Explicit solutions of the Bass and susceptible-infected models on hypernetworks

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We analyze the Bass model and the susceptible-infected model on hypernetworks with three-body interactions. We derive the master equations for general hypernetworks and use them to obtain explicit expressions for the expected adoption-infection level on infinite complete hypernetworks, infinite Erdős-Rényi hypernetworks, and on infinite hyperlines. These expressions are exact, as they are derived without making any approximation.

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I. INTRODUCTION

Spreading processes on networks have been studied in various research areas, including mathematics, physics, marketing, computer science, and sociology [1–3]. In marketing, the study of the diffusion of innovations began with the seminal work of Bass in 1969 [4], which inspired a huge body of theoretical and empirical research [5]. In epidemiology, mathematical models have been used to study the spread of infectious diseases in social networks [6–9]. A key question in these studies has been the role that the network structure plays in the spreading process.

In the Bass and susceptible-infected (SI) models on networks, the overall rate of peer influences on a susceptible individual is the *sum* of the influence rates exerted by their peers which are adopters or infected. This assumption is a reasonable starting point. In some cases, however, it is more realistic to use a *threshold model* in which the decision to adopt the product takes place only if the number of adopters exceeds a certain threshold at which the net benefits for adopting the product begin to exceed the net costs [10]. In other cases, the marginal influence of an adopter may be a decreasing function of the number of adopters who have already influenced the nonadopter.

In order to allow for a nonlinear dependence of the overall rate of peer influence on the individual peer influences, it is natural to model these processes on hypernetworks [11]. Indeed, in recent years, extensive research has been devoted to spreading processes on hypernetworks [12–14]. For example, Palafox-Castillo *et al.* [15] used the mean-field approach to analyze the steady state of the SIR model on simplicial complexes. Matamalas *et al.* [16] used a microscopic Markov chain approximation to find abrupt phase transitions in the SIS model on simplicial complexes. Kim *et al.* [17] used the facet approximation on random nested hypernetworks to compute the steady state in the SIS model and showed that the hyperedge-nestedness affects the phase diagram significantly. Arenas *et al.* [18] used the triadic approximation in the SIS model on hypernetworks to demonstrate the double-edged effect of increased overlap between two- and three-body interactions: it decreases the invasion threshold, but also results in generally smaller outbreaks. Iacopini *et al.* [19] used the mean-field approximation in the SIS model on simplicial complexes and found discontinuous transitions in the steady state and the emergence of bistable regions where both healthy and endemic states coexist. Higham and de Kergorlay [20] used the mean-field approximation in the SIS model on hypernetworks to obtain spectral conditions for the local asymptotic stability of the zero-infection state. Arruda *et al.* [21] used the assumption that the variables describing the infected state of the nodes are independent in the SIS model on hypernetworks to show that the model exhibits a vast parameter space, including first- and second-order transitions, bistability, and hysteresis. Bianconi [22] used the mean-field approximation to derive macroscopic equations in the SIS model on simplicial complexes and found discontinuous transitions and bistability regions.

In this paper, we analyze the Bass and SI models on hypernetworks. To simplify the presentation, we only consider hypernetworks with pure three-body interactions. The extension of our results and methodology to more general hypernetworks is straightforward. We first derive the master equations for general hypernetworks with pure three-body interactions. We then solve these equations explicitly and obtain explicit expressions for the expected adoption-infection level as a function of time, for infinite complete hypernetworks, infinite Erdős-Rényi hypernetworks, and infinite hyperlines. These expressions are exact, as they are derived without making any approximation. In all cases, we present careful numerical simulations that confirm the validity of the explicit expressions.

Our work differs from previous studies in several aspects.

(i) We obtain explicit expressions for the expected adoption-infection level as a function of time, whereas previous studies focused more on the limiting steady state. These expressions allow us to address questions such as, e.g., the time for half of the population to become infected.

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(ii) The availability of explicit solutions simplifies the analysis, since it is much easier to analyze an explicit solution than to analyze the original stochastic network model.

(iii) From a methodological perspective, we start from the full system of the master equations, and solve it exactly, without applying any approximate closure at the level of pairs or triplets. As a result, the explicit solutions that we obtain are exact and not approximate.

(iv) We analyze the SI model and the Bass model using a unified framework.

The paper is organized as follows. In Sec. II we define the Bass and SI models in three-body hypernetworks. In Sec. III we derive the master equations that describe the dynamics of diffusion and infection across hypernetworks. In Sec. IV we derive explicit solutions to the master equations for complete three-body hypernetworks. In Sec. V we explore the initial dynamics of the expected adoption level in the Bass model on arbitrary three-body hypernetworks and prove that the adoption rate initially decreases, regardless of the hypernetwork structure or parameters. This is the only case where we observe a qualitative difference between the spreading dynamics on networks and on hypernetworks. In Sec. VI we examine the spreading dynamics on Erdős-Rényi hypernetworks. In Sec. VII we derive explicit solutions to the master equations for infinite three-body hyperlines, supplemented by numerical simulations that validate the theoretical results. Section VIII presents comparisons between our exact solutions and a mean-field approximation. Section IX concludes with some final remarks and suggestions for extensions of this study.

