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REVENUE EQUIVALENCE OF LARGE ASYMMETRIC AUCTIONS*

GADI FIBICH†, ARIEH GAVIOUS‡, AND NIR GAVISH§

Abstract. One of the most important results in auction theory is that when bidders are symmetric (homogeneous), then under quite general conditions, the seller's expected revenue is independent of the auction mechanism (Revenue Equivalence Theorem). More often than not, however, bidders are asymmetric, and so revenue equivalence is lost. Previously, it was shown that asymmetric auctions become revenue equivalent as $n \to \infty$, where n is the number of bidders. In this paper, we go beyond the limiting behavior and explicitly calculate the revenue to $O(1/n^3)$ accuracy, essentially with no information on the auction payment rules or bidders' equilibrium strategies, for a large class of asymmetric auctions that includes first-price, second-price, and optimal auctions. These calculations show that the revenue differences among asymmetric auctions scale as ϵ^2/n^3, where ϵ is the level of asymmetry (heterogeneity) among the bidders. Therefore, bidders' asymmetry has a negligible effect on revenue ranking of auctions with as few as $n = 6$ bidders.

Key words. auction theory, game theory, asymmetric auction, first-price auction, second-price auction, optimal auction, revenue ranking, revenue equivalence, asymptotic methods, Laplace method for integrals, homogenization, averaging

AMS subject classifications. 91B26, 41A60, 34B15, 34E15

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1. Introduction.

The auction is an important economic mechanism which is central to the modern economy. For example, in 2016 the US treasury auctioned securities in a total sum of 8.1 trillion dollars. Google makes most of its profits by selling sponsored links via online auctions. The first systematic analysis of auctions was done in 1961 by Vickrey [30]. Since then, auctions have been the subject of intense study. A brief theoretical background on auction theory is provided in section 2. For an introduction to auction theory, see, e.g., [16, 18].

When selling an object through an auction, the seller has the prerogative to choose the auction mechanism. In most cases, the key determinant for the seller is her expected revenue. By the Revenue Equivalence Theorem [25, 27], when all n bidders are *symmetric* (homogeneous), i.e., when their private values for the object (i.e., the price at which each bidder values the object) are distributed according to a *common* cumulative distribution function (CDF) $F(v)$ in $[0, 1]$, then under quite general conditions the seller's expected revenue in equilibrium is independent of the auction mechanism. More often than not, however, bidders are *asymmetric* (heterogeneous), i.e., the value of bidder i for the object is distributed according to the CDF $F_i(v)$ in $[0, 1]$, where $i = 1, \dots, n$. In this case, the revenue depends on the auction mechanism. As a result, revenue ranking of asymmetric auctions attracted considerable research effort [12, 3, 23, 15, 19, 7, 2, 6].

Most studies on the revenue of asymmetric auctions have adopted a yes/no

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†School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel (fibich@tau.ac.il).
‡Faculty of Business Administration, Ono Academic College, Kiryat Ono 5545173, Israel, and Department of Industrial Engineering and Management, Ben-Gurion University, Beer Sheva 84105, Israel (ariehg@bgu.ac.il).
§Department of Mathematics, Technion – Israel Institute of Technology, Haifa 69978, Israel (ngavish@tx.technion.ac.il).

approach to the question of revenue equivalence ("symmetric auctions are revenue equivalent," "second-price auctions yield more/less revenue than first-price auctions," etc.) [23, 15, 19, 2, 12, 3]. In this study we adopt a more nuanced approach by quantifying the level of revenue inequivalence. Another difference from most previous studies is that they considered specific auctions for which the equations for the bidding strategies were known, whereas we consider a large class of asymmetric auctions. This approach was previously applied in [4, 7, 20] to weakly asymmetric auctions, where it was shown that if ϵ denotes the level of asymmetry among $\{F_i(v)\}_{i=1}^n$,[1] if A and B are two auction mechanisms, and if R^{A} and R^{B} are the corresponding revenues, then

$$R^{\mathrm{A}}(\epsilon) - R^{\mathrm{B}}(\epsilon) \sim c\,\epsilon^2, \qquad \epsilon \ll 1. \tag{1}$$

While the numerical simulations in [4, 7] confirmed that the revenue differences scale as $O(\epsilon^2)$, it was not noted at the time that the revenue differences were much smaller than what one may expect from (1), i.e., that the constant c in (1) was very small. Similarly, other studies numerically tested the revenue ranking of first-price and second-price auctions [22, 14, 13, 21] but did not pay much attention to the fact that the revenue differences were typically less than 1%, and in most cases in the third or fourth digit, even when ϵ was not small. Until now, these numerical observations had no theoretical explanation.

