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Revenue equivalence in asymmetric auctions

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Abstract

The Revenue Equivalence Theorem is generalized to the case of asymmetric auctions in which each player's valuation is drawn independently from a common support according to his/her distribution function.

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1. Introduction

One of the most fundamental results in auction theory is the Revenue Equivalence Theorem, which states that the expected revenue of the seller in equilibrium is independent of the auction mechanism under quite general conditions. Vickrey [12] established the revenue equivalence of the classical auction mechanisms (first-price auctions, Dutch auctions, English auctions, and second-price auctions). This result was generalized 20 years later by Myerson [10], and independently by Riley and Samuelson [11]. Vickrey [12] and Riley and Samuelson [11] proved the revenue equivalence of symmetric auctions, i.e., auctions in which the valuations of all the players are drawn from the same distribution function. Myerson [10] showed that the Revenue Equivalence Theorem remains true for asymmetric auctions (auctions in which the bidders' valuations are drawn independently from different distributions)

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provided that at any realization of the players' valuations the probability of a player to win the object is independent of the auction mechanism. However, this condition does not usually hold in asymmetric auctions. Indeed, it is well known that asymmetric auctions are not necessarily revenue equivalent. For example, the expected revenue in first-price auctions can be higher or lower than in second-price auctions (see, e.g., Maskin and Riley [8]).

Auction theory has dealt mostly with symmetric auctions, since in this case explicit expression for the equilibrium bidding strategies can be obtained. In many cases, however, bidders' valuations are drawn from different distribution functions. Because explicit expressions for asymmetric equilibrium strategies cannot be obtained except for very simple models, analysis of asymmetric auctions is considerably more complex and relatively little is known about them at present.

In this paper, we study the seller's revenue in asymmetric auctions. Let us consider, for example, a situation where initially the distribution functions of all the players are identical, but then the distribution function of each player undergoes a mild independent change. How does the seller's expected revenue change as a result of this weak asymmetry? Since there are no explicit solutions for the new equilibrium strategies, there is no exact answer to this question. In situations like this where it is difficult or not even possible to obtain exact solutions, much insight can be gained by employing perturbation analysis, whereby one calculates an explicit approximation to the solution. Indeed, perturbation analysis is one of the most powerful tools in applied mathematics, and is extensively used in the analysis of mathematical models in the exact sciences. As we shall see in this study, the approximate expression that we obtain for the revenue in asymmetric auctions makes the sacrifice of "exactness" worthwhile. We note that perturbation analysis has been recently applied to other problems in auction theory, such as risk aversion in auctions [3] and asymmetry in first-price auctions [2].

In Section 2, we analyze the effect of weak asymmetry on the seller's expected revenue by using perturbation analysis. We show that under the same conditions as those of the classical (i.e., symmetric) Revenue Equivalence Theorem, weak asymmetry generates differences in revenues across the auction mechanisms that are only of second order. Formally, let ε be the level of asymmetry among the distribution functions and let $R(\varepsilon)$ be the seller's expected revenue in equilibrium. Then, we show that $R(\varepsilon) = R(0) + \varepsilon R'(0) + O(\varepsilon^2)$, where both $R(0)$ (the seller's expected revenue in the symmetric case) and $R'(0)$ (the leading-order effect of the asymmetry) are independent of the auction mechanism. The value of $R''(0)$, however, does depend on the auction mechanism. Therefore, although asymmetric auctions are not revenue equivalent, the above result shows that weakly asymmetric auctions are "essentially" revenue equivalent (roughly speaking, 10% asymmetry will only result in 1% revenue difference among different auction mechanisms). This observation is relevant to the problem of how to rank the seller's revenue in different auction mechanisms under asymmetry, which has long been an open question (see, e.g., [1,7,8]). Indeed, it implies that in the case of a weak asymmetry the revenue difference can be in the third or fourth digit, in which case the ranking problem is more of academic interest than of practical value.

In Section 3 we apply a powerful concept from perturbation theory known as averaging to asymmetric auctions. We show that the seller's expected revenue in asymmetric auctions can be well-approximated with the seller's expected revenue in the symmetric case with the same number of players whose distribution function is the arithmetic average of the asymmetric distribution functions. Hence, when asymmetry is weak, the revenues in the asymmetric case and in the corresponding symmetric case are essentially identical.

In Section 4 the strength of our approach is illustrated by comparing our analytical approximations of the revenue in first-price and second-price auctions with their exact values obtained by numerical calculations. Although our results are derived under the assumption of weak asymmetry, this comparison suggests that these approximations remain accurate even when the asymmetry is not small. In Section 5 we show the generalization of our results for incentive compatible mechanisms which are not necessarily auctions.