II. BASS-SI MODEL ON THREE-BODY HYPERNETWORKS

The Bass model describes the adoption of new products or innovations within a population. In this framework, all individuals start as nonadopters and can transition to becoming adopters due to two types of influences: external factors, such as exposure to mass media, and internal factors where individuals are influenced by their peers who have already adopted the product. The SI model is used to study the spreading of infectious diseases within a population. In this model, some individuals are initially infected (the "patient zero" cases) and all subsequent infections occur through internal influences, whereby infected individuals transmit the disease to their susceptible peers and infected individuals remain contagious indefinitely. In both models, once an individual becomes an adopter or infected, it remains so at all later times. In particular, she or he remain "contagious" forever. It is convenient to unify these two models into a single model, the Bass-SI model on networks, as follows. The difference between the SI model and the Bass model is the lack of external influences in the former and the lack of "adopters zero" in the latter.

Consider M individuals, denoted by $\mathcal{M} := \{1, \ldots, M\}$. We denote by $X_j(t)$ the state of individual j at time t, so that

$$X_j(t) = \begin{cases} 1, & \text{if } j \text{ is adopter or infected at time } t, \\ 0, & \text{otherwise}, \end{cases} \qquad j \in \mathcal{M}.$$

The initial conditions at $t = 0$ are stochastic, so that

$$X_j(0) = X_j^0 \in \{0, 1\}, \quad j \in \mathcal{M}, \tag{1a}$$

where

$$\mathbb{P}\left(X_j^0 = 1\right) = I_j^0, \quad \mathbb{P}\left(X_j^0 = 0\right) = 1 - I_j^0, \quad I_j^0 \in [0, 1], \quad j \in \mathcal{M}, \tag{1b}$$

and

$$\text{the random variables } \left\{X_j^0\right\}_{j \in \mathcal{M}} \text{ are independent.} \tag{1c}$$

Deterministic initial conditions are a special case where $I_j^0 \in \{0, 1\}$. As long as j is a nonadopter or susceptible, its adoption or infection rate at time t is[1]

$$\lambda_j(t) = p_j + \sum_{k_1, k_2 = 1}^{M} q_{k_1, k_2, j} X_{k_1}(t) X_{k_2}(t), \quad j \in \mathcal{M}. \tag{1d}$$

Here, $p_j \geqslant 0$ is the rate of external influences on j and $q_{k_1,k_2,j} \geqslant 0$ is the rate of internal influences by k_1 and k_2 on j, provided that k_1 and k_2 are already adopters or infected. In addition, $q_{k_1,k_2,j} > 0$ if k_1, k_2 and j are distinct and the directional hyperedge $\{k_1, k_2\} \to j$ exists. Otherwise, $q_{k_1,k_2,j} = 0$.

Once j becomes an adopter or infected, it remains so at all later time.[2] Hence, as $\Delta t \to 0$,

$$\mathbb{P}[X_j(t + \Delta t) = 1 \mid \mathbf{X}(t)] = \begin{cases} \lambda_j(t)\,\Delta t, & \text{if } X_j(t) = 0, \\ 1, & \text{if } X_j(t) = 1, \end{cases} \quad j \in \mathcal{M}, \tag{1e}$$

where $\mathbf{X}(t) := \{X_j(t)\}_{j \in \mathcal{M}}$ is the state of the network at time t and

$$\text{the random variables } \{X_j(t+\Delta t) \mid \mathbf{X}(t)\}_{j \in \mathcal{M}} \text{ are independent.} \tag{1f}$$

In the Bass model there are no adopters when the product is first introduced into the market and so $I_j^0 \equiv 0$ and $p_j > 0$ for $j \in \mathcal{M}$. In the SI model there are only internal influences for $t > 0$ and so $p_j \equiv 0$ and $I_j^0 > 0$ for $j \in \mathcal{M}$.

The quantity of most interest is the expected adoption-infection level

$$f(t) := \frac{1}{M} \sum_{j=1}^{M} f_j(t), \tag{2}$$

where $f_j := \mathbb{E}[X_j]$ is the adoption-infection probability of node j.

[1] For comparison, the adoption-infection rate on two-body networks is $\lambda_j = p_j + \sum_{k=1}^{M} q_{k,j} X_k(t)$.

[2] That is, the only admissible transition is $X_j = 0 \to X_j = 1$.

III. MASTER EQUATIONS

The most important analytic tool for the Bass-SI model on networks are the master equations, which were derived in [23]. In this section, we derive the master equations for general hypernetworks with three-body interactions. Let $\Omega \subset \mathcal{M}$ be a nontrivial subset of the nodes, let $\Omega^c := \mathcal{M} \setminus \Omega$, and let

$$S_\Omega(t) := \{X_m(t) = 0, m \in \Omega\}, \quad [S_\Omega](t) := \mathbb{P}(S_\Omega(t))$$

denote the event that all the nodes in Ω are nonadopters or susceptibles at time t and the probability of this event, respectively. In what follows, we will use the notations

$$\begin{gathered}[S_{\Omega,k_1}] := [S_{\Omega\cup\{k_1\}}], \quad [S_{\Omega,k_1,k_2}] := [S_{\Omega\cup\{k_1,k_2\}}],\\ p_\Omega := \sum_{m\in\Omega} p_m, \quad q_{k_1,k_2,\Omega} := \sum_{m\in\Omega} q_{k_1,k_2,m},\end{gathered} \tag{3}$$

where $k_1, k_2 \in \Omega^c$.