The standard approach in auction theory for analyzing large auctions has been to consider the limit as $n \to \infty$ [31, 26, 28, 17]. Using this approach, it has been shown [1, 29] that under quite general conditions, the seller's revenue approaches the maximal value for the object $v_{\max} = 1$, i.e.,

$$\lim_{n\to\infty} R(n) = 1. \tag{2}$$

Consequently, asymmetric auctions become revenue equivalent as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} R^{\mathrm{A}}(n) = \lim_{n\to\infty} R^{\mathrm{B}}(n). \tag{3}$$

This limiting revenue equivalence result is intuitive. Indeed, as the number of asymmetric bidders increases, the competition that different bidders face becomes more and more similar, since any two bidders face the same $n-2$ bidders and only one different bidder. Hence, there is an $O(1/n)$ difference between the competition that bidders face. Therefore, as $n \to \infty$, asymmetric auctions "become symmetric" and in particular become revenue equivalent. If we continue with the intuitive argument above, the $O(1/n)$ level of asymmetry in the competition that bidders face can be expected to lead to $O(1/n)$ revenue inequivalence among asymmetric auctions. The limit (3), however, does not provide any information on the level of revenue inequivalence among asymmetric auctions for a finite n.

To go beyond limiting behavior, in Theorem 3.1 we write the revenues in second-price, first-price, and optimal auctions as one-dimensional integrals and expand these three integrals for $n \gg 1$ using the *Laplace method for integrals.* These asymptotic calculations yield $R = R_{\mathrm{asymp}} + O\left(\frac{1}{n^3}\right)$, where

$$R_{\mathrm{asymp}} = 1 - \frac{1}{n-1}\frac{2}{f_G(1)} - \frac{1}{(n-1)^2}\left[\frac{3f_G'(1)}{f_G^3(1)} - \frac{4}{f_G(1)} + \frac{\mathrm{Var}[f_1(1), \ldots, f_n(1)]}{f_G^3(1)}\right], \tag{4}$$

[1] E.g., $\epsilon = \max_{\substack{0\le v\le 1\\ 1\le i\le n}} |F_i - \frac{1}{n}\sum_{i=1}^n F_i|$.

$f_g = F'_G$, F_G is the geometric mean of $\{F_i\}_{i=1}^n$, and $f_i = F'_i$. Therefore, we have the unexpected result that the revenue inequivalence between asymmetric first-price, second-price, and optimal auctions is $O(1/n^3)$ small.

The asymptotic computations in Theorem 3.1 make use of the explicit expressions for the corresponding equilibrium bidding strategies. More generally, when asymmetric bidders are in a Nash equilibrium (i.e., when no single bidder will benefit from deviating from her equilibrium strategy), the seller's expected revenue is given by [7]

$$R = -\sum_{i=1}^{n} \int_{r_i}^{1} \left(\prod_{j=1}^{n} F_j(b_j^{-1}(b_i(v)) \right) \frac{1 - F_i(v) - v f_i(v)}{F_i(v)}\, dv,$$

where $b_i(v)$, the equilibrium bidding strategy of bidder i, depends on the auction mechanism and on the CDFs of all other players. In the symmetric case $F_i = F$ and $b_j^{-1}(b_i(v)) \equiv v$, and so $R = -n \int_r^1 F^n(v) \frac{1-F(v)-vf(v)}{F(v)}\, dv$ is independent of the auction mechanism (the Revenue Equivalence Theorem). In the general asymmetric case, there is essentially no information on the auction payment rule, let alone on the bidding strategies. Therefore, it seems unlikely that we can expand this integral beyond the leading order. Nevertheless, we show this integral can be expanded to $O(1/n^3)$, leading to *one of the key results of this study (Theorem* 3.2*), that* $R = R_{\text{asymp}} + O\left(\frac{1}{n^3}\right)$ *under quite general conditions.* The result that all asymmetric auctions that satisfy the conditions of Theorem 3.2 are $O(1/n^3)$ asymptotically revenue equivalent is highly surprising, since a priori one would expect only $O(1/n)$ asymptotic revenue equivalence. Yet, miraculously, being in a Nash equilibrium induces a "universality" on the revenue, which makes it completely auction-independent in the symmetric case, and $O(1/n^3)$ auction-independent in the asymmetric case.

