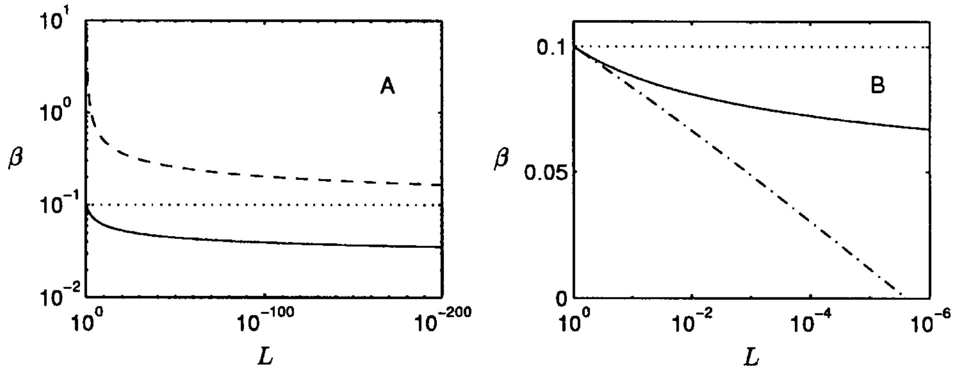

FIG. 3.4. A: *Strict adiabaticity* (3.27; dotted line) *is a better approximation to the evolution of* β *according to the reduced system* (3.25–3.26; solid line) *than the asymptotic approximation* (3.19; dashed line) *that leads to the loglog law, even after amplification by* 200 *orders of magnitude.* B: *Except for the initial stage of self-focusing, strict adiabaticity (dotted line) is a better approximation for the evolution of* β (3.25–3.26; solid line) *than the Euler 'improvement'* (3.33; dash-dot line). *In all cases* $\beta(0) = 0.1\ ,\quad L(0) = 1$.

3.6. Comparison of Fibich's adiabatic law, Malkin's adiabatic law, and the loglog law. The adiabatic law (3.29) can be rewritten in the form

$$L(z) = \sqrt{2\sqrt{\beta}\,(Z_c - z) + C(z)\,(Z_c - z)^2}\ . \tag{3.34}$$

As z approaches the singularity point, the quadratic term becomes negligible (see Appendix E) and (3.34) reduces to Malkin's adiabatic law [47]:

$$L(z) = \sqrt{2\sqrt{\beta}(Z_c - z)}\ . \tag{3.35}$$

Thus, (3.34) and (3.35) agree asymptotically but (3.34) is valid earlier, since in addition to the beam power it also incorporates the focusing angle. Similarly, the asymptotic limit of (3.35) agrees with the loglog law. To see this, note that if in the derivation of the loglog law we use (3.19) instead of (3.22) in (3.21), we get (3.35). Therefore, the three laws are asymptotically equivalent; only their domains of validity differ.

In Figure 3.5 we compare the value of L from numerical simulations of CNLS (solved by the method of dynamic rescaling; see section 6) with the predictions of the three asymptotic laws. The initial conditions used are $\psi_0 = 1.02R(r)$ (power slightly above critical and close to the asymptotic profile) and $\psi_0 = 4\exp(-r^2)$ (large excess power above critical). In both cases, the adiabatic laws become $O(\beta)$ accurate early on and maintain this accuracy, while the loglog law is not valid even after focusing by more than 10 orders of magnitude. The advantage of Fibich's law over Malkin's law during the initial stage can be seen in Figure 3.5(A) where the initial condition is close to the asymptotic one. In Figure 3.5(B) the initial conditions are not close to the asymptotic profile and the two adiabatic laws take longer to become valid, at which point they are already in the domain where they agree.

In order to understand why the adiabatic laws become valid quite early and the loglog law does not, we take a closer look at the point where their derivations become different. For the adiabatic laws to be applicable, β should be moderately small so that $\nu(\beta) \ll 1$. In contrast to this, a necessary condition for the loglog law to be valid is (3.24). To estimate the corresponding beam width, we apply the adiabatic

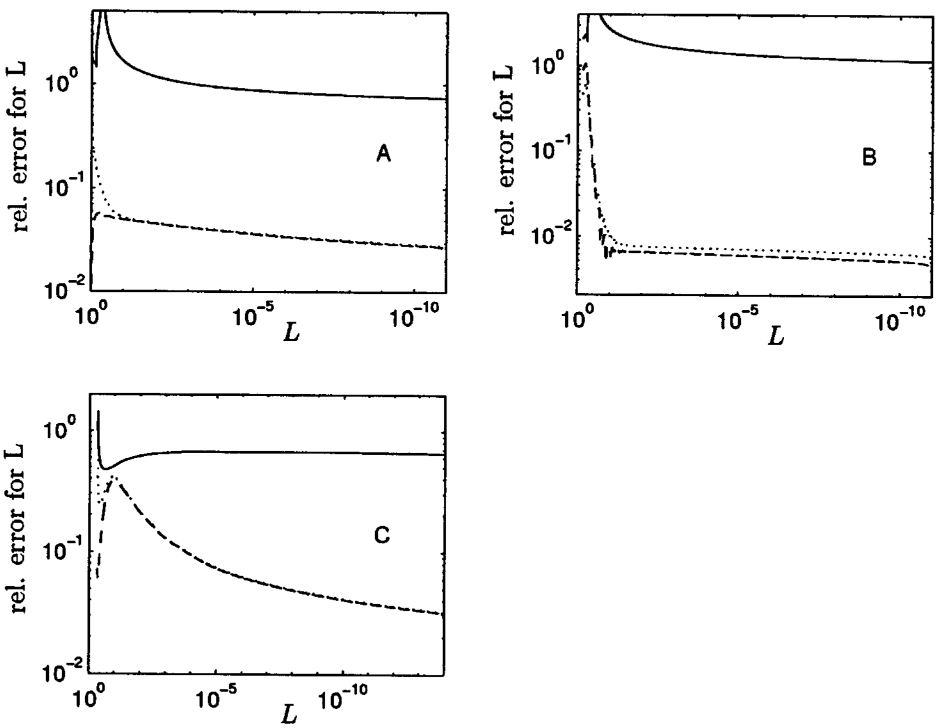

FIG. 3.5. *The relative error in the prediction for L of the adiabatic laws of Fibich* (3.34; dashed line) *and Malkin* (3.35; dotted line) *and of the loglog law* (3.23; solid line) *for the initial conditions* (A) $\psi_0 = 1.02R(r)$ (B) $\psi_0 = 4\exp(-r^2)$.

approximation to $L_\zeta/L \sim -\sqrt{\beta}$ to get $L \sim \exp(-\sqrt{\beta}\zeta)$. Therefore, a necessary condition for the loglog law to hold is

$$L \ll \exp\left(-\frac{\beta_0^2}{\nu(\beta_0)}\right),$$

which shows that for $\beta(0) = 0.1$ the loglog law is not valid even when $L \sim 10^{-90}$. Indeed, in Figure 3.4(A) it can be seen that when $\beta(0) = 0.1$, the approximation (3.19) which is used to derive the loglog law does not become valid even after amplification by 200 orders of magnitude.

As noted above, the adiabatic laws of Fibich and Malkin agree asymptotically near the singularity. However, since Malkin's law is less accurate during the initial stage of focusing, the prediction for Z_c derived from (3.35),

$$Z_c = \frac{L_0^2}{2\sqrt{\beta_0}} \tag{3.36}$$

is less accurate than (3.31). For example, for real initial conditions ($L_z = 0$) the prediction (3.36) is off by a factor of 2 compared with (3.32), which is (3.31) for real initial conditions. In addition, the estimate (3.36) does not satisfy the lens relation (2.28), since it is independent of the initial focusing angle.

In Figure 3.6 we compare the dynamic predictions for the distance from the

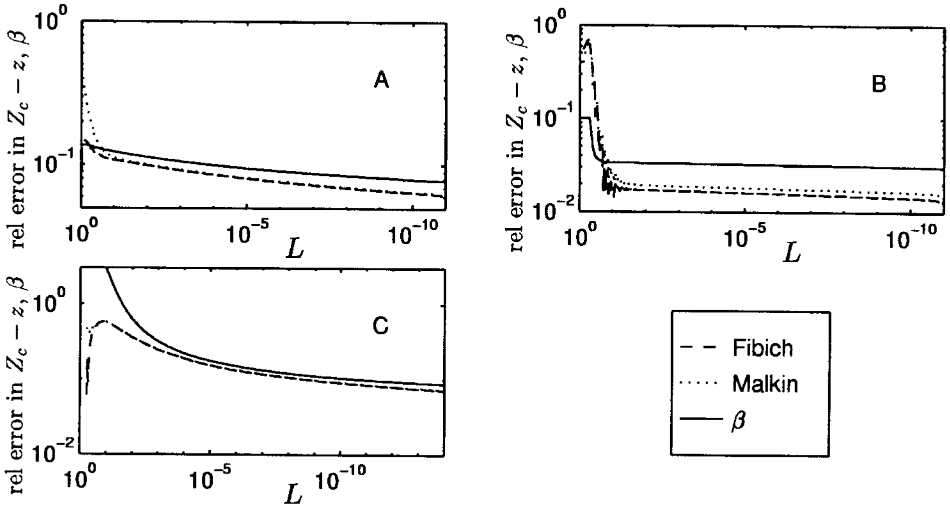

FIG. 3.6. *The relative accuracy of the dynamic prediction for the location of the singularity* $(Z_c - z)$ *based on Fibich's adiabatic law* (3.37; dashed line) *and on Malkin's adiabatic law* (3.38; dotted line). *Initial conditions are as in Figure* 3.5.

singularity of (3.31)

$$Z_c - z = \frac{L^2(z)}{\sqrt{\beta(z)} + a(z)} , \quad a = -LL_z \tag{3.37}$$

and of (3.36)

$$Z_c - z = \frac{L^2(z)}{2\sqrt{\beta(z)}} . \tag{3.38}$$

In the adiabatic regime both predictions have $O(\beta)$ relative accuracy. The advantage of (3.37) during the initial stages is again seen for the initial condition $1.02R$ (Figure 3.6(A)). In addition, only (3.37) will maintain the same relative accuracy for all z if we add a focusing quadratic phase factor to the initial condition.

3.7. Location of the singularity. Equation (3.32) for the location of the singularity was derived under the assumptions that ψ_s is close to the asymptotic form (3.1) and that the excess power above critical is small ($\beta \ll 1$). However, it is desirable to have the value of Z_c for any given power, focusing angle, and radial distribution. To do that, we extrapolate (3.32) outside its stated domain of validity by estimating the value of β from (3.15):

$$\beta \sim \frac{N_c}{M}(p-1) , \quad p = \frac{N}{N_c},$$

even when β is not small. The value of L_0 is determined by looking for the modulated Townes soliton which best approximates the initial radial distribution:

$$|\psi_0(r)| \sim R_{L_0} , \quad R_{L_0} = \frac{1}{L_0} R\left(\frac{r}{L_0}\right) .$$

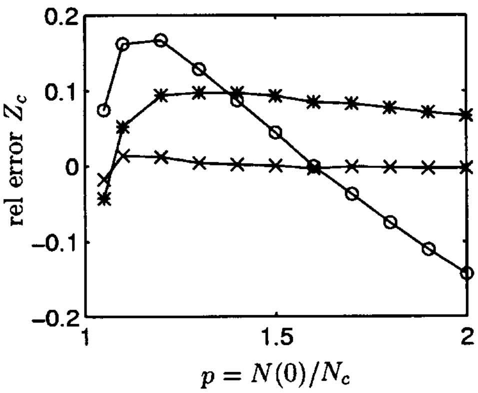

FIG. 3.7. *The relative accuracy of the predictions for the location of the singularity for Gaussian initial conditions of the formula of Dawes and Marburger* (3.41, "*") *and of the theoretical adiabatic formula* (3.42, "o") *is around* 10%. *The new empirical formula* (3.43, "x") *has a* 1% *relative accuracy.*

One possibility for matching is to set $|\psi_0(0)| = R_{L_0}(0)$, which gives $L_0 = R(0)/|\psi_0(0)|$. However, our simulations suggest that better results are obtained if matching is done by setting $\int |\nabla_\perp \psi_0|^2 = \int |\nabla_\perp R_{L_0}|^2$. From this condition and (2.14) we get

$$L_0 = N_c^{1/2} \left(\int_0^\infty |\nabla_\perp \psi_0|^2 \, rdr \right)^{-1/2} \tag{3.39}$$

and

$$Z_c \sim \sqrt{\frac{MN_c}{p-1}} \left(\int_0^\infty |\nabla_\perp \psi_0|^2 \, rdr \right)^{-1} . \tag{3.40}$$

Note that (3.40) takes into account beam power, radial distribution, and initial focusing angle. The validity of (3.40) for various initial conditions is shown in [24].

3.7.1. Gaussian initial conditions. The only available formula for the location of the singularity, with reasonable accuracy, is that of Dawes and Marburger[8] [20],

$$Z_c = 0.184[(p^{1/2} - 0.852)^2 - 0.0219]^{-1/2} . \tag{3.41}$$

This formula was derived for the special case of Gaussian initial conditions $\psi_0 = c\exp(-r^2)$ by curve-fitting values of Z_c obtained from simulations. For comparison, in the case of Gaussian initial conditions $L_0 \sim \sqrt{1/2p}$ and the theoretical formula, (3.40) becomes

$$Z_c \sim \frac{1}{2p} \sqrt{\frac{M}{N_c}} \sqrt{\frac{1}{p-1}} . \tag{3.42}$$

Both (3.41) and (3.42) have a relative accuracy of around 10% in the range $1.05 \le p \le 2$ (Figure 3.7). Based on our numerical simulations we suggest a new empirical formula for Gaussian initial conditions,

$$Z_c = 0.1585 * (p-1)^{-0.6346}, \tag{3.43}$$

which has a relative accuracy of 1% in this range (Figure 3.7).