Theorem 1. The master equations for the Bass-SI model (1) on three-body hypernetworks are

$$\begin{aligned}\frac{d[S_\Omega]}{dt} = &-\left(p_\Omega + \sum_{k_1,k_2\in\Omega^c} q_{k_1,k_2,\Omega}\right)[S_\Omega]\\ &+ \sum_{k_1,k_2\in\Omega^c} q_{k_1,k_2,\Omega}\big([S_{\Omega,k_1}] + [S_{\Omega,k_2}] - [S_{\Omega,k_1,k_2}]\big),\end{aligned} \tag{4a}$$

subject to the initial conditions

$$[S_\Omega](0) = \prod_{m\in\Omega}\big(1 - I_m^0\big), \tag{4b}$$

for all $\emptyset \neq \Omega \subset \mathcal{M}$.

Proof. Consider the average over an infinite number of realizations of the Bass-SI model (1). By definition, the event $S_\Omega(t)$ occurs at a fraction $[S_\Omega](t)$ of these realizations. Since the only allowed transition is $S \to I$, new S_Ω realizations cannot be created and the existing S_Ω realizations are destroyed whenever any of the nodes in Ω adopts. The adoption rate of node $m \in \Omega$ is $\lambda_m(t)$; see (1d).

(i) Thus an existing S_Ω realization is destroyed if node m adopts externally. Since there are $[S_\Omega]$ such realizations, this external influence leads to a reduction in $[S_\Omega]$ at the rate of $p_m[S_\Omega]$.

(ii) An existing S_Ω realization is also destroyed if node m adopts as a result of an internal influence by some nodes $k_1, k_2 \in \Omega^c$. For this to occur, at time t all nodes of Ω should be nonadopters and nodes $k_1, k_2 \in \Omega^c$ should be adopters. Denote this event by $S_\Omega(t) \cap I_{k_1,k_2}(t)$ and the probability of this event by $[S_\Omega \cap I_k](t)$. Since there are $[S_\Omega \cap I_{k_1,k_2}](t)$ such realizations, this external influence leads to a reduction in $[S_\Omega]$ at the rate of $q_{k_1,k_2,m}[S_\Omega \cap I_{k_1,k_2}]$.

Therefore, the rate of change in $[S_\Omega]$ due to external or internal influences on node m is

$$-p_m[S_\Omega] - \sum_{k_1,k_2\in\Omega^c} q_{k_1,k_2,m}[S_\Omega \cap I_{k_1,k_2}].$$

The overall rate of change in $[S_\Omega]$ is the sum of the rates of change in $[S_\Omega]$ due to the adoptions of all nodes $m \in \Omega$. Hence

$$\frac{d[S_\Omega]}{dt} = -p_\Omega[S_\Omega] - \sum_{k_1,k_2\in\Omega^c} q_{k_1,k_2,\Omega}[S_\Omega \cap I_{k_1,k_2}]. \tag{5}$$

In order to express the master equations using only nonadoption probabilities, we first write S_Ω as the union of four disjoint sets,

$$S_\Omega = S_{\Omega,k_1,k_2} \cup (S_{\Omega,k_1} \cap I_{k_2}) \cup (S_{\Omega,k_2} \cap I_{k_1}) \cup (S_\Omega \cap I_{k_1,k_2}).$$

Therefore,

$$[S_\Omega] = [S_{\Omega,k_1,k_2}] + [S_{\Omega,k_1} \cap I_{k_2}] + [S_{\Omega,k_2} \cap I_{k_1}] + [S_\Omega \cap I_{k_1,k_2}].$$

In addition, S_{Ω,k_1} can be written as the union of two disjoint sets,

$$S_{\Omega,k_1} = S_{\Omega,k_1,k_2} \cup (S_{\Omega,k_1} \cap I_{k_2}).$$

Hence

$$[S_{\Omega,k_1}] = [S_{\Omega,k_1,k_2}] + [S_{\Omega,k_1} \cap I_{k_2}].$$

Similarly,

$$[S_{\Omega,k_2}] = [S_{\Omega,k_1,k_2}] + [S_{\Omega,k_2} \cap I_{k_1}].$$

Combining the above, we have that

$$[S_\Omega \cap I_{k_1,k_2}] = [S_\Omega] - [S_{\Omega,k_1}] - [S_{\Omega,k_2}] + [S_{\Omega,k_1,k_2}]. \tag{6}$$

Equation (4a) follows from (5) and (6) and (4b) follows from (1). ■

In general, there are $2^M - 1$ master equations in (4), for all possible subsets $\Omega \subset \mathcal{M}$. Therefore, obtaining an explicit solution for general hypernetworks is not practical. In the following, we will obtain a considerably smaller reduced system of master equations for some special hypernetworks.

IV. COMPLETE HYPERNETWORKS

In [24,25], it was shown that the expected adoption-infection level in the Bass-SI model on infinite complete homogeneous networks is the solution of the equation

$$\frac{df}{dt} = (1 - f)(p + qf), \quad f(0) = I^0, \tag{7}$$

and is given by $f_{\rm Bass}(t) := \frac{1-e^{-(p+q)t}}{1+\frac{q}{p}e^{-(p+q)t}}$ for the Bass model and by $f_{\rm SI}(t) := \frac{I^0}{e^{-qt}+(1-e^{-qt})I^0}$ for the SI model. In this section, we adopt a similar approach and compute explicitly, without making any approximation, the infinite-population limit of the Bass-SI model on complete three-body hypernetworks, where

$$\begin{aligned}&I_j^0 \equiv I^0, \quad p_j \equiv p, \quad q_{k_1,k_2,j} = \frac{q}{\binom{M-1}{2}} \mathbb{1}_{k_1\neq k_2,\, j\neq k_1,\, j\neq k_2},\\ &j, k_1, k_2 \in \mathcal{M},\end{aligned} \tag{8a}$$