2. The model

We consider private-value auction mechanisms with n players that satisfy the same conditions of the (symmetric) Revenue Equivalence Theorem[1] (see Riley and Samuelson [11]):

Condition 1. All players are risk neutral.

Condition 2. Player i's valuation is private information to i and is drawn independently by a continuously differentiable distribution function $F_i(v)$ from a support $[\underline{v}, \bar{v}]$ which is common to all players.

Condition 3. The object is allocated to the player with the highest bid.[2]

Condition 4. Any player with valuation $\underline{v}$ expects zero surplus.

Consider distribution functions of the form[3]

$$F_i(v) = F(v) + \varepsilon H_i(v), \quad i = 1, \ldots, n, \tag{1}$$

where $F(\underline{v}) = 0$, $F(\bar{v}) = 1$, $H_i(\underline{v}) = H_i(\bar{v}) = 0$ and $|H_i| \leq 1$ in $[\underline{v}, \bar{v}]$ for all i. Thus, ε is a measure of the level of asymmetry. We assume that when ϵ is sufficiently small (i.e., weak asymmetry) the auction has equilibrium bids $\{b_i(v; \varepsilon)\}_{i=1}^n$ that are strictly increasing functions of v and continuously differentiable with respect to ϵ.[4] We

[1] We assume that the seller does not impose a binding reserve price or entry fee.

[2] In the symmetric case condition 3 is equivalent to the condition that the object is allocated to the player with the highest valuation. This equivalence, however, does not hold in the asymmetric case since asymmetric auctions are not necessarily efficient.

[3] The assumption that the distribution functions are of the form (1) is not restrictive. Indeed, we can bring any family of distribution functions $\{F_i\}_{i=1}^n$ to this form by defining $F = \frac{1}{n}\sum_{i=1}^n F_i$, $\varepsilon = \max_i \max_v |F_i - F|$ and $H_i = (F_i - F)/\varepsilon$.

[4] Throughout this paper we study auction mechanisms for which an equilibrium does exist. Note that a differentiable equilibrium exists in asymmetric second-price auctions since the players have weakly dominant strategies. Under mild assumptions an equilibrium also exists in asymmetric first-price auctions (Maskin and Riley [9], Lebrun [5,6]). Differentiability with respect to ϵ can be derived from Lebrun [5,6].

denote by $R(\varepsilon)$ the seller's expected revenue as a function of ε when all bidders follow their equilibrium strategies. Then, the following result shows that weak asymmetry generates differences in revenues across the auction mechanisms that are of second order.

Theorem 1. *The seller's expected revenue in any auction mechanism that satisfies Conditions* 1–4 *is given by*

$$R(\varepsilon) = R(0) + \varepsilon R'(0) + O(\varepsilon^2),$$

where

$$R(0) = \bar{v} + (n-1)\int_{\underline{v}}^{\bar{v}} F^n(v)\,dv - n\int_{\underline{v}}^{\bar{v}} F^{n-1}(v)\,dv \tag{2}$$

and

$$R'(0) = -(n-1)\int_{\underline{v}}^{\bar{v}} F^{n-2}(v)(1-F(v))\sum_{i=1}^{n} H_i(v)\,dv. \tag{3}$$

Proof. Let $E_i(v)$, $S_i(v)$ and $P_i(v)$ be the expected payment, the expected surplus, and the probability of winning for bidder i with type v at equilibrium, respectively. Therefore,

$$S_i = vP_i(v) - E_i(v). \tag{4}$$

It is well known (see, e.g., Klemperer [4]) that

$$\frac{dS_i}{dv} = P_i(v). \tag{5}$$

From (4),(5) it follows that

$$E_i'(v) = vP_i'(v). \tag{6}$$

Let R_i be the expected payments of player i averaged across her types. Then,

$$R_i = \int_{\underline{v}}^{\bar{v}} E_i(v)F_i'(v)\,dv = E_i(v)F_i\Big|_{\underline{v}}^{\bar{v}} - \int_{\underline{v}}^{\bar{v}} E_i'(v)F_i(v)\,dv$$
$$= E_i(\bar{v}) - \int_{\underline{v}}^{\bar{v}} vP_i'(v)F_i(v)\,dv.$$

Since $S_i(\underline{v}) = 0$ (Condition 4) we have from (6) that $E_i(\bar{v}) = \int_{\underline{v}}^{\bar{v}} vP_i'(v)\,dv$. Hence,

$$R_i = \int_{\underline{v}}^{\bar{v}} vP_i'(v)(1-F_i(v))\,dv = -\int_{\underline{v}}^{\bar{v}} P_i(v)[v(1-F_i(v))]'\,dv.$$