[8]The value of Z_c here is half of the one given in [20], since the initial condition in [20] is $\psi_0 = c\exp(-r^2/2)$.

4. Modulation theory for self-focusing in the perturbed CNLS. In the previous sections we saw that self-focusing in critical NLS is controlled by the delicate balance between the focusing nonlinearity and defocusing Laplacian. As a result, if a small perturbation is added to CNLS it will have a large effect on self-focusing as soon as it becomes comparable to $(\Delta_\perp\psi + |\psi|^2\psi)$, even though it is small compared with each of these terms separately. This property is unique to critical focusing, which is the borderline case between subcritical self-focusing where diffraction dominates and supercritical self-focusing where nonlinear focusing dominates. Indeed, if the solution of the focusing NLS (2.1) is self-similar, i.e., $\psi \sim V(r/L)/L$, then $\Delta_\perp \sim L^{-3}$ and $|\psi|^{2\sigma}\psi \sim L^{-1-2\sigma}$. Therefore, only when $\sigma = 1$ can nonlinearity and diffraction remain of the same order as $L \searrow 0$. In fact, diffraction and critical nonlinearity exactly balance each other in the special case of the waveguide solution (2.22), where $V = R$ and $\beta \equiv 0$. Therefore, in critical self-focusing, given by (3.1) with $V \sim R$ and $0 < \beta \ll 1$, diffraction and critical nonlinearity almost completely balance each other.

4.1. Modulation theory. We have seen that the adiabatic approach is very effective in the analysis of self-focusing in CNLS. In this section we extend this approach to a modulation theory for analyzing the effects of various small perturbations on self-focusing. We consider a general perturbed critical NLS of the form

$$i\psi_z + \Delta_\perp\psi + |\psi|^2\psi + \epsilon F(\psi, \psi_z, \nabla_\perp\psi, \psi_t, \ \ldots) = 0 \ , \quad |\epsilon| \ll 1 \ , \tag{4.1}$$

where F is an even function in x and y. Using modulation theory, the perturbed CNLS (4.1) is replaced with a system of reduced equations which is much simpler for analysis and simulations because it is independent of the transverse variables. For example, in section 5 we apply modulation theory to the perturbations of CNLS listed in Table 1.1.

Modulation theory is valid when the following three conditions hold.

Condition 1. The focusing part of the solution is close to the asymptotic profile (3.1)–(3.2),

$$\psi_s(z, x, y, \cdot) \sim \frac{1}{L(z,\cdot)} V(\zeta, \xi, \eta, \cdot) \exp\left[i\zeta(z,\cdot) + i\frac{L_z}{L}\frac{r^2}{4}\right], \tag{4.2}$$

where

$$\xi = \frac{x}{L} \ , \quad \eta = \frac{y}{L} \ , \quad \zeta_z = \frac{1}{L^2}$$

and $V = R + O(\beta, \epsilon)$.

Condition 2. The power is close to critical

$$\left| \frac{1}{2\pi} \int |\psi_s(z, x, y, \cdot)|^2 \, dxdy - N_c \right| \ll 1$$

or, equivalently,

$$|\beta(z,\cdot)| \ll 1 \ .$$

Condition 3. The perturbation ϵF is small compared with the other terms in (4.1):

$$|\epsilon F| \ll |\Delta_\perp\psi| \ , \quad |\epsilon F| \ll |\psi|^3 \ .$$

The dots in the arguments of ψ and of the modulation parameters indicate that they may depend on additional variables, such as t in the case of time-dispersion.

In general, at the onset of self-focusing only Condition 3 holds. Therefore, if the power is above critical the solution will initially self-focus as in the unperturbed CNLS. As a result, near the location of the blowup point in the absence of the perturbation, Conditions 1–2 will also be satisfied. It is only at this stage that the Laplacian and the nonlinearity almost completely balance each other, so that the small perturbation can have a significant effect. Therefore, one can identify at least three stages in the evolution of self-focusing in the perturbed CNLS:

- *Nonadiabatic self-focusing.* Self-focusing is as in the nonadiabatic stage of the unperturbed CNLS. Only Condition 1 holds.
- *Unperturbed adiabatic self-focusing.* Self-focusing is as in the adiabatic stage of the unperturbed CNLS. Conditions 1–3 hold.
- *Perturbed adiabatic self-focusing.* The perturbation is small but has a significant effect. Conditions 1–3 hold.

Note that Conditions 1–3 hold in the second and third stages, both of which are therefore covered by modulation theory. In some cases (e.g., nonparaxiality, saturating nonlinearity) one can show that the reduced system remains valid for all z by showing that all three conditions remain satisfied in the reduced system. However, in other cases (e.g., small normal time-dispersion) it is unclear for how long modulation theory remains valid, and self-focusing may enter a new stage which is not covered by modulation theory.

The main result of modulation theory is the following proposition.

PROPOSITION 4.1. *If Conditions* 1–3 *hold, self-focusing in the perturbed CNLS* (4.1) *is given to leading order by the reduced system*

$$\beta_z + \frac{\nu(\beta)}{L^2} = \frac{\epsilon}{2M}(f_1)_z - \frac{2\epsilon}{M} f_2 \ , \quad L_{zz} = -\frac{\beta}{L^3} \ . \tag{4.3}$$

The auxiliary functions f_1 *and* f_2 *are given by*

$$f_1(z,\cdot) = 2L(z,\cdot) Re \left[\frac{1}{2\pi} \int F(\psi_R) \exp(-iS) [R(\rho) + \rho \nabla_\perp R(\rho)] \, dxdy \right], \tag{4.4}$$

$$f_2(z,\cdot) = Im \left[\frac{1}{2\pi} \int \psi_R^* F(\psi_R) \, dxdy \right], \tag{4.5}$$

where

$$\psi_R = \frac{1}{L} R(\rho) \exp(iS), \quad \rho = \frac{r}{L} \ , \quad S = \zeta(z,\cdot) + \frac{L_z}{L}\frac{r^2}{4} \ , \quad \frac{\partial \zeta}{\partial z} = \frac{1}{L^2}. \tag{4.6}$$

We note the following:

- Assuming that we can carry out the transverse integration, f_1 and f_2 are known functions of the modulation variables L, β, ζ, and their derivatives.
- The reduced system (4.3) is much easier for analysis and simulations than (4.1) because it does not depend on the transverse variables (x, y).

A proof of Proposition 4.1 is postponed until section 4.2.

4.1.1. Conservative and nonconservative perturbations. Considerable simplification can be achieved by distinguishing between conservative perturbations, i.e., those for which the power remains conserved in (4.1),

$$\frac{d}{dz} \int |\psi(z,x,y,\cdot)|^2 \, dxdy \equiv 0$$

and nonconservative perturbations.

PROPOSITION 4.2. *Let Conditions* 1–3 *hold.*

1. *If F is a conservative perturbation, i.e.,*

$$Im \int \psi^* F(\psi)\, dxdy \equiv 0 \ ,$$

then $f_2 \equiv 0$, then to leading order (4.3) *reduces to*

$$-L^3 L_{zz} = \beta_0 + \frac{\epsilon}{2M} f_1 \ , \quad \beta_0 = \beta(0,\cdot) - \frac{\epsilon}{2M} f_1(0,\cdot) \ , \tag{4.7}$$

where β_0 is independent of z.

2. *If F is a nonconservative perturbation, i.e.,*

$$Im \int \psi^* F(\psi)\, dxdy \not\equiv 0,$$

then to leading order (4.3) *reduces to*

$$\beta_z = -\frac{2\epsilon}{M} f_2 \ , \quad L_{zz} = -\frac{\beta}{L^3} \ . \tag{4.8}$$

Note that in both cases, nonadiabatic effects disappear from the leading order behavior of (4.3). The proof of Proposition 4.2 is given in Appendix G.

A useful relation which is derived in the proof of Proposition 4.1 in section 4.2.2 is that the power of the focusing part of the beam is given by

$$N_s \sim N_c + \beta M - \frac{\epsilon}{2} f_1 \ .$$

Therefore, in the case of a "purely nonconservative" perturbation (i.e., $f_1 \equiv 0$), relation (3.15) and the interpretation of β as the excess power above critical still hold. Similarly, the Hamiltonian of ψ_s is given by[9] (H.5),

$$H_s \sim \frac{M}{2}(L^2)_{zz} + \frac{\epsilon f_1}{2L^2} \ .$$

4.1.2. Generic effect of conservative perturbations. As we shall see in section 5, for various conservative perturbations f_1 turns out to have the generic form

$$f_1 \sim -\frac{C_1}{L^2}, \ C_1 = \text{constant} \ . \tag{4.9}$$

The following two propositions cover this canonical case. The first deals only with adiabatic effects and the second deals with nonadiabatic effects when the conservative perturbation results in oscillatory focusing-defocusing behavior.

PROPOSITION 4.3. *When self-focusing is given by* (4.7) *and f_1 is given by* (4.9)*, then*

$$y := L^2$$

[9]This is true except when $(f_1)_z = f_2 \equiv 0$, as in the case of the Davey–Stewartson equations (section 5.4).

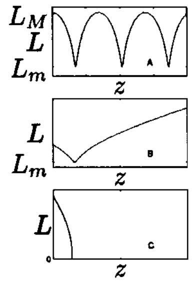

FIG. 4.1. *The leading-order effect of the generic conservative perturbation* (4.9). (A) *Defocusing perturbation and* $H_0 < 0$ (*Proposition* 4.3 part 1(a)(i))*;* (B) *defocusing perturbation,* $H_0 > 0$ *and* $L_z(0) < 0$ (*Proposition* 4.3 part 1(b)(i))*;* (C) *focusing perturbation and* $L_z(0) < 0$ (*Proposition* 4.3 part 2)*. In all cases* $\beta_0 > 0$ *(i.e., power above critical).*

satisfies the generic oscillator equation

$$(y_z)^2 = 4\beta_0 - \frac{\epsilon C_1}{M}\frac{1}{y} + \frac{4H_0}{M}y \tag{4.10}$$

or, equivalently,

$$(y_z)^2 = \frac{-4H_0}{M}\frac{1}{y}(y_M - y)(y - y_m), \tag{4.11}$$

where

$$y_M = \frac{\sqrt{\beta_0^2 + \epsilon C_1 H_0/M^2} + \beta_0}{-2H_0/M} = \frac{M\beta_0}{-H_0}\left[1 + O\left(\frac{\epsilon H_0}{\beta_0^2}\right)\right], \tag{4.12}$$

$$y_m = \frac{\epsilon C_1}{2M}\frac{1}{\sqrt{\beta_0^2 + \epsilon C_1 H_0/M^2} + \beta_0} = \frac{\epsilon C_1}{4M\beta_0}\left[1 + O\left(\frac{\epsilon H_0}{\beta_0^2}\right)\right], \tag{4.13}$$

$$\beta_0 = \beta(0) + \frac{\epsilon C_1}{2ML^2(0)}\ , \quad H_0 \sim H(0) + \frac{\epsilon C_1}{4}\frac{1}{L^4(0)}\ .$$

Let us define

$$L_m := y_m^{1/2}\ , \quad L_M := y_M^{1/2}\ .$$

1. *If the perturbation is* defocusing, *i.e.,*

$$\epsilon C_1 > 0\ , \tag{4.14}$$

then it will arrest blowup in (4.7)*, i.e.,* L *remains positive for all* z*.*

(a) *If in addition to* (4.14)*,* $\beta_0 > 0$ *and* $H_0 < 0$*, then*

$$0 < L_m < L_M$$

and L *goes through periodic oscillations between* L_m *and* L_M *(Figure* 4.1(A)*). The period of the oscillations is*

$$\Delta Z = 2\sqrt{\frac{My_M}{-H_0}}\,E\left(1 - \frac{y_m}{y_M}\right), \tag{4.15}$$

where $E(m) = \int_0^{\pi/2}(1 - m\sin^2\theta)^{1/2}\,d\theta$ *is the complete elliptic integral of the second kind* [2].

(b) *If in addition to* (4.14)*,* $\beta_0 > 0$ *and* $H_0 > 0$*, then*

(i) *if* $L_z(0) < 0$*, self-focusing is arrested when* $L = L_m > 0$*, after which* L *is monotonically defocusing to infinity (Figure* 4.1(B)*).*

(ii) *if* $L_z(0) > 0$*,* L *is monotonically defocusing to infinity.*

2. *If the perturbation is* focusing*, i.e.,*

$$\epsilon C_1 < 0,$$

and if in addition $\beta_0 > 0$ *and one of two conditions holds—*(1) $H_0 > 0$ *and* $L_z(0) < 0$ *or* (2) $H_0 < 0$*—then the solution of* (4.7) *will blow up in a finite distance (Figure* 4.1(C)*), i.e.,*

$$\text{there exists } Z_* \quad \text{such that} \quad 0 < Z_* < \infty \quad \text{and} \quad L(Z_*) = 0.$$

3. *The location* z_0 *of the (first) arrest in Proposition* 4.3 *parts* 1(a) *and* 1(b) (i) *is almost the same as that of the singularity in the unperturbed case with the same initial conditions:*

$$z_0 = \int_{y(0)}^{y_m} z_y \, dy \sim Z_c \ , \qquad Z_c \text{ given by } (3.31) \ .$$

In particular, if ψ_0 *is real, then*

$$z_0 = \frac{1}{2}\Delta Z = Z_c \left(1 + O\left(\frac{\epsilon H_0}{\beta_0^2}\right)\right), \quad Z_c \text{ given by } (3.32) \ .$$

The proof of Proposition 4.3 is given in Appendix H.