Since $R = R_{\text{asymp}} + O\left(\frac{1}{n^3}\right)$, the revenue differences among large asymmetric auctions are at most $O(1/n^3)$; see Corollary 4.1. This result is optimal in the sense that there exist asymmetric auctions for which we can prove analytically (Lemma 4.2) or observe numerically (section 4) that the revenue differences decay as $1/n^3$, i.e.,

$$R^{\text{A}}(n) - R^{\text{B}}(n) \sim \frac{c_3}{n^3}, \qquad n \gg 1.$$

Moreover, if the bidders are weakly asymmetric, it follows from (1) that $c_3 \sim \epsilon^2$. Therefore, we get our main result (Theorem 5.1), that

$$R^{\text{A}}(n,\epsilon) - R^{\text{B}}(n,\epsilon) \sim \tilde{c}_3 \frac{\epsilon^2}{n^3}, \qquad n \gg 1, \quad \epsilon \ll 1. \tag{5}$$

Thus, *there are two forces that simultaneously act to decrease the revenue inequivalence among asymmetric auctions: The large number of players and the weak asymmetry among bidders' CDFs.* This insight provides the first theoretical explanation for the smallness of the revenue differences among asymmetric auctions, which was observed numerically in previous studies (see the paragraph following (1).)

Our numerical simulations (sections 5 and 6) confirm that the revenue difference decays as ϵ^2/n^3 for $n \gg 1$ and $\epsilon \ll 1$ and suggest that relation (5) remains valid even for as few as $n = 6$ players and $\epsilon = 0.5$. Therefore, more often than not, *revenue ranking of asymmetric auctions is only of theoretical interest.* Additional conclusions are given in section 7.

2. Theoretical background.

In what follows we provide a brief introduction to auction theory. For more information, see, e.g., [18].

2.1. Auction mechanisms. Consider n bidders that compete for a single indivisible object. Each player has its own valuation v_i for the object. Player i places a bid $b_i(v_i)$ that depends on her value but also on the auction mechanism and her beliefs about other players; see section 2.2. A bidding strategy is a function $b_i(v)$ for all admissible values of v. The bidding strategies are assumed to be in a Nash equilibrium so that no single bidder can benefit from deviating from her equilibrium strategy. In many auctions, the equilibrium bidding strategies are strictly increasing, and so one can consider the inverse equilibrium strategies $v_i(b) = b_i^{-1}(v)$.

In standard auctions, the player with the highest bid wins the object and pays according to the payment rule of the auction he participated in. The most common mechanisms are the first-price and second-price auctions, in which the highest bidder wins the object and pays the highest or the second-highest bid, respectively, and all other bidders pay nothing. The auctioneer can also set a *reserve price* r, i.e., a lower bound for the admissible bids.

The seller would like to choose the auction mechanism that would maximize her expected revenue. Myerson [25] computed the auction format which yields the highest expected revenue for the seller. Myerson's *optimal mechanism* is not practical to implement, except in the case of symmetric bidders (see section 2.2), where it reduces to a second-price auction with an optimal reserve price. Nevertheless, this mechanism is important theoretically, since it provides an upper bound for the expected revenue under any auction mechanism.

2.2. Assumptions on players. We assume that bidder i has its own private value v_i for the object, which is distributed according to a cumulative distribution function (CDF) $F_i(v_i)$, where $i = 1, \dots, n$. This value is private to i, i.e., it is unknown to all other bidders. The CDFs $\{F_i(v_i)\}_{i=1}^n$, however, are known to all other bidders. We compute the seller's revenue as $n \to \infty$ under the following assumptions:

Condition 1. All players are risk neutral.