The seller's expected revenue is thus given by

$$R = \sum_{i=1}^{n} R_i = -\sum_{i=1}^{n}\int_{\underline{v}}^{\bar{v}} P_i(v)[v(1-F_i(v))]'\,dv. \tag{7}$$

Now let F_i be given by (1). Then $R = R(\varepsilon)$ depends on the asymmetry parameter ε. The Revenue Equivalence Theorem for symmetric auctions states that $R(0)$ is independent of the auction mechanism and is given by (2) (Riley and Samuelson [11]). We now calculate $(dR/d\varepsilon)_{\varepsilon=0}$ and show, in particular, that it is also independent of the auction mechanism. Indeed,

$$\left.\frac{dR}{d\varepsilon}\right|_{\varepsilon=0} = I_1 + I_2, \tag{8}$$

where

$$I_1 = -\sum_{i=1}^{n} \int_{\underline{v}}^{\bar{v}} \left.\frac{dP_i(v)}{d\varepsilon}\right|_{\varepsilon=0} [v(1-F(v))]'\, dv, \quad I_2 = \sum_{i=1}^{n} \int_{\underline{v}}^{\bar{v}} P(v)[vH_i(v)]'\, dv, \tag{9}$$

and where $P(v) = F^{n-1}(v)$ is the probability of winning for a player with type v at equilibrium in the symmetric case $\varepsilon = 0$.

It is clear that I_2 is independent of the auction mechanism. Indeed integration by parts shows that

$$I_2 = -(n-1)\sum_{i=1}^{n} \int_{\underline{v}}^{\bar{v}} vH_i(v)F^{n-2}(v)F'(v)\, dv. \tag{10}$$

In contrast, I_1 appears to depend on the auction mechanism. In order to calculate I_1 (and thus show that it is independent of the auction mechanism) we first prove the following lemma:

Lemma 1.

$$\sum_{i=1}^{n} \left.\frac{dP_i(v)}{d\varepsilon}\right|_{\varepsilon=0} = (n-1)F^{n-2}(v)\sum_{i=1}^{n} H_i(v).$$

Proof. See Appendix 6.1.

Using Lemma 1 we get that

$$\begin{aligned} I_1 &= -\sum_{i=1}^{n} \int_{\underline{v}}^{\bar{v}} (n-1)F^{n-2}(v)H_i(v)[v(1-F(v))]'\, dv \\ &= -(n-1)\sum_{i=1}^{n} \int_{\underline{v}}^{\bar{v}} F^{n-2}(v)H_i(v)[(1-F(v)) - F'(v)v]\, dv. \end{aligned}$$

Combining this with (8), (10) gives (3). □

The classical Revenue Equivalence Theorem states that $R(0)$ is independent of the auction mechanism. Theorem 1 shows that $R'(0)$ is also independent of the auction mechanism. The value of $R''(0)$, however, does depend on the auction mechanism (see Section 4). Since the differences among asymmetric auctions are only of second order, we can conclude that weakly asymmetric auctions are "essentially" revenue equivalent.

3. Averaging

In this section we apply a powerful concept from perturbation theory known as averaging to asymmetric auctions. Let us first introduce the notations $R_{\mathrm{sym}}[F]$ for the expected seller's revenue in symmetric auctions with n bidders with identical distribution function F, and $R[F_1, \ldots, F_n]$ for the expected seller's revenue in asymmetric auctions with distribution functions $\{F_i\}_{i=1}^n$. According to the Revenue Equivalence Theorem

$$R_{\mathrm{sym}}[F] = \bar{v} + (n-1)\int_{\underline{v}}^{\bar{v}} F^n \, dv - n\int_{\underline{v}}^{\bar{v}} F^{n-1} \, dv. \tag{11}$$

Our goal is to approximate the seller's expected revenue $R[F_1, \ldots, F_n]$ in asymmetric auctions using the explicit expression (11), i.e.,

$$R[F_1, \ldots, F_n] \approx R_{\mathrm{sym}}[F_{\mathrm{avg}}], \tag{12}$$

where F_{avg} is an appropriately chosen average of $\{F_i\}_{i=1}^n$. To illustrate, when the functions $\{F_i\}_{i=1}^n$ are given by (1), one possibility is to use the naive approximation $F_{\mathrm{avg}} \approx F$ and approximate $R[F_1, \ldots, F_n] = R(\varepsilon)$ with $R_{\mathrm{sym}}[F] = R(0)$. Theorem 1 shows that this approximation has an $O(\varepsilon)$ accuracy. We now show that considerably better averaging, with $O(\varepsilon^2)$ accuracy, can be achieved if F_{avg} is taken to be the arithmetic mean of the distributions.[5]