$p > 0$, $q > 0$, and $I^0 = 0$ in the Bass model and $p = 0$, $q > 0$, and $0 < I^0 < 1$ in the SI model. The adoption rate of j is, see (1d),

$$\begin{aligned}\lambda_j^{\rm complete}(t) &= p + \frac{q}{\binom{M-1}{2}} \sum_{k_1,k_2=1}^{M} \mathbb{1}_{k_1\neq k_2,\, j\neq k_1,\, j\neq k_2} X_{k_1}(t)X_{k_2}(t)\\ &= p + \frac{q}{\binom{M-1}{2}}\binom{N(t)}{2},\end{aligned} \tag{8b}$$

where $N(t) = \sum_{j=1}^{M} X_j(t)$ is the number of adopters or infected in the population.

Theorem 2. Let $f^{\text{complete}}(t; M)$ denote the expected adoption-infection level in the Bass-SI model [(1) and (8)] on complete three-body hypernetworks. Then $\lim_{M\to\infty} f^{\text{complete}} = f^{\text{compart}}$, where f^{compart} is the solution of the compartmental three-body Bass-SI model

$$\frac{df}{dt} = (1-f)(p+qf^2), \quad f(0) = I^0. \tag{9}$$

Furthermore, $f^{\text{compart}}(t)$ is given by the explicit inverse formula

$$(p+q)t = \sqrt{\frac{q}{p}}\left(\tan^{-1}\left(\sqrt{\frac{q}{p}}f^{\text{compart}}\right) - \tan^{-1}\left(\sqrt{\frac{q}{p}}I^0\right)\right) + \ln\left(\frac{1-I^0}{1-f^{\text{compart}}}\right) + \frac{1}{2}\ln\left(\frac{p+q\left(f^{\text{compart}}\right)^2}{p+q(I^0)^2}\right), \quad \text{if } p>0, \tag{10a}$$

and

$$qt = \ln\left(\frac{f^{\text{compart}}}{1-f^{\text{compart}}}\frac{1-I^0}{I^0}\right) + \frac{1}{I^0} - \frac{1}{f^{\text{compart}}}, \quad \text{if } p=0. \tag{10b}$$

Proof. Because of the symmetry of the hypernetwork, $[S_\Omega]$ only depends on the number of nodes in Ω and not on the specific choices of nodes in Ω. Therefore, we can denote by

$$[S^n] := [S_\Omega \mid |\Omega| = n] \tag{11}$$

the probability that all the nodes in any specific subset of n nodes are nonadopters (susceptibles) at time t. Substituting (8) and (11) in the master Eq. (4a) gives

$$\frac{d[S^n]}{dt} = -n\left(p+q\frac{(M-n)(M-n-1)}{(M-1)(M-2)}\right)[S^n] + nq\frac{(M-n)(M-n-1)}{(M-1)(M-2)}(2[S^{n+1}]-[S^{n+2}]), \quad n=1,\dots,M-2, \tag{12a}$$

$$\frac{d[S^n]}{dt} = -np[S^n], \quad n = M-1, M. \tag{12b}$$

Holding n fixed and letting $M\to\infty$, Eq. (12) approaches

$$\frac{d[S^n]}{dt} = -n(p+q)[S^n] + nq(2[S^{n+1}]-[S^{n+2}]), \quad [S^n](0) = (1-I^0)^n, \quad n=1,2\dots. \tag{13}$$

The substitution $[S^n] = [S]^n$ reduces the infinite system (13) to the single ODE

$$\frac{d[S]}{dt} = -(p+q)[S] + q(2[S]^2-[S]^3) = -[S](p+q(1-2[S]+[S]^2)), \quad [S](0) = 1-I^0. \tag{14}$$

Substituting $f = 1-[S]$ gives (9). Using partial fractions,

$$\frac{1}{(1-f)(p+qf^2)} = \frac{1}{p+q}\left(q\frac{1+f}{p+qf^2} + \frac{1}{1-f}\right).$$

Integrating and using $f(0) = I^0$ gives (10a). Taking the limit $p\to 0^+$ in (10a) gives (10b). ■

In Fig. 1(a) we compute numerically the expected adoption level in the Bass model on a complete three-body hypernetwork with $M = 5000$ nodes. The result is nearly indistinguishable from the explicit solution (10a) on an infinite hypernetwork. A similar agreement is observed in Fig. 1(b) for the expected infection level in the SI model.

We note that, from Eq. (10), it follows that the time for half of the population to become adopters (infected) in the Bass-SI model on infinite complete hypernetworks is

$$T_{\frac{1}{2}} = \frac{1}{p+q}\left(\sqrt{\frac{q}{p}}\left[\tan^{-1}\left(\frac{1}{2}\sqrt{\frac{q}{p}}\right) - \tan^{-1}\left(\sqrt{\frac{q}{p}}I^0\right)\right] + \ln\left(\frac{1-I^0}{1-\frac{1}{2}}\sqrt{\frac{p+\frac{q}{4}}{p+q(I^0)^2}}\right)\right).$$

For example, in the Bass model $I^0 = 0$, and so

$$T_{\frac{1}{2}}^{\text{Bass}} = \frac{1}{p+q}\left(\sqrt{\tilde{q}}\left(\tan^{-1}\left(\frac{1}{2}\sqrt{\tilde{q}}\right)\right) + \ln\left(2\sqrt{1+\frac{1}{4}\tilde{q}}\right)\right), \quad \tilde{q} := \frac{q}{p}.$$