Condition 2. For any given n, player i's valuation for the object is private information to i and is drawn independently by a continuously differentiable, strictly increasing, distribution function $F_i^{(n)}(v)$ from a support $[0, 1]$. Therefore, the n players have the CDFs $\mathcal{F}^{(n)} = \{F_1^{(n)}, \dots, F_n^{(n)}\}$. We assume that $F_i^{(n)}$ is three times continuously differentiable near $v = 1$ and that $0 < f_i^{(n)}(1) \le M$ for $i = 1, \dots, n$, where $f_i^{(n)} = (F_i^{(n)})'$ is the corresponding density function and M is independent of i and n.[2]

Condition 3. The geometric mean $F_G[\mathcal{F}^{(n)}]$ of the CDFs is independent of n, where

$$F_G[F_1(v), \dots, F_n(v)] := \left(\prod_{i=1}^{n} F_i(v) \right)^{1/n}. \tag{6}$$

Condition 3 is needed in order to make sense of the limit of the CDFs $\{\mathcal{F}^{(n)}\}$ as $n \to \infty$. Indeed, we shall see that the geometric mean $F_G[\mathcal{F}^{(n)}]$ arises naturally in the asymptotic analysis of large auctions. Condition 3 holds, for example, when

[2] The first two derivatives at $v = 1$ are needed for computation of the $O(\frac{1}{n-1})$ and $O(\frac{1}{(n-1)^2})$ terms in the asymptotic expansion (see (4)), and the third derivative at $v = 1$ is needed so that the $O(\frac{1}{(n-1)^3})$ error term will be uniformly bounded.

there are K "types" of bidders, such that there are $\beta_k n$ bidders with the CDF $F_k(v)$, where $k = 1, \ldots, K$ and $\{\beta_k\}_{k=1}^K$ are integers.[3,4]

For future reference, we note the following technical identities.

LEMMA 2.1. *Let $f_G(v) := F_G'(v)$ denote the density of geometric mean. Then*

$$f_G(1) = \frac{1}{n}\sum_{i=1}^n f_i(1), \qquad f_G'(1) = f_G^2(1) + \frac{1}{n}\sum_{i=1}^n \left(f_i'(1) - f_i^2(1)\right). \tag{7}$$

In addition, let $\operatorname{Var}[\mathbf{f}] := E[\mathbf{f}^2] - (E[\mathbf{f}])^2$, *where* $\mathbf{f} = [f_1, \ldots, f_n]$ *and* $E[\mathbf{f}] := \frac{1}{n}\sum_{i=1}^n f_i$. *Then*

$$\operatorname{Var}[f_1(1), \ldots, f_n(1)] = \frac{1}{n}\sum_{i=1}^n f_i^2(1) - f_G^2(1). \tag{8}$$

Proof. Differentiation of (6) and substitution of $F_i(1) = 1$ yield relations (7). Relation (8) follows from (7). ∎

2.3. Weakly asymmetric auctions. Consider the case where

$$F_i = F_i(v;\epsilon), \quad F_i \text{ is smooth in } \epsilon, \quad F_i(v;\epsilon = 0) = F(v), \qquad i = 1, \ldots, n. \tag{9}$$

Since

$$F_i(v;\epsilon) = F(v) + \epsilon H_i(v) + O(\epsilon^2), \qquad H_i(v) = \left.\frac{\partial F_i}{\partial \epsilon}\right|_{\epsilon=0},$$

$\epsilon = 0$ is the symmetric case, and $\epsilon \ll 1$ corresponds to a weak asymmetry.

Since there is revenue equivalence when $\epsilon = 0$, and since an $O(\epsilon)$ change in the CDFs leads to an $O(\epsilon)$ change in the revenue, one could expect that weakly symmetric auctions are $O(\epsilon)$ revenue equivalent. Fibich and Gavious [4] showed, however, that

$$R^{\text{1st}}[F_1(v;\epsilon), \ldots, F_n(v;\epsilon)] - R^{\text{2nd}}[F_1(v;\epsilon), \ldots, F_n(v;\epsilon)] = O(\epsilon^2), \qquad \epsilon \ll 1. \tag{10}$$

This result was generalized to any two asymmetric auction mechanisms that satisfy the conditions of the classical Revenue Equivalence Theorem [7]. It is a consequence of the facts that the seller's revenue (i) is differentiable in ϵ and (ii) remains unchanged if we permute the identities (indices) of the bidders [8, 20].