4.1.3. Nonadiabatic effects. Proposition 4.2 shows that the exponentially small term $\nu(\beta)$, which plays such an important role in CNLS self-focusing, disappears from the leading-order behavior of perturbed CNLS. (In the nonconservative case the effect of $\nu(\beta)$ is even smaller than the $(f_1)_z$ term, which is also ignored.) Nevertheless, if the leading-order effect of the perturbation according to Proposition 4.2 results in periodic focusing-defocusing oscillations, $\nu(\beta)$ may provide the only mechanism for the decay of the oscillations. In that case, if the perturbation is conservative, in order to account for the nonadiabatic effects we should use

$$\beta_z + \frac{\nu(\beta)}{L^2} = \frac{\epsilon}{2M}(f_1)_z, \quad L_{zz} = -\frac{\beta}{L^3}. \tag{4.16}$$

If in (4.16) the power loss during one oscillation is small, then the oscillations are slowly decreasing and the effect of $\nu(\beta)$ can be lumped into the change in N_s over one period:

$$\Delta N_s := N_s(z + \Delta Z) - N_s(z) \sim -M \int_z^{z+\Delta Z} \frac{\nu(\beta)}{L^2} \, dz.$$

In the conservative case when f_1 is given by (4.9), $\beta_0 > 0$ and $H_0 < 0$ (Proposition 4.3 part 1(a)), the following proposition provides an estimate for ΔN_s. (A detailed analysis of nonadiabatic effects is given in [47].)

PROPOSITION 4.4. *If nonadiabatic effects are included in the case of Proposition* 4.3 part 1(a) *and if* $|\Delta N_s| \ll N_s - N_c$*, the oscillations are slowly decreasing and* *after each cycle there is an overall power loss due to radiation of* ΔN_s*. Most radiation occurs when* $y \sim y_M$ *and*

$$\Delta N_s \sim -M\nu(\beta_M)\beta_M^{-1/4}\left(\frac{y_M}{y_m}\right)^{1/2}, \tag{4.17}$$

where

$$\beta_M := \beta(y = y_M) = -\frac{H_0}{M}(y_M - y_m) .$$

For a proof of Proposition 4.4 see Appendix I. Note that nonadiabatic effects lead to slowly decaying focusing-defocusing oscillations only when the change in N_s over one oscillation is small compared with the excess power above critical ($|\Delta N_s| \ll N_s - N_c$). From Proposition 4.4 we see that this holds when ϵ is moderately small, but not for a very small ϵ since $\Delta N_s \sim \epsilon^{-1/2}$ as $\epsilon \to 0$.

4.1.4. Modulation theory for multiple perturbations. In some cases, one is interested in the combined effect of several small perturbations, e.g., randomness and quintic nonlinearity (section 5.6) or time-dispersion and nonparaxiality (section 5.8). Modulation theory can easily handle these cases, since the modulation equations are linear in F. Therefore, one simply adds the contribution of each perturbation to the modulation equations.

4.2. Proof of Proposition 4.1. In this section we derive the reduced equation (4.3) of Proposition 4.1. This derivation generalizes the one for CNLS (section 3.1).

4.2.1. Perturbation analysis. When Condition 1 is satisfied, the focusing part of the solution is described by (4.2) and the corresponding equation for V is

$$iV_\zeta + \Delta_\perp V - V + |V|^2 V + \frac{1}{4}\beta\rho^2 V + \epsilon L^3 F\left(\frac{V(\rho,\zeta)}{L(z)}\exp(iS)\right)\exp(-iS) = 0. \tag{4.18}$$

As in the case of CNLS, we expand V asymptotically for β and ϵ small

$$V \sim V_0^\epsilon + V_1^\epsilon + \cdots, \tag{4.19}$$

where V_0^ϵ is quasi-steady in z, satisfying

$$\Delta_\perp V_0^\epsilon - V_0^\epsilon + |V_0^\epsilon|^2 V_0^\epsilon + \frac{1}{4}\beta\rho^2 V_0^\epsilon - i\frac{M}{2N_c}\nu^\epsilon(\beta)V_0^\epsilon + \epsilon w(V_0^\epsilon) = 0, \tag{4.20}$$
$$w(V_0^\epsilon) := L^3 \mathrm{Re}\left[F\left(\frac{V_0^\epsilon(\rho)}{L(z)}\exp(iS)\right)\exp(-iS)\right].$$

When $\epsilon = 0$ this is (3.8), which determines V_0 and $\nu(\beta)$. We have now added the *dispersive* part of the perturbation and we assume that there is a perturbed pair V_0^ϵ and $\nu^\epsilon \sim \nu$ that satisfies (4.20). If we expand V_0^ϵ in the two small parameters ϵ and β we have

$$V_0^\epsilon \sim R(\rho) + \beta g(\rho) + \epsilon h(\zeta,\xi,\eta) + o(\beta,\epsilon). \tag{4.21}$$

The equations for R and g are (2.11) and (3.10) and the equation for h is

$$\Delta_\perp h + 3R^2 h - h = -w(R), \quad (\partial_\xi, \partial_\eta)\, h(\zeta,0,0) = 0, \quad h(\zeta,\rho=\infty) = 0. \tag{4.22}$$

Integration by parts shows that (Lemma A.1)

$$\frac{1}{2\pi}\int Rh\,d\xi d\eta = -\frac{1}{4}f_1. \tag{4.23}$$

Equation (4.3) is obtained from the solvability condition for the next-order term V_1^ϵ. The equation for V_1^ϵ is (4.18)–(4.20)

$$\Delta_\perp V_1^\epsilon - V_1^\epsilon + 2|V_0^\epsilon|^2 V_1^\epsilon + (V_0^\epsilon)^2 (V_1^\epsilon)^* + \frac{1}{4}\beta\rho^2 V_1^\epsilon$$
$$= -i\left[(V_0^\epsilon)_\zeta + \frac{M}{2N_c}\nu^\epsilon(\beta)V_0^\epsilon\right] - i\epsilon L^3 \mathrm{Im}\left[F(\psi)\exp(-iS)\right].$$

Using (4.21), to principal order in β and ϵ this equation reduces to

$$\Delta_\perp V_1 - V_1 + 2R^2 V_1 + R^2 V_1^* = -i\left[g\beta_\zeta + \epsilon h_\zeta + \frac{M}{2N_c}\nu(\beta)R\right] - i\epsilon L^3 \mathrm{Im}\left[F(\psi_R)\exp(-iS)\right]. \tag{4.24}$$

From the solvability theory of Appendix F, the equation for the real part of V_1 is always solvable when h is even, and the solvability condition for the imaginary part of V_1 is that R is perpendicular to the imaginary part of the right-hand side of (4.24) (Lemma F.1):

$$\int R\left[g\beta_\zeta + \epsilon h_\zeta + \frac{M}{2N_c}\nu(\beta)R + \epsilon L^3 \mathrm{Im}\left[F(\psi_R)\exp(-iS)\right]\right] d\xi d\eta = 0.$$

Using (3.2), (3.12), and (4.23), we see that this relation is (4.3).

4.2.2. Derivation of the reduced equation (4.3) from balance of power. As in the case of CNLS (section 3.2), we can also derive the reduced equation (4.3) from balance of power. To do that, we multiply (4.1) by ψ^*, subtract the conjugate equation, and integrate over the transverse variables to get an equation for the balance of power in (4.1):

$$\frac{\partial}{\partial z}\int |\psi|^2\,dxdy = -2\epsilon\,\mathrm{Im}\int \psi^* F(\psi)\,dxdy. \tag{4.25}$$

The left-hand side has two components, the focusing part ψ_s and the nonfocusing one (3.3):

$$\int |\psi|^2 = \int |\psi_s|^2 + \int |\psi_{back}|^2.$$

The focusing part can can be approximated using

$$\frac{1}{2\pi}\int |\psi_s|^2\,dxdy \sim \frac{1}{2\pi}\int_{0\le\rho\le\rho_c} |V_0|^2\,d\xi d\eta$$
$$= \int_0^\infty R^2\,\rho d\rho + 2\beta\int_0^\infty Rg\,\rho d\rho + \frac{\epsilon}{\pi}\int Rh\,d\xi d\eta + o(\beta,\epsilon)\ ,$$

which can be rewritten as (3.12), (4.23):

$$N_s \sim N_c + \beta M - \frac{\epsilon}{2}f_1\ .$$

In addition, to leading order, the radiation rate is still given by (3.16). If we combine all the above and approximate ψ by ψ_R on the right-hand side, (4.25) reduces to (4.3).

4.2.3. Derivation of the reduced equation (4.3) from a variational principle. If the perturbed CNLS equation has a Lagrangian density, then we can derive a Lagrangian density for the modulation equations by substituting the ansatz (4.2) in the action integral and integrating over the transverse variables. For example, this has already been done for the case of time-dispersive CNLS (see Figure 1 in [27]).

There are several problems with this approach, which is why we do not pursue it here. For one thing, it can be applied only to perturbations of CNLS which have a variational formulation. In addition, with this approach we can analyze only the adiabatic effects of the perturbation, because the nonadiabatic term in (4.3) does not appear in the averaged Lagrangian. Finally we note that when this approach is applied to the perturbed CNLS with the wrong ansatz (typically a Gaussian or a sech), the reduced equation fails to capture the delicate balance of critical self-focusing and can lead to erroneous predictions.

5. Applications of modulation theory. In this section we apply modulation theory to various perturbations of CNLS. We include several new applications and present previous applications within the framework of modulation method.

5.1. Self-focusing in fiber arrays. In the last few years it has been suggested that faster transmission in optical fibers may be achieved by using an array of coupled optical waveguides arranged on a line in which the pulses undergo 2D self-focusing. The model equation for the nth fiber is given by

$$i\psi_z^n - \beta_2\psi_{tt}^n + 2\gamma|\psi^n|^2\psi^n + \delta(\psi^{n+1} - 2\psi^n + \psi^{n-1}) = 0, \tag{5.1}$$

where $\psi^n(z,t)$ is the electric field envelope in the nth fiber, δ is the coupling coefficient between neighboring fibers, β_2 is the group velocity dispersion, and γ is the nonlinear coefficient. For theoretical and numerical studies of (5.1), see, e.g., [3, 4, 5, 6, 37, 79].

Let

$$\psi^n = \psi(z, t, x = nh) \tag{5.2}$$

and assume that the optical field is slowly varying over a number of fibers in the x direction, i.e., $h \ll 1$. If time-dispersion is anomalous ($\beta_2 < 0$), substitution of the change of variables

$$\tilde{z} = \delta h^2 z\ , \quad \tilde{\psi} = \frac{1}{h}\left(\frac{2\gamma}{\delta}\right)^{1/2}\psi\ , \quad \tilde{y} = \left[h\left(\frac{\delta}{|\beta_2|}\right)^{1/2}\right]t,$$

and (5.2) in (5.1) yields (after dropping the tildes)

$$i\psi_z + \psi_{yy} + |\psi|^2\psi + \frac{\psi(\cdot, x+h) - 2\psi(\cdot, x) + \psi(\cdot, x-h)}{h^2} = 0. \tag{5.3}$$

In order to apply modulation theory to (5.3), we rewrite it as

$$i\psi_z + \Delta_\perp\psi + |\psi|^2\psi + \left[\frac{\psi(\cdot, x+h) - 2\psi(\cdot, x) + \psi(\cdot, x-h)}{h^2} - \psi_{xx}\right] = 0,$$

which in the notation of (4.1) corresponds to $\epsilon = h^2/12$ and

$$F = \frac{12}{h^2}\left[\frac{\psi(\cdot, x+h) - 2\psi(\cdot, x) + \psi(\cdot, x-h)}{h^2} - \psi_{xx}\right]. \tag{5.4}$$

It is easy to see that F is conservative. In addition, we can expand

$$F = \psi_{xxxx} + \frac{2\epsilon}{5}\psi_{xxxxxx} + \cdots .$$

Since $\epsilon \ll 1$, let us begin by considering the perturbed nonlinear Schrödinger (PNLS)

$$i\psi_z + \Delta_\perp \psi + |\psi|^2\psi + \epsilon\psi_{xxxx} = 0, \tag{5.5}$$

i.e., the effect of the conservative perturbation

$$F = \psi_{xxxx}. \tag{5.6}$$

The evaluation of

$$f_1(z) = \frac{1}{\pi}\mathrm{Re}\int [R(\rho)\exp(iS)]_{xxxx}\exp(-iS)[R + \rho\nabla_\perp R]\, dxdy$$

can be simplified if we note that $[R\exp(iS)]_x \sim R_x \exp(iS)$, because $R = R(x/L)$, $S = S(x\sqrt{L_z/L})$, and $LL_z \sim \beta^{1/2} \ll 1$. Therefore,

$$f_1(z) = \frac{1}{\pi}\left(\int [R(\rho)]_{xxxx}[R + \rho\nabla_\perp R]\, dxdy\right)(1 + O(\beta^{1/2})) .$$

Since

$$\frac{1}{\pi}\left(\int [R(\rho)]_{xxxx}[R + \rho\nabla_\perp R]\, dxdy\right) = \frac{2}{\pi L^2}\int (R_{\xi\xi})^2\, d\xi d\eta ,$$

$$\int (R_{\xi\xi})^2\, d\xi d\eta = \frac{3}{8}\int (\Delta_\perp R)^2\, d\xi d\eta = \frac{3\pi}{4}\int (\Delta_\perp R)^2\, \rho d\rho ,$$

and (2.11), (2.14)

$$\int (\Delta_\perp R)^2\, \rho d\rho = I_6 - 3N_c,$$

where[10]

$$I_6 = \int_0^\infty R^6\, rdr ,$$

we have that

$$f_1 \sim -\frac{C_1}{L^2} , \quad C_1 = -\frac{3}{2}(I_6 - 3N_c) \cong -\frac{9N_c}{2} .$$

Thus, self-focusing in (5.5) is covered by Proposition 4.3. Since $\epsilon C_1 < 0$, we are in the case of a *focusing* perturbation (case 2 of the proposition) that can result in a finite-z blowup if the initial power is above critical.

Another way to see that $\epsilon\psi_{xxxx}$ is indeed a focusing perturbation is from the Hamiltonian identity for (5.5): We multiply (5.5) by ψ_z^*, add the conjugate equation, and integrate to get

$$\int |\nabla_\perp\psi|^2 - \frac{1}{2}\int |\psi|^4 - \epsilon\int |\psi_{xx}|^2 \equiv \text{constant}.$$

[10]Numerical computations show that $I_6 \cong 6N_c$. However, this is not an exact identity.