Theorem 2. *Let $\varepsilon = \max_i \max_v |F_i - F_{\mathrm{avg}}|$ be small. Then the seller's expected revenue in any auction mechanism satisfying Conditions* 1–4 *is*

$$R[F_1, \ldots, F_n] = R_{\mathrm{sym}}[F_{\mathrm{avg}}] + O(\varepsilon^2),$$

where F_{avg} is given by

$$F_{\mathrm{avg}} = \frac{1}{n}\sum_{i=1}^{n} F_i.$$

Proof. Apply Theorem 1 with $F = F_{\mathrm{avg}} = (1/n)\sum_{i=1}^n F_i$ and $H_i = (F_i - F_{\mathrm{avg}})/\varepsilon$. Note that since $\sum_{i=1}^n H_i(v) = 0$ we have $R'(0) = 0$. $\square$

The result of Theorem 2 can be explained as follows. Theorem 1 shows that the leading-order effect of asymmetry on the revenue R depends only on the sum of all

[5] Cantillon [1] used the geometric mean of $\{F_i\}_{i=1}^n$ to compare the expected revenue in asymmetric first- and second-price auctions. Taking F_{avg} as the geometric mean of $\{F_i\}_{i=1}^n$ also yields $O(\varepsilon^2)$ accuracy.

the H_i's. Hence, this leading-order effect would be the same if all players experience an identical (i.e., symmetric) change $F_i = F + \varepsilon H$ (for $i = 1, \dots, n$), provided that

$$H = \frac{1}{n} \sum_{i=1}^{n} H_i.$$

4. Simulations

In this section we illustrate the results obtained in Theorems 1 and 2 by comparing asymmetric first-price and second-price auctions.

It is well known that in second-price auctions players have weakly dominant strategies $b_i = v_i$ (see [12]). Therefore, the seller's expected revenue is

$$R^{2\text{nd}} = E[\text{second}\max\{v_1, \dots, v_n\}].$$

A simpler expression for $R^{2\text{nd}}$ is derived in the following lemma:

Lemma 2. *The expected revenue in a second-price auction is given by*

$$R^{2\text{nd}} = \bar{v} - \int_{\underline{v}}^{\bar{v}} \prod_{i=1}^{n} F_i(v)\, dv - \sum_{i=1}^{n} \int_{\underline{v}}^{\bar{v}} (1 - F_i(v)) \prod_{\substack{j=1 \\ j \neq i}}^{n} F_j(v)\, dv. \tag{13}$$

Proof. See Appendix 6.2.

When there are only two players expression (13) reduces to

$$R^{2\text{nd}} = \int_{\underline{v}}^{\bar{v}} (1 - F_1)(1 - F_2)\, dv.$$

In this case, substituting $F_i = F + \varepsilon H_i$ gives the exact expression

$$R^{2\text{nd}}(\varepsilon) = \int_{\underline{v}}^{\bar{v}} (1 - F)^2\, dv - \varepsilon \int_{\underline{v}}^{\bar{v}} (1 - F)(H_1 + H_2)\, dv + \varepsilon^2 \int_{\underline{v}}^{\bar{v}} H_1 H_2\, dv. \tag{14}$$

Unlike second-price auctions, explicit expressions of the equilibrium bids in first-price auctions exist only for the symmetric case. The equilibrium bids and the expected revenue, however, can be calculated numerically (see [2,7]). In this section we provide two examples that illustrate the results obtained in Theorems 1 and 2.

Example 1. Consider the case of two bidders 1 and 2 whose valuations are distributed on $[0, 1]$ according to the distribution functions $F_1 = v + \varepsilon v(1 - v)$ and $F_2 = v - \varepsilon v(1 - v)$, respectively. Note that in this case $F_{\text{avg}} = v$ and $\sum_i H_i = 0$.

According to Theorem 1 we have that $R(0) = R_{\text{sym}}[v] = 1/3$, $R'(0) = 0$, and thus the seller's expected revenue is $R(\varepsilon) = 1/3 + O(\varepsilon^2)$. This result also follows from Theorem 2, since in this case $F_{\text{avg}} = v$. The difference between the values of $R(\varepsilon)$ and

Table 1
Seller's expected revenue in asymmetric first-price R^{1st} and second-price R^{2nd} auctions as functions of ε compared to the seller's expected revenue in the symmetric auction $R_{sym}[F_{avg}] = 1/3$.