In the SI model $p = 0$, and so

$$T_{\frac{1}{2}}^{\text{SI}} = \frac{1}{q}\left(\ln\left(\frac{1-I^0}{I^0}\right) + \frac{1}{I^0} - 2\right).$$

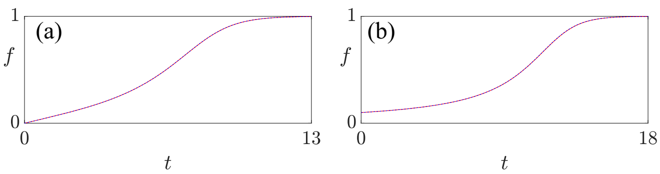

FIG. 1. (a) Expected adoption level $f^{\text{complete}}(t)$ in the Bass model (1, 8) on a complete three-body hypernetwork with $M = 5000$ nodes (solid red line) is nearly indistinguishable from the explicit expression (10) for f^{compart} (blue dashed line). Here $p = 0.05$, $q = 1$, and $I^0 = 0$. (b) Same as (a) for the SI model with $p = 0$ and $I^0 = 0.1$.

V. INITIAL DYNAMICS (BASS MODEL)

The expected adoption level in the Bass model on infinite complete hypernetworks is given by (9), with $p, q > 0$. Therefore, $\frac{df}{dt}$ *initially decreases* from $\frac{df}{dt}(0) = p$ to a local minimum, then increases to a global maximum, and finally decays to zero [Fig. 2(a)]. This initial dynamics is qualitatively different from that on two-body infinite complete networks, where $\frac{df}{dt}$ is an inverted parabola in f; see (7). In particular, if $q > p$, $\frac{df}{dt}$ increases from $\frac{df}{dt}(0) = p$ to a global maximum and then decays to zero [Fig. 2(b)].

The initial decline of the adoption rate is not limited to complete hypernetworks. Rather, it occurs for all three-body hypernetworks.

Theorem 3. Consider the Bass model (1) on a three-body hypernetwork. Then

$$f'(0) > 0, \quad f''(0) < 0.$$

Proof. Substituting $\Omega = \{j\}$ in the master Eq. (4a) gives

$$\frac{d[S_j]}{dt} = -\,p_j[S_j] - \sum_{k_1,k_2=1}^{M} q_{k_1,k_2,j}\big([S_j] - [S_{k_1,j}] - [S_{k_2,j}] + [S_{k_1,k_2,j}]\big). \tag{15}$$

Substituting $t = 0$ in (15) and using the initial conditions $[S_j](0) = [S_{k_1,j}](0) = [S_{k_2,j}](0) = [S_{k_1,k_2,j}](0) = 1$ gives

$$\frac{d[S_j]}{dt}(0) = -p_j. \tag{16}$$

Similarly,

$$\frac{d[S_{k_1,j}]}{dt}(0) = -p_{k_1} - p_j, \quad \frac{d[S_{k_2,j}]}{dt}(0) = -p_{k_2} - p_j,$$
$$\frac{d[S_{k_1,k_2,j}]}{dt}(0) = -p_{k_1} - p_{k_2} - p_j. \tag{17}$$

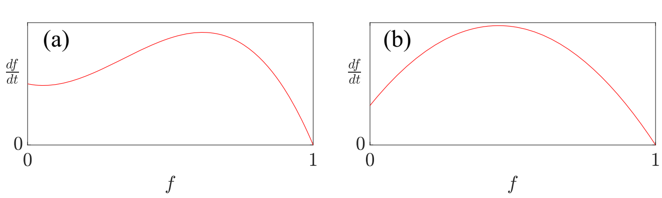

FIG. 2. (a) $\frac{df}{dt}$ as a function of f for the Bass model (1,8) on an infinite complete three-body hypernetwork. Here, $\frac{q}{p} = 20$ and $I^0 = 0$. (b) Same as (a) on an infinite complete two-body network.

Differentiating (15), substituting $t = 0$, and using (16) and (17) yields

$$\begin{aligned}\frac{d^2[S_j]}{dt^2}(0) &= -p_j\frac{d[S_j]}{dt}(0) - \sum_{k_1,k_2=1}^{M} q_{k_1,k_2,j}\frac{d}{dt}\big([S_j] - [S_{k_1,j}] \\ &\quad - [S_{k_2,j}] + [S_{k_1,k_2,j}]\big)\big|_{t=0} \\ &= p_j^2 - \sum_{k_1,k_2=1}^{M} q_{k_1,k_2,j}\big(-p_j + p_{k_1} + p_j + p_{k_2} \\ &\quad + p_j - p_{k_1} - p_{k_2} - p_j\big) = p_j^2.\end{aligned}$$

Since $f = 1 - \frac{1}{M}\sum_{j=1}^{M}[S_j]$,

$$\frac{df}{dt}\bigg|_{t=0} = -\frac{1}{M}\sum_{j=1}^{M}\frac{d}{dt}[S_j]\bigg|_{t=0} = \frac{1}{M}\sum_{j=1}^{M} p_j > 0$$

and

$$\frac{d^2 f}{dt^2}\bigg|_{t=0} = -\frac{1}{M}\sum_{j=1}^{M}\frac{d^2}{dt^2}[S_j]\bigg|_{t=0} = -\frac{1}{M}\sum_{j=1}^{M} p_j^2 < 0.$$

■

Since $f'(0) > 0$ and $f''(0) < 0$, the adoption rate decreases initially on all three-body hypernetworks, regardless of the hypernetwork structure or of the ratio $\frac{q}{p}$.