Therefore, from the relative signs we see that when $\epsilon > 0$ the perturbation acts with the focusing nonlinearity and against diffraction.

If the initial conditions are such that there is indeed blowup in (5.5), then it is not justified to approximate (5.4) with (5.6) and we need to add the next-order term in (5.4),

$$F = \psi_{xxxx} + \frac{2}{5}\epsilon\psi_{xxxxxx},$$

corresponding to the PNLS

$$i\psi_z + \Delta_\perp\psi + |\psi|^2\psi + \epsilon\psi_{xxxx} + \frac{2}{5}\epsilon^2\psi_{xxxxxx} = 0. \tag{5.7}$$

We can see immediately that the $O(\epsilon^2)$ term is a defocusing perturbation by observing the relative signs in the Hamiltonian identity for (5.7):

$$\int |\nabla_\perp\psi|^2 - \frac{1}{2}\int |\psi|^4 - \epsilon\int |\psi_{xx}|^2 + \frac{2\epsilon^2}{5}\int |\psi_{xxx}|^2 \equiv \text{constant}.$$

In order to evaluate f_1 for (5.7) we note that as before

$$\int [R(\rho)\exp(iS)]_{xxxxxx}\exp(-iS)[R+\rho\nabla_\perp R]\,dxdy \sim \int [R(\rho)]_{xxxxxx}[R+\rho\nabla_\perp R]\,dxdy$$

and

$$\int R_{\xi\xi\xi\xi\xi\xi}(R+\rho\nabla R)\,d\xi d\eta = -3\int (R_{\xi\xi\xi})^2\,d\xi d\eta.$$

Therefore

$$f_1 \sim \frac{|C_1|}{L^2} - \frac{C_2\epsilon}{L^4} \tag{5.8}$$

with

$$C_2 = \frac{6}{5\pi}\int (R_{\xi\xi\xi})^2\,d\xi d\eta > 0.$$

Plugging (5.8) into (4.7) gives that self-focusing in (5.7) is given to leading order by

$$-L^3L_{zz} = \beta_0 + \frac{\epsilon}{2M}\left[\frac{|C_1|}{L^2} - \frac{C_2\epsilon}{L^4}\right].$$

Integrating this equation twice, as in the derivation of (4.10) in Appendix H, gives

$$(y_z)^2 = \frac{4H_0}{M}y + 4\beta_0 + \frac{\epsilon|C_1|}{My} - \frac{2\epsilon^2C_2}{3My^2}\ . \tag{5.9}$$

From (5.9) we see that there is no blowup (y cannot go to zero) in (5.9). In addition, we can estimate the minimum value of y from the balance of the third and fourth terms on the right-hand side of (5.9):

$$y_m \sim \frac{2C_2\epsilon}{3|C_1|}\ .$$

Likewise, if $H_0 < 0$ the solution of (5.9) is oscillatory and we can estimate the maximum value of y from the balance of the first and second terms on the right-hand side of (5.9),

$$y_M \sim \frac{M\beta_0}{-H_0} \; .$$

Therefore, we can rewrite (5.9) as

$$(y_z)^2 = \frac{-4H_0}{M}\frac{1}{y^2}(y_M - y)(y - y_m)(y - y_3),$$

where

$$y_3 = \frac{\epsilon^2 C_2}{6H_0 y_m y_M} \sim -\frac{\epsilon|C_1|}{4M\beta_0} < 0 \; .$$

Thus, self-focusing in (5.9) is very similar qualitatively to the generic case of Proposition 4.3. In particular, when $\beta_0 > 0$ and $H_0 < 0$ the solution will oscillate between y_M and y_m.

Equation (5.9) captures the leading-order behavior for (5.7), hence also for (5.3) and (5.1). In the case of periodic oscillations in (5.9), nonadiabatic effects (which were neglected so far) will gradually cause the oscillations in (5.7), (5.3), and (5.1) to decay, in a manner similar to the one covered by Proposition 4.4. This qualitative picture agrees with the simulations of (5.1) of Aceves et al. [5, 6], where it was observed that the initial collapse towards the central fiber is arrested, followed by oscillations of power between the central fiber and its neighbors.

5.2. Small defocusing fifth-power nonlinearity. The case of small dispersive fifth-power nonlinearity

$$i\psi_z + \Delta_\perp \psi + |\psi|^2\psi - \epsilon|\psi|^4\psi = 0 \; , \quad 0 < \epsilon \ll 1 \tag{5.10}$$

was analyzed by Malkin [47]. In the notation of modulation theory we have

$$F = -|\psi|^4\psi \; ,$$

which is conservative ($f_2 \equiv 0$) and

$$f_1 \sim -\frac{4I_6}{3L^2} \; .$$

Therefore, self-focusing is covered by Proposition 4.3 with $C_1 = 4I_6/3$:

$$(y_z)^2 = \frac{-4H_0}{M}\frac{1}{y}(y_M - y)(y - y_m), \quad y_M \sim \frac{M\beta_0}{-H_0}, \quad y_m \sim \frac{\epsilon I_6}{3M\beta_0}. \tag{5.11}$$

5.3. Dispersive saturating nonlinearities. The use of (5.10) to model dispersive saturation of the nonlinearity is sometimes criticized because as $|\psi|$ increases the nonlinearity changes its sign and becomes defocusing. For this reason (5.10) is often replaced by

$$i\psi_z + \Delta_\perp \psi + \frac{1 - \exp(-2\epsilon|\psi|^2)}{2\epsilon}\psi = 0, \quad 0 < \epsilon \ll 1 \tag{5.12}$$

or by

$$i\psi_z + \Delta_\perp \psi + \frac{|\psi|^2}{1+\epsilon|\psi|^2}\psi = 0, \quad 0 < \epsilon \ll 1. \tag{5.13}$$

Equations (5.12) and (5.13) can be viewed as regularizations of (5.10): the nonlinearity is approximately the same as in (5.10) when $\epsilon|\psi|^2 \ll 1$, but it has a finite and positive limit as $|\psi|$ goes to infinity. It turns out that these regularizations have essentially the same effect on self-focusing as the unregularized case (5.10). This is true only for critical NLS and to the best of our knowledge its articulation is due to Malkin [45].

PROPOSITION 5.1. *Self-focusing in* (5.12) *and* (5.13) *is the same to leading order as in* (5.10).

Proof. The perturbation functions F corresponding to (5.12) and (5.13) are conservative and they satisfy

$$F = -\epsilon|\psi|^4(1 + O(\epsilon|\psi|^2), \quad \text{provided that} \quad \epsilon|\psi|^2 \ll 1.$$

Thus, as long as $|\psi|^2 \ll \epsilon^{-1}$, the leading-order behavior of (5.12) and (5.13) is still given by (5.11). Since for (5.11)

$$y \geq y_m \sim \frac{\epsilon}{\beta} \ ,$$

throughout the focusing-defocusing cycle $\epsilon|\psi|^2 = O(\beta) \ll 1$ and

$$F = -\epsilon|\psi|^4(1 + O(\beta)) \ .$$

We have, therefore, the important result that all small dispersive regularizations of critical NLS lead to the same canonical focusing-defocusing effect.

The oscillatory behavior of solutions of (5.12) and (5.13), in accordance with (5.11), was observed in numerical simulations of LeMesurier et al. [42]. In [74], special attention was given to the nonadiabatic power radiation in (5.12).

5.4. Davey–Stewartson equations. The Davey–Stewartson equations

$$i\psi_z + \Delta_\perp \psi + |\psi|^2\psi - \epsilon\phi_x\psi = 0 \ , \quad \alpha\phi_{xx} + \phi_{yy} = -(|\psi|^2)_x \tag{5.14}$$

arise in the study of gravity-capillary surface waves [1]. When $0 < \epsilon \ll 1$, the system (5.14) can be viewed as a perturbation of CNLS with

$$F = -\phi_x\psi \ .$$

This is a conservative perturbation and

$$f_1 = -\frac{1}{\pi}\mathrm{Re}\left[\int (\phi_R)_x R(\rho)(R + \rho\nabla_\perp R)\, dxdy\right],$$

where ϕ_R is the solution of

$$\alpha(\phi_R)_{xx} + (\phi_R)_{yy} = -(|\psi_R|^2)_x \ .$$

Let $\tilde{\phi}_R(\xi, \eta)$ be the solution of

$$\alpha(\tilde{\phi}_R)_{\xi\xi} + (\tilde{\phi}_R)_{\eta\eta} = -(R^2)_\xi(\xi, \eta) \ .$$

Then $\phi_R(x,y) = L^{-1}\tilde{\phi}_R(\xi,\eta)$ and

$$f_1 = -\frac{1}{\pi}\int (\tilde{\phi}_R)_\xi R(\rho)(R+\rho\nabla_\perp R)\, d\xi d\eta = \text{constant}.$$

Therefore, to leading order (4.16) reduces to

$$\beta_z = -\frac{\nu(\beta)}{L^2}\ ,$$

as in the case of self-focusing in the unperturbed CNLS (3.13). It follows that self-focusing in the Davey–Stewartson equations follows the adiabatic law for CNLS self-focusing (3.34), which ultimately reduces to the loglog law. It is remarkable that this perturbation has no effect on the blowup rate, as was first shown by Papanicolaou et al. [56], who derived the asymptotically equivalent equation (3.14) for self-focusing in the Davey–Stewartson equations and concluded that self-focusing in the Davey–Stewartson equations is given by the loglog law.

5.5. Nonparaxiality. In the standard derivation of CNLS as the model equation for laser beam propagation through a Kerr medium, the vectorial Maxwell equations for the propagation of a laser beam are reduced to the vectorial Helmholtz equations in the time-harmonic case. These equations are further reduced to the scalar Helmholtz equation for the electric field E

$$\left(\Delta_\perp + \frac{\partial^2}{\partial z^2}\right)E + k^2 E = 0\ ,\quad k^2 = k_0^2\left(1+\frac{2n_2}{n_0}|E|^2\right)$$

by neglecting vectorial effects [19]. Introducing the slowly varying envelope form $E = \psi\exp(ik_0 z)$ for the electric field leads to the nondimensional form of the Helmholtz equation [25]

$$\epsilon\psi_{zz} + i\psi_z + \Delta_\perp\psi + |\psi|^2\psi = 0\ ,\quad \epsilon = \left(\frac{\lambda}{4\pi r_0}\right)^2\ . \tag{5.15}$$

Since the beam wavelength λ is much smaller than the initial beam radius r_0,

$$0 < \epsilon \ll 1.$$

This suggests that $\epsilon\psi_{zz}$ can be neglected, in which case (5.15) reduces to CNLS.

Neglecting $\epsilon\psi_{zz}$ is called the *paraxial approximation* or the *parabolic approximation* and it is a valid approximation for rays which propagate almost parallel to the z-axis. Mathematically, this is a problematic approximation, because a boundary value problem (Helmholtz) is replaced with an initial value problem (NLS). Moreover, the paraxial approximation breaks down near the focal point, as was already pointed out by Kelley [32]. Indeed, from the asymptotic form of CNLS self-focusing solution (3.1), we see that the magnitudes of $\Delta_\perp\psi$ and $|\psi|^2\psi$ are $O(L^{-3})$ and that of the nonparaxial term is $O(\epsilon L^{-5})$. This suggests that the paraxial approximation breaks down when $L = O(\sqrt{\epsilon})$. In fact, we will now show that the nonparaxial term does not even get to be of the same size as the other terms because it arrests self-focusing when it is still $O(\beta)$-small compared with the CNLS terms.

We analyze the effect of small beam nonparaxiality by applying modulation theory with the perturbation

$$F = \psi_{zz}.$$

This perturbation is nonconservative and

$$f_2 \sim N_c \left(\frac{1}{L^2}\right)_z .$$

Therefore, (5.15) reduces to [25]

$$\beta_z = -\frac{2\epsilon N_c}{M}\left(\frac{1}{L^2}\right)_z . \tag{5.16}$$

Although this is a nonconservative perturbation, in light of (5.16) we can still apply the results of Proposition 4.3 with $C_1 = 4N_c$ to get

$$(y_z)^2 = \frac{-4H_0}{M}\frac{1}{y}(y_M - y)(y - y_m), \quad y_M \sim \frac{M\beta_0}{-H_0}, \quad y_m \sim \frac{2\epsilon N_c}{M\beta_0}. \tag{5.17}$$

It is remarkable that this nonconservative perturbation leads to the same generic reduced equation as in the previous examples of conservative perturbations.

From (5.17) and Proposition 4.3 we see that even when $\beta_0 > 0$ (i.e., initial power above critical) the solution of (5.17) does not blow up. If in addition $H_0 < 0$, the behavior is given by focusing-defocusing oscillations which gradually decay because of nonadiabatic effects. Note that throughout the focusing-defocusing cycle the relative magnitude of the nonparaxial term is

$$\frac{[\epsilon\psi_{zz}]}{[|\psi|^2\psi]} = \frac{\epsilon}{L^2} \leq \frac{\epsilon}{y_m} = O(\beta) ,$$

providing an a posteriori justification for treating it as a small perturbation.

The prediction of modulation theory of decaying focusing-defocusing oscillations is in qualitative agreement with the simulations of Feit and Fleck of the nonlinear Helmholtz equation [22] and with the studies of [8, 68]. This suggests that the answer to the open question

Is there blowup in the nonlinear Helmholtz equation?

is *no*, or that if there is blowup in the nonlinear Helmholtz equation, it is completely different from that of CNLS. This is an important question, since the singularity formation in CNLS is clearly nonphysical, indicating that some small terms that were neglected in the derivation of CNLS should be included in a model of physical self-focusing which is valid at and beyond the blowup point. Since the paraxial approximation is the last approximation in the derivation, if indeed there is no blowup in the nonlinear Helmholtz equation, it may prove to be *the* physically regularizing term, analogous to viscosity in fluid dynamics.