ε	$\frac{R^{1st} - R_{sym}[F_{avg}]}{R_{sym}[F_{avg}]}$	$\frac{R^{2nd} - R_{sym}[F_{avg}]}{R_{sym}[F_{avg}]}$
0.05	−0.0036%	−0.025%
0.1	−0.015%	−0.1%
0.2	−0.054%	−0.4%
0.4	−0.172%	−1.6%

the approximation $R_{sym}[F_{avg}]$ can be seen in Table 1. Even for non-small values of ε, $R_{sym}[F_{avg}]$ is an excellent approximation of the expected revenue $R(\varepsilon)$ for both first-price and second-price auctions. The simulation results also confirm that the error in the approximation of $R(\varepsilon)$ with $R_{sym}[F_{avg}]$ scales like ε^2, as Theorem 2 predicts. Since the expected revenues in first-price and second-price auctions are not identical, the 'strict' revenue equivalence does not hold in this example. Indeed, whereas $R(0)$ and $R'(0)$ are identical among first-price and second-price auctions, the numerical results show that $R''(0)$ is different for first-price and second-price auctions as $\frac{d^2}{d\varepsilon^2}R^{2nd}|_{\varepsilon=0} \approx -0.066 \neq \frac{d^2}{d\varepsilon^2}R^{1st}|_{\varepsilon=0} \approx -0.008$.

Example 2. Consider the case of two bidders with distribution functions $F_1 = v + \varepsilon v(1-v)$ and $F_2 = v - 2\varepsilon v(1-v)$. Note that in this case $\sum_i H_i \neq 0$ and thus $F_{avg} \neq v$ except for the case when $\varepsilon = 0$.

In this example the bidders' distributions are not symmetric with respect to v, and the average distribution $F_{avg} = v - 0.5\varepsilon v(1-v)$ depends on ε. From Theorem 1 we have that $R(\varepsilon) = R(0) + \varepsilon R'(0) + O(\varepsilon^2)$, where $R(0) = R_{sym}[v] = \frac{1}{3}$ and $R'(0) = \frac{1}{12}$. Alternatively, from Theorem 2 we have that $R(\varepsilon) = R_{sym}[v - 0.5\varepsilon v(1-v)] + O(\varepsilon^2)$ and by (11), $R_{sym}[v - 0.5\varepsilon v(1-v)] = 1/3 + \varepsilon/12 + O(\varepsilon^2)$. Thus, the prediction for both R^{1st} and R^{2nd} is $R(\varepsilon) = 1/3 + \varepsilon/12$, with an $O(\varepsilon^2)$ error.

In Table 2 we compare the expected revenue in the asymmetric case $R(\varepsilon)$ with $R_{sym}[F_{avg}]$ and also with the naive approximation $R(0) = R_{sym}[v]$. As predicted in Section 3, the approximation of R^{1st} and R^{2nd} with $R_{sym}[F_{avg}]$ is considerably better than with $R_{sym}[F]$. In conclusion, the above examples demonstrate that when asymmetry is weak, the difference in revenue between a first-price auction and a second-price auction is negligible. Therefore, weakly asymmetric first-price auctions and second-price auctions remain "essentially" revenue equivalent.

5. Asymmetric mechanisms

The results of this paper can be generalized to the case of incentive compatible mechanisms which are not necessarily auctions. In light of the revelation principle, it is sufficient to consider direct mechanisms. Let $Q_i(v_i, \mathbf{v}_{-i})$ be the probability of

Table 2
Seller's expected revenue in asymmetric first-price R^{1st} and second-price R^{2nd} auctions as a function of ε compared with the seller's expected revenue in the symmetric auctions $R_{sym}[F_{avg}]$ and $R_{sym}[v]$.

ϵ	$R^{1st} - R_{sym}[F_{avg}]$	$R^{1st} - R_{sym}[v]$	$R^{2nd} - R_{sym}[F_{avg}]$	$R^{2nd} - R_{sym}[v]$
0.1	-0.10×10^{-3}	8.3×10^{-3}	-0.75×10^{-3}	7.67×10^{-3}
0.2	-0.34×10^{-3}	16.0×10^{-3}	-3.0×10^{-3}	14.0×10^{-3}
0.3	-0.53×10^{-3}	25.0×10^{-3}	-6.75×10^{-3}	19.0×10^{-3}
0.4	-0.21×10^{-3}	34.0×10^{-3}	-12.0×10^{-3}	22.0×10^{-3}

winning for bidder i when the n bidders announce their types $(v_1, \dots, v_n)$ and $\mathbf{v}_{-\mathrm{i}}$ is the vector of types excluding v_i. The expected probability that i wins when his type is v_i is given by

$$P_i(v_i) = \int_{\mathbf{v}_{-\mathrm{i}}} Q_i(v_i, \mathbf{v}_{-\mathrm{i}}) \prod_{j \neq i} F_j'(v_j)\, d\mathbf{v}_{-\mathrm{i}}. \tag{15}$$