2.4. Asymmetric second-price, first-price, and optimal auctions. In asymmetric second-price auctions, the optimal bidding strategy for any player is to bid his true value, i.e., $b_i^{\text{2nd}}(v_i) = v_i$, and so the seller's revenue can be written as [4]

$$R^{\text{2nd}}[F_1, \ldots, F_n] = 1 - (1-n)\int_0^1 \prod_{i=1}^n F_i(v)\,dv - \sum_{i=1}^n \int_0^1 \prod_{j\neq i} F_j(v)\,dv. \tag{11}$$

In Myerson's optimal auction,[5] the seller's revenue can be written as

$$R^{\text{opt}} = 1 - \int_0^1 \left(\prod_{i=1}^n F_i\left(\Psi_i^{-1}(v)\right)\right) dv, \qquad \Psi_i(v) := v - \frac{1 - F_i(v)}{f_i(v)}, \tag{12}$$

[3] This setup is motivated by a tradition in economics to consider what happens when an economy is "replicated," i.e., when one proportionally adds "clones" to the population.

[4] In this case $F_G[\mathcal{F}^{(n)}] = (\prod_{k=1}^K F_k^{\beta_k})^{1/\sum_{k=1}^K \beta_k}$ is indeed independent of n.

[5] We only consider the regular case where the virtual valuations are strictly monotonically increasing functions.

where $\Psi_i(v)$ is the *virtual valuation* of bidder i. Finally, in asymmetric first-price auctions, the revenue can be written as [4]

$$R^{\text{1st}}[F_1, \ldots, F_n] = \bar{b} - \int_0^{\bar{b}} \prod_{i=1}^{n} F_i(v_i(b))\, db, \tag{13}$$

where $v_i(b)$ is the inverse equilibrium strategy of the ith bidder and $\bar{b}$ is the common maximal bid of all bidders. Therefore, obtaining an asymptotic expansion for R^{1st} requires having an explicit expression for $\{v_i(b)\}_{i=1}^n$. These functions satisfy the nonstandard, free-boundary, nonlinear boundary value problem [20, 24]

$$v_i'(b) = \frac{F_i(v_i(b))}{f_i(v_i(b))}\left[\left(\frac{1}{n-1}\sum_{j=1}^{n}\frac{1}{v_j(b)-b}\right) - \frac{1}{v_i(b)-b}\right], \qquad i = 1, \ldots, n, \tag{14}$$

for $0 < b < \bar{b}$, subject to the $2n$ boundary conditions $v_i(0) = 0$ and $v_i(\bar{b}) = 1$ for $i = 1, \ldots, n$, where the location $\bar{b}$ of the right boundary is unknown. Recently, Fibich and Gavish [10] showed that when $n \gg 1$, $\{v_i'(b)\}_{i=1}^n$ (but not $\{v_i(b)\}_{i=1}^n$) have a *boundary layer* of $O(1/n^2)$ width near the common maximal bid $\bar{b}$. Although the width of the boundary layer shrinks to zero as $n \to \infty$, one has to resolve the behavior of $v_i(b)$ in the boundary layer region in order to compute the revenue (13) to $O(1/n^3)$, since most of the revenue comes from bids near $\bar{b}$ (i.e., from bidders with $v \approx 1$).

3. Asymptotic calculation of the revenue.

In the section, we expand the seller's revenue (18) in large asymmetric auctions asymptotically using the Laplace method for integrals.

3.1. R^{2nd}, R^{1st}, and $R^{\text{opt.}}$.

We begin with the two most common auction mechanisms: First-price and second-price. We also consider Myerson's optimal mechanism which provides an upper bound to the seller's revenue.

THEOREM 3.1. *Consider a sequence $\{\mathcal{F}^{(n)}\}$, with an increasing n, of sets of n players with CDFs $\mathcal{F}^{(n)} = \{F_1^{(n)}, \ldots, F_n^{(n)}\}$, such that Conditions* 1–3 *hold. Let R denote the revenue in a second-price auction, a first-price auction, or in Myerson's optimal mechanism. Then*

$$R[F_1^{(n)}, \ldots, F_n^{(n)}] = R_{\text{asymp}}[F_1^{(n)}, \ldots, F_n^{(n)}] + O\left(\frac{1}{n^3}\right), \qquad n \gg 1, \tag{15}$$

where

$$R_{\text{asymp}}[F_1, \ldots, F_n] = 1 - \frac{1}{n-1}\frac{2}{f_G(1)} - \frac{3f_G'(1) - 4f_G^2(1) + \operatorname{Var}[f_1(1), \ldots, f_n(1)]}{(n-1)^2 f_G^3(1)} \tag{16}$$

and F_G, $f_G(1)$, and $\operatorname{Var}[f_1(1), \ldots, f_n(1)]$ are given by (6), (7), *and* (8), *respectively.*