At present, a full picture of self-focusing in the nonlinear Helmholtz equation is still lacking. In particular, the effect of backscattering is unclear. In addition, there are no rigorous analytic results on singularity formation in the nonlinear Helmholtz equation.

5.6. Effect of randomness. The propagation of a narrow laser beam in a medium with impurities can be modeled by

$$i\psi_z + \Delta_\perp\psi + |\psi|^2\psi + \epsilon_1(x^2 + y^2)h(z)\psi = 0 , \quad 0 < \epsilon_1 \ll 1 , \tag{5.18}$$

where $h(z)$ is a real-valued random function. The perturbation

$$F = (x^2 + y^2)h(z)\psi$$

is conservative and

$$f_1 = 8ML^4 h(z).$$

Therefore, in this case the reduced equation (4.7) becomes

$$L_{zz} = -\frac{\beta_0}{L^3} - 4\epsilon L(z)h(z),$$

showing that the effect of this random perturbation becomes negligible as $L \searrow 0$.

Random inhomogeneities can become important if they act in conjunction with an additional defocusing perturbation which leads to oscillatory behavior. One example is defocusing quintic nonlinearity and randomness

$$i\psi_z + \Delta_\perp \psi + |\psi|^2\psi + \epsilon_1(x^2 + y^2)h(z)\psi - \epsilon_2|\psi|^4\psi = 0, \tag{5.19}$$

whose reduced equation is

$$-L^3 L_{zz} = \beta_0 + 4\epsilon_1 L^4 h(z) - \frac{4\epsilon_2 N_c}{M}\frac{1}{L^2} . \tag{5.20}$$

Random inhomogeneities have, in general [15], the form $h(z, x, y)\psi$. However, when the beam is narrow we can expand h about the beam axis

$$h = h_0(z) + (x, y) \cdot \nabla_\perp h + \frac{1}{2}(x, y) \cdot \nabla\nabla h \cdot (x, y) + \cdots .$$

The linear terms can be eliminated by preliminary transformations of the transverse coordinates and the phase [49]. If we also assume, for simplicity, that the inhomogeneities are transversely isotropic then we get (5.19). We will also assume that $h(z)$ is stationary with mean zero $\langle h(z)\rangle = 0$, where $\langle\rangle$ is ensemble average.

The reduced equation (5.20) can be written as a nonlinear oscillator equation with a parametrically random, linear term:

$$L_{zz} + 4\epsilon_1 h(z)L(z) + U'(L) = 0 , \quad U(L) = \frac{\epsilon_2 N_c}{ML^4} - \frac{\beta_0}{2L^2} . \tag{5.21}$$

The effects of randomness in (5.21) are not easy to assess and will be analyzed elsewhere. In the following we present some preliminary results. The potential $U(L)$ has a minimum at $L_{\min} = 2\sqrt{\epsilon_2 N_c/M\beta_0}$. For small oscillations about this minimum we can linearize (5.21) by writing $L = L_{\min} + \delta L$, with $0 \le \delta L \ll L_{\min}$, to get for δL the randomly forced linear oscillator equation

$$\delta L_{zz} + \omega^2 \delta L = \tilde{h}(z), \tag{5.22}$$

where the frequency ω is given by

$$\omega = \frac{\beta_0^{3/2} M}{2^{3/2}\epsilon_2 N_c}$$

and the random forcing by

$$\tilde{h}(z) = -8\epsilon_1 \sqrt{\frac{\epsilon_2 N_c}{M\beta_0}} h(z) \ .$$

Note that the frequency of the small oscillations decreases with β_0 but increases as $\epsilon_2 \to 0$. The random forcing will make the energy of the small oscillations increase on the average as z increases:

$$\frac{d}{dz} \left\langle \frac{1}{2}(\delta L)_z^2 + \frac{\omega^2}{2}(\delta L)^2 \right\rangle = \int_0^z \cos(\omega s)\tilde{R}(s)ds,$$

where $\tilde{R}(z) = \langle \tilde{h}(z+s)\tilde{h}(s)\rangle$ is the covariance of the random force $\tilde{h}(z)$. For large z the energy of the small oscillations grows linearly,

$$\left\langle \frac{1}{2}(\delta L)_z^2 + \frac{\omega^2}{2}(\delta L)^2 \right\rangle \ \sim \ \frac{z}{2}\hat{R}(\omega),$$

where $\hat{R}(\omega) \geq 0$ is the power spectral density [57] of the random forcing $\tilde{h}$,

$$\hat{R}(\omega) = \int_{-\infty}^{\infty} e^{i\omega s}\tilde{R}(s)ds \ .$$

Ultimately, the growth of the energy will make the linearization invalid and the full nonlinear equation (5.21) should be considered. An important issue is to estimate the probability of escape (i.e., $L \to +\infty$) by the random inhomogeneities. This could be done in a manner similar to the one used in [33]. The main result should be that the amplitude of the focusing-defocusing oscillations grows until there is no more focusing.

5.7. Temporal effects. In nonlinear optics CNLS (1.1) is derived for time-harmonic laser beams propagating in a medium with an instantaneous nonlinear polarization response. However, temporal effects, such as time-dispersion and Debye relaxation, can become important in the propagation of ultrashort laser pulses. Since in these nonstationary cases the initial condition is given at the medium interface $z = 0$ for all (x, y, t), time behaves like a third spatial variable and z plays the role of "time." As a result, the reduced equations and the modulation variables L, β, and ζ depend on both z ("time") and t ("space").

5.7.1. Small time-dispersion. The nonlinear Schrödinger equation with small time-dispersion

$$i\psi_z + \Delta_\perp\psi - \epsilon\psi_{tt} + |\psi|^2\psi = 0 \ , \quad \psi(z=0,x,y,t) = \psi_0(x,y,t) \ , \quad |\epsilon| \ll 1 \tag{5.23}$$

arises in the study of the propagation of ultrashort laser pulses in media with an instantaneous Kerr nonlinearity. The correct expression for ϵ is[11]

$$\epsilon = \frac{r_0^2 k_0 k_{\omega\omega}}{T^2},$$

where r_0 is the initial pulse radius, $k = \omega n_0(\omega)/c$ is the wavenumber, n_0 is the linear index of refraction, c is speed of light, and T is the pulse duration. Time-dispersion is called normal if $\epsilon > 0$ and anomalous if $\epsilon < 0$.

[11] We would like to thank B. Rockwell [63] for pointing out to us the error in the expression for ϵ in [27].

If time-dispersion is anomalous, (5.23) is supercritical NLS, which has solutions that undergo 3D collapse. However, the dynamics in the case of normal time-dispersion are more complicated because of the opposite sign of diffraction and time-dispersion. In particular, a new phenomenon occurs in the presence of small normal time-dispersion: a temporal splitting of the pulse into two components (see Figure 5.1). Pulse splitting and the possibility of multisplitting has attracted considerable interest over the last decade, as it may provide a physical mechanism which prevents the singularity formation in CNLS.

Zharova et al. [81] were the first to show that in the case of small normal time-dispersion self-focusing is arrested at t_m, the t cross section (i.e., the plane $(x, y, t = t_m)$ of the initial peak. As a result, the pulse undergoes a temporal split into two components. They authors went on to conjecture that the new peaks would go on splitting into "progressively smaller-scale." Although in their simulations they observed two splitting events, the reliability of their simulations is unclear. Indeed, in subsequent numerical simulations of (5.23) [18, 27, 44, 64] secondary pulse splitting was not observed. In [44], Luther, Newell, and Moloney derived reduced equations for the evolution of the pulse at the t_m cross section which show the arrest of self-focusing there. The validity of their reduced system was supported by a direct comparison with the numerical solution of (5.23).

In [27], Fibich, Malkin, and Papanicolaou derived the reduced system (5.24) for self-focusing in (5.23), using for the first time the systematic approach of modulation theory. Here

$$F = -\psi_{tt}$$

is nonconservative and

$$f_2 = -\frac{1}{2\pi}\mathrm{Im}\int \psi^* \psi_{tt}\, dxdy \sim -N_c \zeta_{tt} \ .$$

Therefore, from (4.8), we see that (5.23) reduces to [27]

$$\beta_z = \frac{2\epsilon N_c}{M}\zeta_{tt} \ , \quad L_{zz} = -\frac{\beta}{L^3} \ , \quad \zeta_z = \frac{1}{L^2} \ . \tag{5.24}$$

The numerical agreement of (5.24) with (5.23) was demonstrated in [27]. The reduced system (5.24) agrees with that of Luther, Newell, and Moloney [44] at t_m. However, (5.24) is valid for all t cross sections, not just at t_m. In fact, analysis of (5.24) shows that while self-focusing is arrested in an exponentially small neighborhood of t_m it continues elsewhere. Analysis of (5.24) also shows that peak splitting is associated with the transition from self-similar 2D collapse to full 3D dynamics. Therefore, it was suggested in [27] that the new peaks would not necessarily split again.

The effect of normal time-dispersion on the nonadiabatic radiation $\nu(\beta)$ was calculated in [12, 13, 14].

At present, it is still unknown whether multiple splitting occurs. A related open question is whether the solution of (5.23) can become singular. At present, these questions cannot be investigated numerically, since current numerical simulations cannot go much further beyond the first pulse-splitting. In addition, the validity of the reduced system (5.24) after the formation of the two new self-focusing peaks is unclear. Indeed, the large values of β and the large t gradients in Figure 5.1 after the pulse splitting violate the assumptions under which (5.24) was derived. *This may indicate*

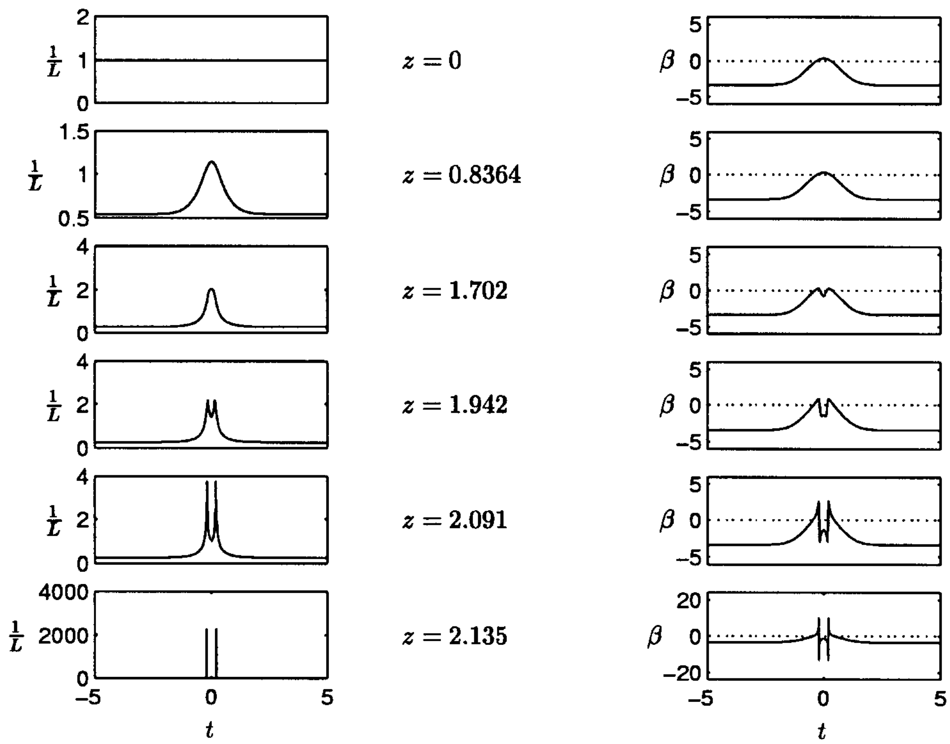

FIG. 5.1. *As a result of small normal time-dispersion the power (β) radiates away from the center, leading to the formation of two symmetrical peaks which continue to self-focus. Results are shown for the reduced system* (5.24) *with the initial conditions $L(0,t) \equiv 1$, $L_z(0,t) \equiv 0$ and $\beta(0,t) = (1.1\exp(-t^2)-1)N_c/M$ and $\epsilon = 0.01$. Although we do not observe secondary peak splitting, the large values of β and the sharp t gradients suggest that the validity of* (5.24) *breaks down at some point.*

that after the first splitting ψ_R ceases to serve as an attractor for ψ_s. If so, this nullifies the whole argument of multisplitting "by induction."

Recently, pulse splitting was observed experimentally by Ranka, Schirmer, and Gaeta [60], some 10 years after its theoretical prediction.

5.7.2. Debye relaxation. In models for the propagation of an intense laser beam, the nonlinear cubic term in CNLS represents an instantaneous nonlinear material polarization response. If the mechanism for the induced nonlinear polarization is molecular orientation, then for sufficiently long pulses the frictional drag between the molecules tends to make the rotation lag behind the torque induced by the electric field. The resulting model equation in this case is CNLS with Debye relaxation:

$$i\psi_z + \Delta_\perp \psi + N\psi = 0, \tag{5.25}$$

$$\epsilon N_t + N = |\psi|^2 \ , \quad \epsilon = \frac{\tau_D}{T} > 0 \ , \tag{5.26}$$

where t is retarded time $(t - z/c_g)$, τ_D is the characteristic response time for dipole reorientation ($\sim\ 10^{-11}$sec for water), and T is pulse duration.