In order to understand the difference in the initial dynamics in the Bass model between networks and hypernetworks, we note that as the adoption level f increases, the adoption rate $\frac{df}{dt}$ is influenced by two opposing mechanisms: (1) the rate of internal influences increases and (2) the rate of external influences decreases (since there are fewer nonadopters).

The initial decrease of external adoptions occurs on both two-body and three-body networks and is captured by $f''(0)$. On two-body networks, the initial increase of internal adoptions is also captured by $f''(0)$; see [23, Eq. (20)]. Therefore, the sign of $f''(0)$ depends on $\frac{q}{p}$. On three-body hypernetworks, however, a hyperedge becomes active only after two nodes adopt. Therefore, the initial increase of internal influence is only captured by $f'''(0)$. As a result, $f''(0)$ is always negative.

To elucidate the underlying reasons for this difference between networks and hypernetworks, let us revisit the spreading dynamics on infinite complete networks. In the case of two-body networks $\frac{df}{dt} = p - pf + qf - qf^2$, see (7), and for three-body hypernetworks $\frac{df}{dt} = p - pf + qf^2 - qf^3$, see (9). Hence the internal adoptions are captured by $f''(0)$ on networks and by $f'''(0)$ on hypernetworks. To the best of our knowledge, the phenomenon of an initial decline in the adoption rate on hypernetworks has not been observed theoretically or empirically. This study shows that if this phenomenon will be observed empirically, this will suggest the presence of a hypernetwork structure.

VI. ERDŐS-RÉNYI HYPERNETWORKS

In Erdős-Rényi (ER) three-body hypernetworks with M nodes, for any three distinct nodes $k_1, k_2, k_3 \in \mathcal{M}$, *the hyperedge* $\{k_1, k_2, k_3\}$ *exists with probability* α, independently of

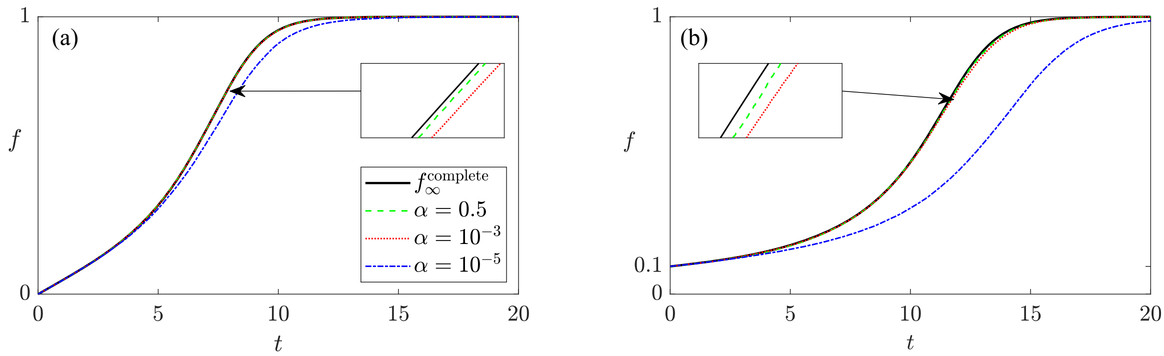

FIG. 3. (a) Expected adoption level f^{ER} in the Bass model (1) on ER hypernetworks with $\alpha = 0.5$ (dashed green line), $\alpha = 10^{-3}$ (red dashed-dotted line), and $\alpha = 10^{-5}$ (blue dotted line). The explicit solution $f^{\text{compart}}(t; p, \alpha q)$ (black solid line), see (10), is nearly identical to f^{ER} with $\alpha = 0.5$ and $\alpha = 10^{-3}$. Here, $M = 1500$, $q = \frac{1}{\alpha}$, $p = 0.05$, and $I^0 = 0$. (b) Same as (a) for the SI model with $p = 0$ and $I^0 = 0.1$.

all other hyperedges. Consider the Bass-SI model (1) on ER hypernetworks, such that

$$I^0_{k_1} = I^0, \quad p_{k_1} \equiv p, \quad q_{k_1,k_2,k_3} = \frac{q}{\binom{M-1}{2}} e_{k_1,k_2,k_3},$$
$$k_1, k_2, k_3 \in \mathcal{M}, \tag{18a}$$

where $\mathbf{E} = (e_{k_1,k_2,k_3})$ is the adjacency tensor, such that $e_{k_1,k_2,k_3} = 1$ if there is a hyperedge connecting $\{k_1, k_2, k_3\}$ and 0 otherwise. The adoption rate of j is thus

$$\lambda_j^{\text{ER}}(t) = p + \frac{q}{\binom{M-1}{2}} \sum_{\{k_1,k_2\}\subset\mathcal{M}} e_{k_1,k_2,j} X_{k_1}(t) X_{k_2}(t). \tag{18b}$$

Lemma 1. Let f^{ER} denote the expected adoption-infection level in the Bass-SI model [(1) and (18)] on infinite ER three-body hypernetworks. Then

$$f^{\text{ER}}(t; p, q, \alpha, I^0) = f^{\text{compart}}(t; p, \alpha q, I^0), \tag{19}$$

where f^{compart} is the solution of (9).