Let $Q_i^{\mathrm{sym}}(v_i, \mathbf{v}_{-\mathrm{i}}) = Q_i(v_i, \mathbf{v}_{-\mathrm{i}})|_{\varepsilon=0}$ be the probability that i wins in the symmetric case. In order to generalize from auction mechanisms to general mechanisms we replace Condition 3 with the following two conditions:

Condition 3A. $Q_i^{\mathrm{sym}}(v_i, \mathbf{v}_{-\mathrm{i}})$ is independent of the mechanism.[6,7]

Condition 3B. The mechanism rule is anonymous, i.e., switching the identities of players does not affect their probabilities of winning. In other words, there exists a function $\mathscr{G}$ such that

$$Q_i(v_i, \mathbf{v}_{-\mathrm{i}}) = \mathscr{G}(v_i, \mathbf{v}_{-\mathrm{i}}; F_i, \mathbf{F}_{-\mathrm{i}}), \quad i = 1, \dots, n.$$

In the symmetric case, Condition 3B implies that there exists a function $\mathscr{G}^{\mathrm{sym}}$ such that

$$Q_i(v_i, \mathbf{v}_{-\mathrm{i}}) = \mathscr{G}^{\mathrm{sym}}(v_i, \mathbf{v}_{-\mathrm{i}}), \quad i = 1, \dots, n.$$

Therefore, in the symmetric case we can drop the i subscript, i.e., the probability and expected probability that i wins when his type is v_i are given by $Q^{\mathrm{sym}}(v_i, \mathbf{v}_{-\mathrm{i}})$ and by

$$P(v_i) = \int_{\mathbf{v}_{-\mathrm{i}}} Q^{\mathrm{sym}}(v_i, \mathbf{v}_{-\mathrm{i}}) \prod_{j \neq i} F'(v_j)\, d\mathbf{v}_{-\mathrm{i}}, \tag{16}$$

respectively.

The generalization of Theorem 1 is as follows:

[6] For example, for any auction mechanism satisfying Condition 3, $Q_i^{\mathrm{sym}}(v_i, \mathbf{v}_{-\mathrm{i}})$ is equal to 1 if $v_i > \max_{j \neq i} v_j$ and to 0 otherwise.

[7] Note that $Q_i(v_i, \mathbf{v}_{-\mathrm{i}})$ is, in general, not independent of the mechanism. Otherwise, by Myerson [10], there is revenue equivalence.

Theorem 3. *The seller's expected revenue in any incentive compatible mechanism that satisfies Conditions* 1, 2, 3A, 3B, *and* 4 *is given by*

$$R(\varepsilon) = R(0) + \varepsilon R'(0) + O(\varepsilon^2),$$

where $R(0)$ *and* $R'(0)$ *are independent of the mechanism.*

Proof. See Appendix 6.3.

Remark. Clearly, $R(0)$ and $R'(0)$ can be different for mechanisms for which $Q^{\mathrm{sym}}(v_i, \mathbf{v}_{-\mathrm{i}})$ are different.

The generalization of Theorem 2 is also straightforward:

Theorem 4. *Let* $\varepsilon = \max_i \max_v |F_i - F_{\mathrm{avg}}|$ *be small. Then the seller's expected revenue in any mechanism that satisfies Conditions* 1, 2, 3A, 3B, *and* 4 *is given by*

$$R[F_1, \ldots, F_n] = R_{\mathrm{sym}}[F_{\mathrm{avg}}] + O(\varepsilon^2),$$

where R_{sym} *is the expected revenue in the symmetric case and* $F_{\mathrm{avg}} = \frac{1}{n}\sum_{i=1}^{n} F_i$.

Proof. See Appendix 6.4.

6. Appendix

6.1. *Proof of Lemma 1*

Let $b_j(v, \varepsilon)$ be the equilibrium strategy of player j. Differentiating the identity $v = b_j^{-1}(b_j(v;\varepsilon);\varepsilon)$ with respect to ε and substituting $\varepsilon = 0$ gives

$$0 = \left.\frac{\partial b_j^{-1}}{\partial \varepsilon}\right|_{\varepsilon=0} + (b_j^{-1})' \left.\frac{\partial b_j}{\partial \varepsilon}\right|_{\varepsilon=0}. \tag{17}$$