In this section we use modulation theory to address the question of whether Debye relaxation can arrest self-focusing when $0 < \epsilon \ll 1$. The Debye perturbation

$\epsilon F = (N - |\psi|^2)\psi$ is conservative ($f_2 = 0$). From (5.26), we get that

$$N \sim |\psi|^2 - \epsilon(|\psi|^2)_t \ .$$

Therefore, we approximate (5.25)–(5.26) by (4.1) with

$$F = -(|\psi|^2)_t \psi \ .$$

Evaluation of f_1 yields

$$f_1 \sim \frac{C_D L_t}{L} \ , \quad C_D = \int (\nabla_\perp R^2)^2 \rho^3 \, d\rho \cong 6.43 \ .$$

Substitution in (4.7) shows that self-focusing in the presence of Debye relaxation is given by

$$-L^3 L_{zz} = \beta_0 + \frac{\epsilon C_D}{2M} \frac{L_t}{L} \ . \tag{5.27}$$

From this equation we see that Debye relaxation slows focusing at times earlier than the pulse peak ($L_t \leq 0$) and enhances it at later times ($L_t \geq 0$). As a result, self-focusing becomes temporally asymmetrical, with the peak moving toward later times (Figure 5.2), as can be expected from a delay mechanism and as was observed in numerical simulations of (5.25)–(5.26) [67].

In order to further analyze the initial effect of Debye relaxation, we note that during the nonadiabatic self-focusing and unperturbed adiabatic self-focusing stages (see section 4.1), the effect of Debye relaxation is negligible and each t cross section (i.e., the plane t = constant in the (x, y, t) space) focuses independently in a 2D self-similar fashion,

$$L(z, t) = L(Z_c(t) - z)$$

with $Z_c(t)$ given by (3.31). If we use this self-similar form in (5.27), we get

$$-L^3 L_{zz} = \beta_0 - \frac{\epsilon C_D \dot{Z}_c(t)}{2M} \frac{L_z}{L} \ , \quad \dot{} := \frac{d}{dt} \ .$$

Making a change of variable, we can rewrite this equation as

$$A_{\zeta\zeta} = \beta_0 A + \frac{\epsilon C_D \dot{Z}_c(t)}{6M} (A^3)_\zeta \ , \quad A = \frac{1}{L} \ .$$

If the peak power is initially at $t = t_0$, then $\dot{Z}_c(t) > 0$ for $t > t_0$ and $\dot{Z}_c(t) < 0$ for $t < t_0$. Therefore, we see that if the power is above critical ($\beta_0 > 0$), there is blowup ($A \nearrow +\infty$) for $t > t_0$ and arrest of blowup for $t < t_0$. However, one cannot apply this conclusion to (5.25)–(5.26) or even to (5.27), because as self-focusing starts to deviate from that the unperturbed CNLS, the validity of the 2D self-similar argument breaks down and the dynamics become fully 3D (i.e., (x, y, t)), as manifested by the shift of the peak toward later times. At present, the question whether solutions of (5.25)–(5.26) can become singular is still open.

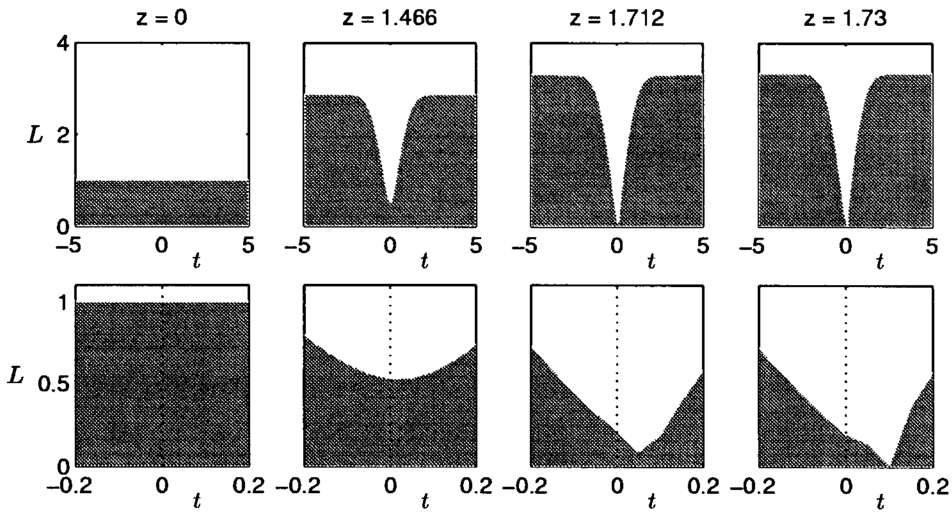

FIG. 5.2. *Self-focusing in the presence of Debye relaxation according to the reduced equation* (5.27) *with the initial conditions* $L_0 \equiv 1$ *and* $\beta(0) = (1.1\exp(-t^2) - 1)Nc/M$. *While most of the pulse is defocusing (top), asymmetric self-focusing takes place in the center, with the peak moving towards later times.*

5.8. Time-dispersion and nonparaxiality. We have seen that both normal time-dispersion and nonparaxiality can lead to self-focusing arrest. This raises the question of determining which of these two mechanisms is dominant in self-focusing of ultrashort pulses. Similarly, if time-dispersion is anomalous, it is enhancing self-focusing as nonparaxiality is slowing it down, and we would like to know which of the two effects will ultimately prevail. Therefore, we are interested in analyzing self-focusing in the presence of both time-dispersion and nonparaxiality.

It may seem that all we need to do is add the separate contribution of each mechanism in the corresponding reduced equation (5.16) and (5.24). However, more careful examination of the derivation of CNLS shows that if one retains both time-dispersion and nonparaxiality in the model, then the model equation contains additional terms [28]:

$$i\psi_z + \Delta_\perp \psi + |\psi|^2\psi + \epsilon_1 \psi_{zz} + \epsilon_2 \left[2i\frac{n_0 c_g}{c}(|\psi|^2\psi)_t - \psi_{zt}\right] - \epsilon_3 \psi_{tt} = 0, \tag{5.28}$$

where

$$\epsilon_1 = \frac{1}{4r_0^2 k_0^2}\ , \quad \epsilon_2 = \frac{1}{c_g k_0 T} = \frac{1}{\omega_0 T}\frac{c}{n_0 c_g}\ , \quad \epsilon_3 = \frac{k_0 r_0^2 k_{\omega\omega}}{T^2}, \tag{5.29}$$

and c_g is the group velocity. The dimensionless parameter $\epsilon_1 \sim$ (wavelength/radial pulse width)^2, $\epsilon_2 \sim$ (period of one oscillation/pulse duration), and ϵ_3 is a dimensionless measure of group velocity dispersion (GVD). Note that ϵ_2 is proportional to the geometric mean of ϵ_1 and ϵ_2:

$$\epsilon_2^2 = \epsilon_1 \epsilon_3 q\ , \quad q = \frac{4}{c_g^2 k_0 k_{\omega\omega}}\ .$$

Therefore, if one retains time-dispersion and nonparaxiality, the mixed term and the shock term (ψ_{zt} and $(|\psi|^2\psi)_t$, respectively) should also be included in the model.

Moreover, in the visible spectrum $q \gg 1$, and the ϵ_2 terms can dominate over both time-dispersion and nonparaxiality [28].

The reduced system corresponding to (5.29) is

(5.30)
$$\beta_z(z,t) = -\gamma_1 \left(\frac{1}{L^2}\right)_z - \gamma_2 \left(\frac{1}{L^2}\right)_t + \gamma_3 \zeta_{tt} \ , \quad \zeta_z(z,t) = \frac{1}{L^2} \ , \quad L_{zz}(z,t) = -\frac{\beta(z,t)}{L^3},$$

where

$$\gamma_1 = 2\epsilon_1 N_c/M, \quad \gamma_2 = \epsilon_2(6c_g n_0/c - 2)N_c/M, \quad \gamma_3 = 2\epsilon_3 N_c/M.$$

Following [27], we can analyze the initial effect of the three terms in (5.28) by looking at special solutions of (5.30). Away from the focal point, the three perturbing terms in (5.28) are small and each t cross section of the pulse (i.e., the 2D plane t = constant in the (x, y, t) space) focuses independently with

$$L(z,t) = L(Z_c(t) - z), \quad \beta(z,t) = \beta(Z_c(t) - z), \quad \zeta(z,t) = \zeta(Z_c(t) - z). \tag{5.31}$$

Here $Z_c(t)$ is the location of the focus in the (z,t) plane when $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$ (3.31). Therefore, (5.30) becomes

$$\beta_z = -\gamma_1 \left(\frac{1}{L^2}\right)_z + \gamma_2 \dot{Z}_c \left(\frac{1}{L^2}\right)_z + \gamma_3(-\ddot{Z}_c \zeta_z + \dot{Z}_c^{\,2} \zeta_{zz}) \ , \quad \dot{} = \frac{d}{dt} \ . \tag{5.32}$$

This equation can be transformed into a nonlinear Airy equation [27]

$$g_{ss} = sg + \kappa g^3, \quad \text{with} \quad g = L^{-1} > 0. \tag{5.33}$$

Here

$$s = (\beta_0 - \gamma_3 \ddot{Z}_c \zeta)(\gamma_3 \ddot{Z}_c)^{-2/3} \ , \quad \beta_0 \sim \beta(0,t),$$
$$\kappa = -(\gamma_1 - \gamma_2 \dot{Z}_c - \gamma_3 \dot{Z}_c^{\,2})(\gamma_3 \ddot{Z}_c)^{-2/3}.$$

The initial conditions for (5.33) are given at

$$s_0(t) := s(z = 0, t) \sim \beta(0,t)(\gamma_3 \ddot{Z}_c)^{-2/3} \ .$$

At the time t_0 of the initial peak power of the pulse, $Z_c(t)$ attains its minimum, $\dot{Z}_c(t_0) = 0$, and the evolution is given by (5.33) with $\kappa = -\gamma_1(\gamma_3 \ddot{Z}_c)^{-2/3} < 0$. Because $\ddot{Z}_c(t_0) > 0$, as $z \to Z_c$ and $\zeta \to +\infty$, $s \to -\infty$ for normal time-dispersion ($\epsilon_3 > 0$), and both time-dispersion and nonparaxiality (first and second terms on the right-hand side of (5.33), respectively) contribute to the arrest of the blowup by preventing g from becoming infinite. When time-dispersion is anomalous ($\epsilon_3 < 0$), it enhances blowup ($s \to +\infty$) while nonparaxiality opposes it. Eventually, as $s \to +\infty$ nonparaxiality prevails and the solution of (5.33) will decay (no blowup).

In the case of normal time-dispersion and $\epsilon_1 = \epsilon_2 = 0$, blowup is arrested only in an exponentially small neighborhood of t_0 [27], where pulse splitting occurs. In order to assess the added effects of nonparaxiality and the mixed term, we note that the condition for blowup [27] in (5.33) as $s \to -\infty$ is $\kappa > 2L^2(0,t)\mathrm{Ai}^2(s_0)$ or

$$\gamma_3 \dot{Z}_c^{\,2} > \gamma_1 - \gamma \dot{Z}_c + 2L^2(0,t)\mathrm{Ai}^2(s_0)(\gamma_3 \ddot{Z}_c)^{2/3},$$

where Ai(s) is the Airy function. Therefore, if nonparaxiality dominates, arrest of blowup occurs over a much larger region (possible everywhere). If the ϵ_2 term dominates, blowup will occur when $\epsilon_3 > -\epsilon_2/\dot{Z}_c$, i.e., only for $t > t_0$. Note that as the solution starts to deviate from that of the unperturbed CNLS, the 2D self-similar structure (5.31) will gradually break down. Therefore, for later z this 2D self-similar argument becomes invalid and the full 3D nature of (5.30) has to be considered.

From (5.32) we see that the effect of the ϵ_2 term on a self-focusing pulse is a temporal power transfer toward later times (recall that β is proportional to the excess power above critical). This will result in an asymmetric temporal development of the pulse, with a greatly enhanced trailing portion and a suppressed leading part, in agreement with previous results on the effect of the shock term [10] and of the linear component of the ϵ_2 term [65].

6. Numerical methods. Numerical integration of self-focusing in CNLS (1.1) requires a code that can handle the ever-increasing gradients near the singularity. In the method of *dynamic rescaling* [50], the independent variables and the function are dynamically rescaled in a way which is based on the asymptotic form of the solution (3.1). In the rescaled variables the function is smooth and the problem can be solved on a fixed grid using standard techniques. Then, the solution of CNLS is recovered from that of the rescaled problem. Subsequent improvements to this method include the use of approximate boundary conditions [35] and extension to the nonisotropic case [40]. The CNLS simulations in this paper were performed using dynamic rescaling with approximate boundary conditions (for more details, see [23]). The power of this method can be seen, for example, in Figure 3.5(C), where focusing factors of 10^{15} were reached.

Dynamic rescaling was also applied to perturbed CNLS: saturating nonlinearity [42], the Davey–Stewartson equations [56], and small normal time-dispersion [27]. However, in these cases the method becomes less successful, since it is inherently based on the special rescaling of CNLS self-focusing. Some of the difficulties which arise are instabilities due to the use of the approximate boundary conditions during defocusing stages and the need for (a yet unknown) additional rescaling in the t direction in the nonstationary cases.

Another approach is to apply a split-step method (e.g., [59]): the linear parts are solved by a Fourier transform in space and the nonlinear part is solved by an appropriate nonlinear solver. A different approach was taken in [9], where CNLS was solved by a Galerkin finite-element method.