Informal proof. When the ER hypernetwork is large, we can apply the *mean-field* approximation and replace $e_{k_1,k_2,j}$ with its expected value $\mathbb{E}(e_{k_1,k_2,j}) = \alpha$, i.e.,

$$\sum_{\{k_1,k_2\}\subset\mathcal{M}} e_{k_1,k_2,j} X_{k_1}(t) X_{k_2}(t)$$
$$\approx \sum_{\{k_1,k_2\}\subset\mathcal{M}} \alpha X_{k_1}(t) X_{k_2}(t) = \alpha \binom{N(t)}{2}.$$

Therefore, using (8b) and (18b),

$$\lambda_j^{\text{ER}}(t; p, q) \approx \lambda_j^{\text{complete}}(t; p, \alpha q).$$

Hence the result follows from Theorem 2. ■

In Fig. 3 we compare the expected adoption-infection level f^{ER} in the Bass and SI models on ER hypernetworks with $M = 1500$ nodes and the theoretical prediction (19). We let $q := \frac{1}{\alpha}$, so that $f^{\text{compart}}(t; p, \alpha q, I^0)$ remains unchanged as we vary α. These calculations show that f^{ER} closely matches the theoretical prediction (19) for $\alpha = 0.5$, but not for $\alpha = 10^{-5}$. This is to be expected, since the mean-field approximation is derived for dense networks and not for sparse ones. At $\alpha = 10^{-3}$ the hypernetwork is not dense, yet f^{ER} is still in good agreement with (19). This can be attributed to the average hyperdegree $\langle k \rangle = \alpha\binom{M-1}{2} = 10^{-3}\binom{1499}{2} \approx 1100$, which is still large.

VII. INFINITE HYPERLINES

The expected adoption-infection level in the Bass-SI model on an infinite line satisfies [25,26]

$$\frac{df}{dt} = (1 - f)(p + q(1 - e^{-pt})), \quad f(0) = I^0,$$

and is given by the explicit formula $f^{\text{1D}}_{\text{Bass}}(t; p, q) := 1 - e^{-(p+q)t + q\frac{1-e^{-pt}}{p}}$ for the Bass model and by $f^{\text{1D}}_{\text{SI}}(t; q, I^0) := 1 - (1 - I^0)e^{-I^0 qt}$ for the SI model. In this section, we derive the corresponding expressions for the Bass-SI model on the infinite homogeneous three-body hyperline, where

$$I^0_j = I^0, \quad p_j \equiv p, \quad q_{k_1,k_2,j} = \begin{cases} q^{\text{L}}, & \text{if } j = k_1 - 1 = k_2 - 2, \\ q^{\text{R}}, & \text{if } j = k_1 + 1 = k_2 + 2, \\ 0, & \text{otherwise}, \end{cases} \quad k_1, k_2, j \in \mathbb{Z}. \tag{20a}$$

The adoption rate of j is, see (1d),

$$\lambda_j^{\text{1D}}(t) = p + q^L X_{j-1}(t) X_{j-2}(t) + q^R X_{j+1}(t) X_{j+2}(t). \tag{20b}$$

Note that, when $q^{\text{L}} \neq q^{\text{R}}$, the hyperline is anisotropic.

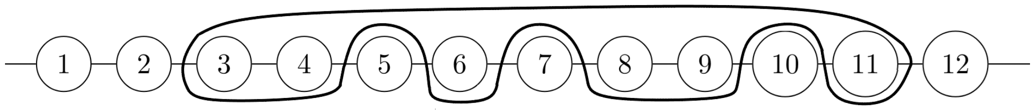

FIG. 4. The set $\Omega_6 = \{3, 4, 6, 8, 9, 11\}$, where $n = 6$, $m_1 = 3$, and $m_6 = 11$. The nodes 5, 7, and 10 cannot affect $[S_{\Omega_6}]$.

Theorem 4. Let f^{1D} denote the expected adoption-infection level in the Bass-SI model (1,20) on the infinite three-body hyperline. Then f^{1D} is the solution of

$$\frac{df^{1D}}{dt} = (p + q[1 - (1 - I^0)e^{-pt}]^2)(1 - f^{1D}), \quad f^{1D}(0) = I^0, \tag{21}$$

where $q = q^L + q^R$, and is given explicitly by

$$f^{1D} = \begin{cases} 1 - [S^0]\exp\left(-(p+q)t + \frac{q}{2p}(1 - e^{-pt})(4 - [S^0] - [S^0]e^{-pt})[S^0]\right), & \text{if } p > 0, \\ 1 - [S^0]e^{-(1-[S^0])^2 qt}, & \text{if } p = 0, \end{cases} \tag{22}$$

and $[S^0] = 1 - I^0$.

Proof. Let $\Omega_n = \{m_1, \ldots, m_n\}$ be a "cluster" of n nodes, such that $m_1 < \cdots < m_n$ and $m_{i+1} - m_i \in \{1, 2\}$ for $i = 1, \ldots, n - 1$. Note that if k is a node such that $m_i < k < m_{i+1}$, then k cannot affect the adoption rate of the nodes in Ω_n (see Fig. 4 for an illustration). Therefore, $[S_{\Omega_n}]$ only depends on n and not on the specific choices of nodes in Ω_n. The master equation for Ω_n reads, see (4) and (20),

$$\frac{d[S_{\Omega_n}]}{dt} = -(np + q^L + q^R)[S_{\Omega_n}] + q^L([S_{\Omega_n, m_1-1}] + [S_{\Omega_n, m_1-2}] - [S_{\Omega_n, m_1-2, m_1-1}]) + q^R([S_{\Omega_n, m_n+1}] + [S_{\Omega_n, m_n+2}] - [S_{\Omega_n, m_n+1, m_n+2}]),$$

subject to $[S_{\Omega_n}](0) = (1 - [I^0])^n$. Therefore, if we denote $[S^n] := [S_{\Omega_n}]$, then

$$\frac{d[S^n]}{dt} = -(np + q)[S^n] + q(2[S^{n+1}] - [S^{n+2}]), \quad [S^n](0) = (1 - I^0)^n, \quad n = 1, 2, \ldots. \tag{23}$$