Since in equilibrium

$$P_i(v) = P\left(b_i(v) > \max_{j \neq i} b_j\right) = \prod_{\substack{j=1 \\ j \neq i}}^{n} F_j(b_j^{-1}(b_i(v)),$$

we have that

$$\begin{aligned}
\left.\frac{dP_i(v)}{d\varepsilon}\right|_{\varepsilon=0} &= F^{n-2}(v)\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\partial}{\partial\varepsilon}\Big[F_j(b_j^{-1}(b_i(v))\Big]_{\varepsilon=0}\\
&= F^{n-2}(v)\sum_{\substack{j=1\\ j\neq i}}^{n}\left\{\left.\frac{\partial F_j}{\partial\varepsilon}\right|_{\varepsilon=0}(v) + F'(v)\frac{\partial}{\partial\varepsilon}\Big[b_j^{-1}(b_i(v))\Big]_{\varepsilon=0}\right\}\\
&= F^{n-2}(v)\sum_{\substack{j=1\\ j\neq i}}^{n}\left\{H_j(v) + F'(v)\left[\left.\frac{\partial b_j^{-1}}{\partial\varepsilon}\right|_{\varepsilon=0} + (b^{-1})'\left.\frac{\partial b_i(v)}{\partial\varepsilon}\right|_{\varepsilon=0}\right]\right\}\\
&= F^{n-2}(v)\sum_{\substack{j=1\\ j\neq i}}^{n}\left\{H_j(v) + F'(v)(b^{-1})'\left[-\left.\frac{\partial b_j(v)}{\partial\varepsilon}\right|_{\varepsilon=0} + \left.\frac{\partial b_i(v)}{\partial\varepsilon}\right|_{\varepsilon=0}\right]\right\},
\end{aligned}$$

where in the last stage we use (17). Summing this equation over $i = 1, \ldots, n$ and using the identities

$$\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} a_i = \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} a_j = (n-1)\sum_{i=1}^{n} a_i$$

completes the proof of the Lemma.

6.2. Proof of Lemma 2

Let $\tilde{v}$ be the second highest number of the values $\{v_1, v_2, \ldots, v_n\}$. The distribution function of $\tilde{v}$ is given by

$$\begin{aligned}
F^{\text{2nd}}(v) &= \Pr(\tilde{v}\leq v) = \Pr(v_1\leq v, \ldots, v_n\leq v) + \sum_{i=1}^{n}\Pr(v_j\leq v, j\neq i, v_i > v)\\
&= \prod_{i=1}^{n} F_i(v) + \sum_{i=1}^{n}(1-F_i(v))\prod_{\substack{j=1\\ j\neq i}}^{n} F_j(v).
\end{aligned}$$

The expectation of $\tilde{v}$ is

$$\begin{aligned}
R^{\text{2nd}} &= E(\tilde{v}) = \int_{\underline{v}}^{\bar{v}} v\,dF^{\text{2nd}}(v) = vF^{\text{2nd}}(v)\big|_{\underline{v}}^{\bar{v}} - \int_{\underline{v}}^{\bar{v}} F^{\text{2nd}}(v)\,dv\\
&= \bar{v} - \int_{\underline{v}}^{\bar{v}}\left(\prod_{i=1}^{n} F_i(v) + \sum_{i=1}^{n}(1-F_i(v))\prod_{\substack{j=1\\ j\neq i}}^{n} F_j(v)\right)dv,
\end{aligned}$$

which gives (13).

6.3. Proof of Theorem 3

The condition of incentive compatibility implies (5). Therefore, as before, the seller's expected revenue is given by (7). Let $\{F_i\}_{i=1}^n$ be given by (1). Consider $R = R(\varepsilon)$ as a function of the asymmetry parameter ε. The Revenue Equivalence Theorem for symmetric mechanisms states that $R(0)$ is independent of the mechanism [10]. In order to show that $R'(0)$ is also independent of the mechanism, we note that $R'(0) = I_1 + I_2$, where I_1 and I_2 are given by (9), and $P(v)$ is given by (16). Therefore, condition 3A implies that I_2 is independent of the mechanism.

Lemma 3.

$$\sum_{i=1}^n \left.\frac{dP_i(v_i)}{d\varepsilon}\right|_{\varepsilon=0} = \sum_{i=1}^n \int_{\mathbf{v}_{-\mathrm{i}}} Q^{\mathrm{sym}}(v_i, \mathbf{v}_{-\mathrm{i}}) \sum_{j\neq i}\left[H_j'(v_j)\prod_{k\neq j,i} F'(v_k)\right] d\mathbf{v}_{-\mathrm{i}}.$$