6.1. Numerical comparison of CNLS and adiabatic theory. In order to compare the numerical solution of a perturbed CNLS with its corresponding reduced system, one must be able to recover the values of L, β, and ζ from ψ. In the case of dynamic rescaling, one solves for the rescaled function u and for $\bar{L}$, which are related to ψ through

$$\psi = \frac{u(\bar{\zeta}, \bar{\rho})}{\bar{L}} \ , \quad u \sim \exp\left(i\lambda^2(\bar{\zeta})\bar{\zeta} + i\frac{\bar{L}_z}{\bar{L}}\frac{r^2}{4}\right)\lambda R(\lambda\bar{\rho}) \ .$$

The bars denote the (numerical) values of L, ζ, etc., in dynamic rescaling. In general, these values are different from the ones used in the asymptotic theory, where $\lambda \equiv 1$. The modulation variables can be recovered using [23]

$$\zeta = \arg u(\rho = 0) \ , \quad L = \bar{L}\frac{R(0)}{|u(0)|} \ , \quad a \sim \bar{a}\left(\frac{R(0)}{|u(0)|}\right)^2 \ , \quad L_z \sim -\frac{\bar{a}}{\bar{L}}\frac{R(0)}{|u(0)|}$$

and

$$H_s \sim \frac{1}{\bar{L}^2} \int_0^{\bar{\rho}_c} \left(|u_{\bar{\rho}}|^2 - \frac{1}{2}|u|^4 \right) \rho d\rho \ .$$

Although β can be recovered by using (3.5), a better way, which does not involve numerical z derivatives, is to use

$$\beta \sim \frac{1}{M} \left(\int_0^{\bar{\rho}_c} |u|^2 \, \rho d\rho - N_c \right). \tag{6.1}$$

Note, however, that this approximation has only $O(\beta)$ accuracy. Therefore, near the singularity a more accurate approximation is

$$\beta \sim a^2 \ , \tag{6.2}$$

which has a theoretical exponential accuracy in β. However, since the last approximation is not valid at the early stages of self-focusing, in the numerical comparison of the adiabatic laws with NLS simulations (Figures 3.5–3.6), we recover β using (6.1) at the early stages of self-focusing and switch to (6.1) for the advanced stages of the blowup.

When we apply modulation theory for nonstationary perturbations of CNLS, the question arises as to how to represent t cross sections whose power is much smaller than N_c, since modulation theory was derived for $|\beta| \ll 1$. The simplest approach is to use (3.15) for all t cross sections. If we do that, then

$$\lim_{t \to \pm\infty} \beta(t) = \frac{-N_c}{M} \cong -3.38.$$

Fortunately, this approximation is quite reasonable, since as $t \to \pm\infty$ the propagation is determined only by linear diffraction, in which case

$$L_{zz} = \frac{4}{L^3}$$

(see, for example, (39) of [7]), corresponding to

$$\lim_{t \to \pm\infty} \beta(t) = -4.$$

A related question is which value to use for $L_0(t)$ for $|t|$ large. We cannot use (3.39), since then

$$\lim_{t \to \pm\infty} L_0(t) = \infty.$$

One possibility is to set $L_0(t) \equiv 1$.

Appendix A. Perturbation analysis for $P \sim R + \epsilon h$. In Lemma A.1 we use regular perturbations to evaluate several integrals which arise when we average over the transverse variables. In the case of perturbed CNLS, the results of this lemma are applied with

$$P = V_0(\xi, \eta; \beta = 0, \epsilon).$$

In Appendix B we apply this lemma for the case of unperturbed CNLS with $P = V_0(\rho; \beta)$, but in that case we have to be more careful with the domains of integration.

Lemma A.1.

1. *Let $R(\rho)$ be the solution of* (2.11). *Then the following identities hold:*

$$\int_0^\infty R^2\,\rho d\rho = \int_0^\infty (\nabla_\perp R)^2\,\rho d\rho = \frac{1}{2}\int_0^\infty R^4\,\rho d\rho. \tag{A.1}$$

2. *Let $P(\xi,\eta)\in H^1$ satisfy the equation*

$$\Delta P - P + P^3 + \epsilon w(P) = 0 \tag{A.2}$$

with $w(P)$ real. Then

$$H(P) = \frac{\epsilon}{2\pi}\int w(P)[P + (\xi,\eta)\cdot\nabla_\perp P]\,d\xi d\eta. \tag{A.3}$$

In addition, if we expand

$$P(\xi,\eta) = R(\rho) + \epsilon h(\xi,\eta) + O(\epsilon^2), \quad |\epsilon| \ll 1,$$

then the equation for h is (4.22) *and*

(A.4)

$$\int Rh\,\rho d\xi d\eta = \int (R^3 h - \nabla_\perp R\nabla_\perp h)\,d\xi d\eta = -\frac{1}{2}\int w(R)[R + \rho\nabla_\perp R]\,d\xi d\eta.$$

Therefore,

$$H(P) = \frac{\epsilon}{2\pi}\int w(R)[R + (\xi,\eta)\cdot\nabla_\perp R]\,d\xi d\eta + O(\epsilon^2).$$

Proof. If we multiply (A.2) by P and integrate by parts, we get

$$-\int(\nabla_\perp P)^2 - \int P^2 + \int P^4 + \epsilon\int w(P)P = 0. \tag{A.5}$$

Similarly, if we multiply (A.2) by $(\xi,\eta)\cdot\nabla_\perp P$ and integrate by parts, we get

$$\int P^2 - \frac{1}{2}\int P^4 + \epsilon\int w(P)(\xi,\eta)\cdot\nabla_\perp P = 0. \tag{A.6}$$

Adding (A.5) and (A.6) gives (A.3).

If we multiply (2.11) by P and integrate by parts, we get

$$-\int \nabla_\perp R\nabla_\perp P - \int PR + \int PR^3 = 0. \tag{A.7}$$

The $O(1)$ and $O(\epsilon)$ equations in (A.5) are, respectively,

$$-\int(\nabla_\perp R)^2 - \int R^2 + \int R^4 = 0, \tag{A.8}$$

$$-2\int\nabla_\perp R\nabla_\perp h - 2\int Rh + 4\int R^3 h = -\int w(R)R. \tag{A.9}$$

The $O(1)$ and $O(\epsilon)$ equations in (A.6) are, respectively,

$$\int R^2 - \frac{1}{2}\int R^4 = 0, \tag{A.10}$$

$$2\int Rh - 2\int R^3 h = -\int w(R)(\xi,\eta)\cdot\nabla_\perp R\,. \tag{A.11}$$

The $O(1)$ and $O(\epsilon)$ equations in (A.7) are, respectively,

$$-\int (\nabla_\perp R)^2 - \int R^2 + \int R^4 = 0, \tag{A.12}$$

$$-\int \nabla_\perp R \nabla_\perp h - \int Rh + \int R^3 h = 0. \tag{A.13}$$

From (A.8)–(A.13), we get (A.1) and (A.4).

Appendix B. Perturbation analysis for $V_0 \sim R + \beta g$. In this appendix we derive the modulation approximations to various integrals which arise when we derive the reduced equations for CNLS from balance of power. Modulation theory for CNLS is based on the ansatz for the focusing part of the solution

$$\psi_s(r,z) \sim \frac{1}{L} V_0(\rho;\beta) \exp\left(i\zeta + i\frac{L_z}{L}\frac{r^2}{4}\right), \quad \rho = \frac{r}{L} \ ,$$

where V_0 is quasi-steady. As we have seen, if V_0 is defined by (3.7), then to the right of the turning point at $\rho_b = 2\beta^{-1/2}$ (see section C), V_0 is oscillatory:

$$V_0 \sim \frac{1}{\rho} \cos\left(\frac{\beta\rho^2}{4}\right).$$

Therefore, with this definition it is not possible to match ψ_s with ψ_{back}. Moreover, V_0 is not even in L^2, as $\int_0^\infty |V_0|^2 \rho d\rho$ diverges.

In order to take care of this problem we need to redefine V_0. One possibility is to have V_0 defined for all ρ, in which case one has to add a small term to (3.7) which will correct its behavior for large ρ. In this case, the equation for V_0 is (3.8). Alternatively, we can consider V_0 to be the solution of (3.7), restricted to the domain $0 \le \rho \le \rho_c$, where $1 \ll \rho_c < \rho_b$ (e.g., $\rho_c = \beta^{-1/2}$). If we adopt this approach, then ψ_s is also defined only for $0 \le \rho \le \rho_c$, as in (3.3).

LEMMA B.1. *Let $V_0(\rho)$ be the solution of* (3.7). *Then*

$$\begin{aligned} H(V_0) &:= \int_0^{\rho_c} |\nabla V_0|^2 \rho d\rho - \frac{1}{2}\int_0^{\rho_c} |V_0|^4 \rho d\rho \\ &= -\frac{\beta}{4}\int_0^{\rho_c} \rho^2 V_0^2 \rho d\rho + \{\textit{terms exponentially small in } \beta\} \ . \end{aligned}$$

In addition, if we expand

$$V_0 \sim R(\rho) + \beta g(\rho) + O(\beta^2), \quad |\beta| \ll 1, \tag{B.1}$$

the equations for R and g are (2.11) *and* (3.10) *and*

$$\int_0^\infty Rg\, \rho d\rho = \frac{M}{2}, \quad N(V_0) := \int_0^{\rho_c} |V_0|^2 \rho d\rho = N_c + \beta M + O(\beta^2).$$

Proof. Use

$$\int_0^{\rho_c} V_0^2 = \int_0^\infty R^2 + 2\beta \int_0^\infty Rg + O(\beta^2)$$

and Lemma A.1 with $P = V_0$, $\epsilon = \beta$, $w = (1/4)\rho^2 V_0$, and $h = g$. Note that in the domain $[0\ \rho_c]$ the expansion (3.9) is uniform in ρ and V_0, R, and g are all exponentially decreasing. Therefore, the error of replacing ρ_c with infinity in integrals is exponentially small in β.

LEMMA B.2. *Let*

$$\psi_s(r,z) = \frac{1}{L} V_0(\rho;\beta) \exp\left(i\zeta + i\frac{L_z}{L}\frac{r^2}{4} \right), \quad \rho = \frac{r}{L} \ .$$

Then

$$N(\psi_s) := \int_0^{L\rho_c} |\psi_s|^2 \, r dr = N_c + \beta M + O(\beta^2) \ . \tag{B.2}$$

Proof. This follows from Lemma B.1.

Appendix C. WKB calculation of the rate of power and Hamiltonian radiation. In this appendix we derive (3.16) and (3.17). Let us rewrite (3.4) as

$$iV_\zeta + \Delta_\perp V - UV = 0 \ , \quad U = 1 - |V|^2 - \frac{1}{4}\beta\rho^2 \ . \tag{C.1}$$

The radiation rates for the power and Hamiltonian of ψ_s are given by

$$\begin{aligned} \frac{d}{dz}N(\psi_s) &= \frac{d}{dz}\int_0^{L\rho_c} |\psi|^2 \, r dr, \\ \frac{d}{dz}H(\psi_s) &= \frac{d}{dz}\int_0^{L\rho_c} \left[|\psi_r|^2 - \frac{1}{2}|\psi|^4 \right] r dr \ . \end{aligned} \tag{C.2}$$

When $0 < \beta \ll 1$,

$$V \sim R(\rho) \ , \qquad 0 \le \rho \ll \beta^{-1/2} \ , \tag{C.3}$$

and the potential U has two turning points: $\rho_a = O(1)$ and $\rho_b \sim 2/\sqrt{\beta}$. Since in the classically inaccessible region $[\rho_a, \ \rho_b]$ the solution V has an exponential decay, if we set ρ_c in (C.2) to be just past the second turning point to the right, i.e., $0 < \rho_c - \rho_b \ll 1$ (rather than $1 \ll \rho_c < \rho_b$, as in Appendix B), this would result only in an exponentially small change in the values of N_s and H_s.

If we differentiate (C.2), use (1.1), and integrate by parts, we get

$$\begin{aligned} \frac{d}{dz}N(\psi_s) &= |\psi|^2 L L_z \rho_c^2 + (i\psi^*\psi_r L\rho_c + c.c.) \ , \\ \frac{d}{dz}H(\psi_s) &= |\psi_r|^2 L L_z \rho_c^2 - \frac{1}{2}|\psi|^4 L L_z \rho_c^2 + [iL\rho_c(\psi_r^*\psi_{rr} - |\psi|^2\psi^*\psi_r) + c.c.] \ . \end{aligned}$$

Using (3.1), these equations can be rewritten in terms of V:

$$\frac{d}{dz}N(\psi_s) = \frac{1}{L^2}(i\rho_c V^* V_\rho + c.c.) \tag{C.4}$$

and

(C.5)
$$\begin{aligned} \frac{d}{dz}H(\psi_s) = &-\frac{L_z\rho_c^2}{L^3}|V_\rho|^2 - \frac{L_z^2\rho_c^3}{4L^2}(iVV_\rho^* + c.c.) + \frac{L_z\rho_c^2}{2L^3}|V|^4 + \frac{\rho_c}{L^4}(iV_\rho^* V_{\rho\rho} + c.c.) \\ &-\frac{L_z\rho_c}{2L^3}(|V|^2)_\rho + \frac{L_z\rho_c^2}{2L^3}(V^* V_{\rho\rho} + c.c.) - \frac{\rho_c}{L^4}(i|V|^2 V^* V_\rho + c.c.). \end{aligned}$$

In order to find the asymptotic behavior of V for $\rho > b$, we rewrite (C.1) as

$$iV_\zeta + \delta^2 \Delta_s V - UV = 0 \ , \quad U = 1 - |V|^2 - s^2 \ , \quad s = \delta\rho \ , \quad \delta = \frac{\beta^{1/2}}{2} \ll 1. \tag{C.6}$$

Since for CNLS $V_\zeta = o(\beta)$, we can use the stationary version of (C.6):

$$\delta^2 \Delta_s V - UV = 0 \ . \tag{C.7}$$

In terms of the new independent variable s, the turning points are at $s_a = O(\delta)$ and $s_b \sim 1$. Using (C.3) and

$$R(\rho) \sim A_R \exp(-\rho)\rho^{-1/2}, \qquad \rho \gg 1,$$

we get that

$$V \sim A_R \exp(-s/\delta)(s/\delta)^{-1/2}, \qquad \delta \ll s \ll 1. \tag{C.8}$$

When $s \gg \delta$, the nonlinearity becomes negligible. Application of WKB to (C.7) shows that

(C.9)

$$V \sim \frac{C_w}{s^{1/2} p^{1/2}} \exp\left(+\frac{i}{\delta}\int_1^s p(r)\, dr\right), \quad p = (-2U)^{1/2} \sim \sqrt{s^2 - 1}, \qquad \delta^{2/3} \ll s - 1,$$

from which it follows that

$$V \sim \frac{C_w}{s} \exp\left(+i\frac{s^2}{2\delta}\right), s \gg 1. \tag{C.10}$$