We introduce the ansatz

$$[S^n] = e^{-(n-1)pt}(1 - I^0)^{n-1}[S^{1D}], \quad n = 1, 2, \ldots,$$

and substitute it in (23) to reduce the infinite system (23) to the single ODE

$$\frac{d[S^{1D}]}{dt} = -(p + q(1 - e^{-pt}(1 - I^0))^2)[S^{1D}], \quad [S^{1D}](0) = 1 - I^0. \tag{24}$$

Substituting $f^{1D} = 1 - [S^{1D}]$ in (24) gives (21). Solving (21) gives (22) for $p > 0$. Taking the limit $p \to 0^+$ of (22) yields (21) for $p = 0$. ■

Figure 5 confirms that the expected adoption level in the Bass and SI models on the three-body hyperline agrees with the theoretical prediction f^{1D}; see (22).

VIII. MEAN-FIELD APPROXIMATION

Consider a homogeneous three-body hypernetwork, where

$$I_j^0 \equiv I^0, \quad p_j \equiv p, \quad q_{k_1,k_2,j} \equiv \frac{q}{\langle k_j \rangle} e_{k_1,k_2,j}, \quad k_1, k_2, j \in \mathcal{M},$$

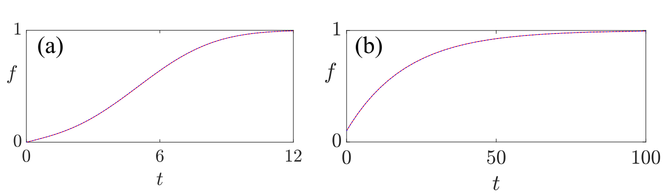

FIG. 5. Expected adoption level (solid red line) in the Bass-SI model [(1) and (20)] on a three-body hyperline is indistinguishable from the theoretical prediction f^{1D} (blue dashed line); see (22). In contrast, the mean-field approximation (black dashed line) provides a poor fit. Here, $M = 5000$, $q^L = 2$, $q^R = 3$, and $q = q^L + q^R$. (a) Bass model: $p = 0.05$ and $I^0 = 0$. (b) SI model: $p = 0$ and $I^0 = 0.1$.

and $\langle k_j \rangle$ is the hyperdegree of j. By (4a), the nonadoption probability of node j is

$$\frac{d[S_j]}{dt} = -\left(p + \frac{q}{\langle k_j \rangle} \sum_{k_1,k_2 \neq j} e_{k_1,k_2,j}\right)[S_j] + \frac{q}{\langle k_j \rangle} \sum_{k_1,k_2 \neq j} e_{k_1,k_2,j}\left(\left[S_{j,k_1}\right] + \left[S_{j,k_2}\right] - \left[S_{j,k_1,k_2}\right]\right), \quad [S_j](0) = 1 - I^0. \tag{25}$$

We apply the mean-field approximation by making the following assumptions.

(1) The states of different nodes are independent, so that $[S_{j,k}] \sim [S_j][S_k]$ and $[S_{j,k_1,k_2}] \sim [S_j][S_{k_1}][S_{k_2}]$.

(2) $[S_k] \sim [S]$ for $k \neq j$.

Applying these assumptions to (25) gives

$$\frac{d[S_j]}{dt} = -(p+q)[S_j] + q(2[S_j][S] - [S_j][S]^2), \quad [S_j](0) = 1 - I^0.$$

Since this equation is independent of j, then $[S_j] \equiv [S]$, where

$$\frac{d[S]}{dt} = -(p+q)[S] + q(2[S]^2 - [S]^3), \quad [S](0) = 1 - I^0.$$

This is the equation for the compartmental three-body Bass-SI model; see (14). Therefore, the mean-field approximation is given by f^{compart}. Therefore, the mean-field approximation is exact on infinite complete hypernetworks and on dense ER hypernetworks, but it performs poorly on sparse networks such as hyperlines, as shown in Fig. 5.

IX. FINAL REMARKS

In this study we derived the master equations for the Bass and SI models on general hypernetworks with three-body interactions. We then used these equations to obtain explicit exact solutions for several types of hypernetworks. In general, both the properties of explicit solutions and the techniques used to derive them mimic those on two-body networks. In fact, the only qualitative difference between the two cases is the initial decline of the adoption rate.

Solving the master equations for large-scale general hypernetworks can be challenging, since the number of master equations grows exponentially with the number of nodes. A potential approach to mitigate these computational challenges is to consider infinite hypernetworks with inherent symmetries. This approach has been successfully applied in this study to the hyperline, complete hypernetworks, and dense ER hypernetworks. We believe that it can be extended to other types of networks, e.g., scale-free, regular, and sparse ER hypernetworks.

Extending our approach to hypernetworks with N-body interactions is straightforward. It is also natural to apply our approach to combinations of higher-order interactions (e.g., two- and three-body interactions). Finally, our methodology can be extended to other models on hypernetworks, such as SIS, SIR, and Bass-SIR; see [8,27,28].

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