Proof. From (15),

$$\sum_{i=1}^n \left.\frac{dP_i(v_i)}{d\varepsilon}\right|_{\varepsilon=0} = \sum_{i=1}^n \int_{\mathbf{v}_{-\mathrm{i}}} \left[\frac{d}{d\varepsilon} Q_i(v_i, \mathbf{v}_{-\mathrm{i}})\right]_{\varepsilon=0} \prod_{j\neq i} F'(v_j)\, d\mathbf{v}_{-\mathrm{i}}$$
$$+ \sum_{i=1}^n \int_{\mathbf{v}_{-\mathrm{i}}} Q^{\mathrm{sym}}(v_i, \mathbf{v}_{-\mathrm{i}}) \left[\frac{d}{d\varepsilon}\prod_{j\neq i} F_j'(v_j)\right]_{\varepsilon=0} d\mathbf{v}_{-\mathrm{i}}.$$

Condition 3B implies that switching the identities of players 1 and i will not change their probability of winning. Thus, for example, $Q_i(v_i, \mathbf{v}_{-\mathrm{i}})$, the probability of winning of player i whose type v_i is drawn by a distribution function F_i, is equal to his probability of winning after he becomes player 1 whose type v_1 is drawn by the same distribution function F_i. Therefore, we can make the following change of variables:

$$\int_{\mathbf{v}_{-\mathrm{i}}} \left[\frac{d}{d\varepsilon} Q_i(v_i, \mathbf{v}_{-\mathrm{i}})\right]_{\varepsilon=0} \prod_{j\neq i} F'(v_j)\, d\mathbf{v}_{-\mathrm{i}}$$
$$= \int_{\mathbf{v}_{-1}} \left[\frac{d}{d\varepsilon} Q_i(v_1, \mathbf{v}_{-\mathbf{1}})\right]_{\varepsilon=0} \prod_{j\neq 1} F'(v_j)\, d\mathbf{v}_{-\mathbf{1}}. \tag{18}$$

Since (18) holds for every i, we have,

$$\sum_{i=1}^n \int_{\mathbf{v}_{-\mathrm{i}}} \left[\frac{d}{d\varepsilon} Q_i(v_i, \mathbf{v}_{-\mathrm{i}})\right]_{\varepsilon=0} \prod_{j\neq i} F'(v_j)\, d\mathbf{v}_{-\mathrm{i}}$$
$$= \int_{\mathbf{v}_{-1}} \sum_{i=1}^n \left[\frac{d}{d\varepsilon} Q_i(v_1, \mathbf{v}_{-1})\right]_{\varepsilon=0} \prod_{j\neq 1} F'(v_j)\, d\mathbf{v}_{-1}$$
$$= \int_{\mathbf{v}_{-1}} \prod_{j\neq 1} F'(v_j) \left[\frac{d}{d\varepsilon}\underbrace{\sum_{i=1}^n Q_i(v_1, \mathbf{v}_{-1})}_{\equiv 1}\right]_{\varepsilon=0} d\mathbf{v}_{-\mathbf{1}} = 0.$$

Therefore, the result of Lemma 3 follows. □

Using (9) and Lemma 3 we get that

$$I_1 = -\int_{\underline{v}}^{\bar{v}} \sum_{i=1}^{n} \int_{\mathbf{v}_{-\mathrm{i}}} Q^{\mathrm{sym}}(v_i, \mathbf{v}_{-\mathrm{i}}) \sum_{j \neq i} \left[H_j'(v_j) \prod_{k \neq j,i} F'(v_k) \right] \times [v_i(1-F(v_i))]' \, d\mathbf{v}_{-\mathrm{i}} \, dv_i. \quad (19)$$

Hence, by Condition 3A, I_1 is also independent of the mechanism.

6.4. Proof of Theorem 4

As in the proof of Theorem 2, it is sufficient to show that $R'(0)$ vanishes when $\sum_{i=1}^{n} H_i(v) \equiv 0$. Clearly, $I_2 = 0$ when $\sum_{i=1}^{n} H_i(v) \equiv 0$, see (9). In order to see this for I_1 we rewrite (19) as

$$I_1 = \sum_{j=1}^{n} \int_{v_j} H_j'(v_j) A_j(v_j) \, dv_j,$$

$$A_j(v_j) = -\sum_{i \neq j} \int_{\mathbf{v}_{-\mathrm{j}}} Q^{\mathrm{sym}}(v_i, \mathbf{v}_{-\mathrm{i}}) \prod_{k \neq j,i} F'(v_k) [v_i(1-F(v_i))]' \, d\mathbf{v}_{-\mathrm{j}}.$$

Since $A_j(v)$ is independent of j, then

$$I_1 = \int_{\underline{v}}^{\bar{v}} \sum_{j=1}^{n} H_j'(v) A_j(v) \, dv = \int_{\underline{v}}^{\bar{v}} A_1(v) \sum_{j=1}^{n} H_j'(v) \, dv.$$

Therefore, $I_1 = 0$ when $\sum_{i=1}^{n} H_i(v) \equiv 0$.

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