Only the term with the plus sign in the exponent was used in (C.9) and (C.10) in order to ensure that $\psi_s \sim V/L \exp(ir^2 L_z/4L)$ has no rapid oscillations as it connects to ψ_{back}. The connection formula for (C.9) beyond the turning point at s_b (e.g., [11, Chapter 10], [38, Chapter 7]) gives

$$V \sim \frac{C_w \exp(-i\pi/4)}{s^{1/2}|p|^{1/2}} \exp\left(-\frac{1}{\delta}\int_1^s |p(s')|\, ds'\right), \quad |p| \sim \sqrt{1 - s^2},$$
$$\delta \ll s < 1 \ , \quad s - 1 \ll \delta^{2/3}.$$

In particular, when $\delta \ll s \ll 1$, $|p| \sim 1$ and

$$V \sim \frac{C_w \exp(-i\pi/4)}{s^{1/2}} \exp\left(-\frac{1}{\delta}\left(\int_1^0 \sqrt{1 - s^2}\, ds + s\right)\right), \qquad \delta \ll s \ll 1. \tag{C.11}$$

The value of C_w is determined by matching (C.11) with (C.8) and using $\int_0^1 \sqrt{1 - s^2}\, ds = \pi/4$:

$$C_w = A_R \delta^{1/2} \exp\left(-\frac{\pi}{4\delta}\right) \exp\left(+i\frac{\pi}{4}\right). \tag{C.12}$$

Combining (C.6), (C.10), and (C.12) gives

$$V \sim 2^{1/2} A_R \beta^{-1/4} \rho^{-1} \exp\left(-\frac{\pi}{2\sqrt{\beta}} + i\frac{\pi}{4} + i\frac{\beta^{1/2}}{4}\rho^2\right), \quad \rho \gg \beta^{-1/2} \ . \tag{C.13}$$

If we substitute (C.13) into (C.4), we get that in the domain of validity of (C.13) the rate of power radiation is independent of ρ:

$$\frac{d}{dz}N_s \sim -\frac{2A_R^2}{L^2}\exp(-\pi/\sqrt{\beta}),$$

which is (3.16). However, if we substitute (C.13) into (C.5), the result will depend on ρ. Therefore, in order to estimate (C.5) we need the asymptotic behavior of V just to the right of the second turning point $s_b = 1$ (C.9):

$$\text{(C.14)}\quad V \sim \frac{C_w}{2^{1/4}(s-1)^{1/4}}\exp\left(+i\frac{2^{1/2}}{\delta}\int_1^s (s'-1)^{1/2}\,ds'\right),\quad \delta^{2/3} \ll s-1 \ll 1.$$

If we substitute (C.14) into (C.5) and use $\delta \sim -LL_z/2$ and $\delta^{2/3} \ll s-1$, we get that for leading order

$$\frac{d}{dz}H_s \sim -\frac{2A_R^2}{L^4}\exp(-\pi/\sqrt{\beta}),$$

which is (3.17).

Appendix D. Asymptotic growth of H_s. In order to estimate the rate at which H_s grows in the adiabatic regime, we use (3.17) and the adiabaticity of β to write

$$H_s \sim -M\nu(\beta)\int^z \frac{1}{L^4(z')}\,dz'.$$

If we use (3.35) and integrate, we get

$$H_s \sim -\frac{M\nu(\beta)}{4\beta}\frac{1}{Z_c - z}$$

or

$$H_s \sim -\frac{M\nu(\beta)}{2\sqrt{\beta}}\frac{1}{L^2}.$$

Appendix E. Derivation of (3.35). We first note that by (3.13), (3.28), (J.1), and using $\beta \sim a^2$, we have that

$$C(z) = \frac{a^2-\beta}{L^2} = \frac{-a_\zeta}{L^2} \sim \frac{\nu(\beta)}{2\sqrt{\beta}L^2}.$$

Using this and (3.35), we have that

$$\frac{C(z)(Z_c-z)^2}{2\sqrt{\beta}(Z_c-z)} \sim \frac{\nu(\beta)(Z_c-z)}{4\beta L^2} \sim \frac{\nu(\beta)}{8\beta^{3/2}} \ll 1,$$

showing the consistency of the adiabatic law (3.35) being the limit of the adiabatic law (3.34) near the singularity.

Appendix F. Solvability conditions for V_1. In the derivation of the reduced equations from a solvability condition for V_1 we use the following result.

LEMMA F.1. *Let* $V_1 = S + iT$ *be the solution of*

$$\Delta_\perp V_1 - V_1 + 2R^2 V_1 + R^2 V_1^* = p(x,y) + iq(x,y), \tag{F.1}$$

where S, T, p, q *are real and* $R(r)$ *is the positive solution of* $\Delta_\perp R + R^3 - R = 0$. *Then the solvability condition for* S *is that* $\int p \nabla_\perp R = 0$ *and the solvability condition for* T *is that* $\int qR = 0$.

From Lemma F.1, Corollary F.2 immediately follows.

COROLLARY F.2. *If* p *is an even function, the equation for the real part of* V_1 *in* (F.1) *is always solvable.*

The proof of Lemma F.1 follows from the following result, which is given in [77] but not proved there for L_+ for the 2D case. Here we give a proof which can be generalized to all dimensions and powers of nonlinearity.

LEMMA F.3. *Let*

$$L_+ = (\Delta_\perp + 3R^2 - 1), \quad L_- = (\Delta_\perp + R^2 - 1),$$

where

$$\Delta_\perp = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr},$$

be operators on

$$B = \left\{ f \in C^2[0\ \infty) \mid f_r(0) = 0,\ f(\infty) = 0 \right\}.$$

Then

1. L_+ *is a self-adjoint operator with null space* $N(L_+) = span\{R_r\}$;
2. L_- *is a self-adjoint operator with null space* $N(L_-) = span\{R\}$.

Proof. We can easily see that R_r is in the null space of L_+ by differentiating the equation for R. Hence, we can use R_r to find the second independent solution u by considering

$$u = vR_r.$$

From $L_+ u = 0$, the equation for v is

$$2v_r(R_r)_r + v_{rr}R_r + \frac{d-1}{r} v_r R_r = 0.$$

This equation can be solved easily, and we get that

$$u = R_r \int^r \frac{1}{(r')^{d-1}[R_r]^2}\, dr'.$$

For large r, $R \sim r^{-1/2} e^{-r}$ and u diverges. Hence, u is not in $N(L_+)$.

The proof for L_- is similar.

Appendix G. Proof of Proposition 4.2. When $f_2 \not\equiv 0$, dimensional argument shows that

$$\frac{[(f_1)_z]}{[f_2]} = \frac{[L^2]}{[Z]} \sim \beta^{1/2} \ll 1.$$

Therefore, the leading order behavior of (4.3) is given by (4.8). Since in this case the accuracy of the approximation is $O(\beta^{1/2})$, there is no point in keeping the $\nu(\beta)$ radiation term.

When $f_2 \equiv 0$ (4.3) becomes (4.16). For leading order we can neglect the $\nu(\beta)$ term and integrate (4.16) to get (4.7).

Appendix H. Proof of Proposition 4.3. If (4.9) holds, we can multiply (4.7) by $-2L_z/L^3$ and integrate to get

$$L_z^2 = \frac{\beta_0}{L^2} - \frac{\epsilon C_1}{4M}\frac{1}{L^4} + D, \quad D = \text{constant}, \tag{H.1}$$

or

$$y_z^2 = 4\beta_0 - \frac{\epsilon C_1}{M}\frac{1}{y} + 4Dy. \tag{H.2}$$

Although the value of D can be obtained directly from (H.1), it is more instructive to obtain it by deriving (H.2) from Hamiltonian balance. To do so, we multiply (4.1) by ψ_z^*, add the conjugate equation, and integrate to get an equation for balance of Hamiltonian in (4.1):

$$\frac{\partial}{\partial z}H(\psi) = \frac{\epsilon}{2\pi}\int [\psi_z^* F(\psi) + c.c.]\, dxdy. \tag{H.3}$$

The right-hand side of (H.3) can be approximated using (3.12), (3.17), and (4.23):

$$\frac{1}{2\pi}\int [\psi_z^* F(\psi) + c.c.]\, dxdy \sim \left(\frac{1}{2L^2}\right)_z f_1 + \frac{2}{L^2} f_2\ .$$

Therefore, in the generic conservative case (4.9), (H.3) reduces to

$$H_z \sim -\frac{\epsilon C_1}{4}\left(\frac{1}{L^4}\right)_z .$$

Simple integration gives

$$H = H_0 - \frac{\epsilon C_1}{4}\frac{1}{L^4}\ , \quad H_0 \sim H(0) + \frac{\epsilon C_1}{4}\frac{1}{L_0^4}\ . \tag{H.4}$$

We note that

$$H(\psi) = H(\psi_s) + H(\psi_{back}), \quad H(\psi_s) \sim ML_z^2 + \frac{H(V_0)}{L^2}.$$

In addition, from Lemmas A.1 and B.1 we have

$$H(V_0) \sim -\beta M + \frac{1}{2}\epsilon f_1.$$

Therefore,[12]

$$H_s \sim \frac{M}{2}(L^2)_{zz} + \frac{\epsilon f_1}{2L^2}, \tag{H.5}$$

[12] This is true except when $(f_1)_z = f_2 \equiv 0$, in which case $(L^2)_{zz}$ is as small as the terms neglected in Lemma B.1.

which in the case of (4.9) becomes

$$H(\psi_s) \sim \frac{M}{2}(L^2)_{zz} - \frac{\epsilon C_1}{2L^4}. \tag{H.6}$$

Substituting (H.6) in (H.4), multiplying by $4y_z/M$, and integrating again gives

$$y_z^2 = -\frac{\epsilon C_1}{M}\frac{1}{y} + \frac{4H_0}{M}y + \text{constant}.$$

Comparison of this equation with (H.2) gives (4.10), which can be rewritten as (4.11).

1. From (4.10) we see that if $\epsilon C_1 > 0$, then y (or L) cannot go to zero.
   (a) If $\beta_0 > 0$ and $H_0 < 0$, then $0 < y_m < y_M$. To evaluate ΔZ we note that

   $$\Delta Z = \sqrt{\frac{M}{-H_0}} \int_{y_m}^{y_M} \sqrt{\frac{y}{(y_M - y)(y - y_m)}}\, dy.$$

   Substituting $(y - y_m)/(y_M - y_m) = \cos^2 u$ gives (4.15).
   (b) When $\beta_0 > 0$ and $H_0 > 0$, (4.10) can be written as

   $$y_z^2 = \frac{4|H_0|}{M}\frac{1}{y}(y + |y_M|)(y - y_m).$$

2. In this case $y_m < 0$.
3. The location of (first) arrest is

$$z_0 = -\int_{y(0)}^{y_m} \frac{1}{2}\sqrt{\frac{M}{-H_0}}\sqrt{\frac{y}{(y_M - y)(y - y_m)}}\, dy \sim \frac{1}{2}\int_0^{y(0)} \left(\beta_0 + \frac{H_0}{M}y\right)^{-1/2} dy = Z_c.$$

   When ψ_0 is real, then $L_z(0) = 0$, and $y(0) = y_M$.

Appendix I. Proof of Proposition 4.4. In this appendix we estimate the value of

$$\Delta N_s \sim -M \int_z^{z+\Delta Z} \frac{\nu(\beta)}{L^2}\, dz,$$

following [25]. (Actually, the expression derived here is somewhat more accurate than the one in [25].) We first note that from (J.1) and (4.11) we have

$$\beta = \frac{1}{4}(y_z)^2 - \frac{1}{2}yy_{zz} = \beta_M \left(1 - 2\frac{y_M/y - 1}{y_M/y_m - 1}\right), \tag{I.1}$$

where

$$\beta_M := \beta(z_M) = -\frac{H_0}{M}(y_M - y_m)$$

and z_M is the location such that

$$y(z_M) = y_M \qquad \text{and} \qquad z \le z_M \le z + \Delta Z.$$

Let us rewrite ΔN_s as

$$\Delta N_s \sim -M\nu(\beta_M) \int_z^{z+\Delta Z} \frac{1}{y(z)} \exp[\lambda_M h(z)]\, dz,$$

where

$$\lambda_M := \frac{\pi}{\sqrt{\beta_M}}\ , \quad h(z) = 1 - \sqrt{\beta_M/\beta}.$$

Since $\lambda_M \gg 1$, we can approximate the integral using Laplace's method for integrals (e.g., [11, 54]):

$$\int_z^{z+\Delta Z} \frac{1}{y(z)} \exp[\lambda_M h(z)]\, dz \sim \frac{1}{y_M} \lambda_M^{-1/2} \sqrt{\frac{-\pi}{2h_{zz}(z_M)}}.$$

Since $\beta_z(z_M) = 0$,

$$h_{zz}(z_M) = \frac{1}{2\beta_M} \beta_{zz}(z_M).$$

Similarly, since $y_z(z_M) = 0$, from (I.1) we have

$$\beta_{zz}(z_M) = \frac{2\beta_M y_m}{y_M^2} y_{zz}(z_M).$$

Differentiating (4.11) gives

$$y_{zz}(z_M) = \frac{2H_0}{M} \left(\frac{y_M - y_m}{y_M} \right) = -\frac{2\beta_M}{y_M}.$$

Therefore,

$$h_{zz}(z_M) = -\frac{2\beta_M y_m}{y_M^3}$$

and ΔN_s is given by (4.17).

Appendix J. Useful relations. The following relations are useful in analysis of the reduced modulation equations:

(J.1) $$\beta = -L^3 L_{zz} = \frac{1}{4}(y_z)^2 - \frac{1}{2} y y_{zz} = \frac{A_{\zeta\zeta}}{A} = a^2 + a_\zeta,$$

(J.2) $$\beta_z = -\frac{1}{2} y y_{zzz},$$

where

$$y = L^2, \quad A = \frac{1}{L}, \quad a = -LL_z = \frac{-L_\zeta}{L}.$$

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