Proof. The results for second-price auctions and the optimal mechanism will follow from Theorem 3.2 below. For completeness, we also provide direct calculations for second-price auctions (supplementary SM1 and for the optimal mechanism supplementary SM2). The calculations for the first-price auction are presented in Appendix A. ∎

Since the asymptotic approximation in Theorem 3.1 has an $O(1/n^3)$ accuracy, one can expect it to become accurate already at moderate values of n. We illustrate this in the following example.

Example 1. Consider $n/2$ players whose CDF is $F_1(v)$, and $n/2$ players whose CDF is $F_2(v)$, where

$$F_1 = v^2, \qquad F_2 = \frac{1-e^{v/2}}{1-e^{1/2}}. \tag{17}$$

Since $F_G = \sqrt{v^2 \frac{1-e^{v/2}}{1-e^{1/2}}}$, then $f_G(1) \approx 1.635$, $f_G'(1) \approx 1.184$, and $\mathrm{Var}[f_1(1), \dots, f_n(1)] \approx 0.132$ for all n. Therefore, by (16),

$$R_{\text{asymp}}[F_1, \dots, F_n] = 1 - \frac{c_1}{n-1} + \frac{c_2}{(n-1)^2}, \qquad c_1 \approx 1.223, \quad c_2 \approx 1.603.$$

To check the accuracy of approximating R with R_{asymp}, we computed R^{2nd} and R^{opt} from (11) and (12), respectively, using standard quadrature methods, and computed the inverse equilibrium strategies (14) and R^{1st} from (13) using the boundary-value method from [9]. These calculations (see Figure 1) show that

$$R_{\text{asymp}} - R^{\text{2nd}} \sim \frac{2.34}{n^3}, \qquad R_{\text{asymp}} - R^{\text{opt}} \sim \frac{2.30}{n^3}, \qquad R_{\text{asymp}} - R^{\text{1st}} \sim \frac{2.34}{n^3},$$

thus confirming the $O(1/n^3)$ accuracy predicted in Theorem 3.1. Notably, the approximation error is less than 0.01 already for $n = 6$ bidders, and below 0.001 for $n = 12$.

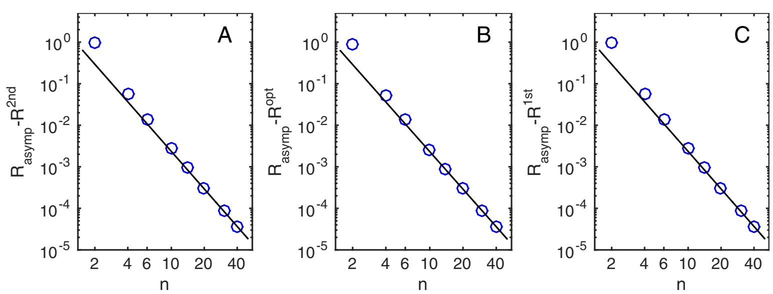

FIG. 1. *The revenue difference $R_{\text{asymp}} - R$ (circles) as a function of n for $n = 2, 4, 6, 10, 14, 20, 30,$ and 40 players from Example 1. Both axes are on a log scale.* A: $R = R^{\text{2nd}}$. *The solid curve is* $2.34/n^3$. B: $R = R^{\text{opt}}$. *The solid curve is* $2.30/n^3$. C: $R = R^{\text{1st}}$. *The solid curve is* $2.34/n^3$.

3.2. The general case. Since the asymptotic expansions in first-price, second-price, and optimal auctions turned out to be identical up to $O(1/n^3)$, it is natural to ask whether this result can be extended to a broader class of auctions, e.g., all-pay auctions in which all players pay their bid, k-price auctions in which the winner pays the kth largest bid, auctions which introduce a reserve price, etc. The following theorem suggests that this is indeed the case.

THEOREM 3.2. *Consider a sequence $\{\mathcal{F}^{(n)}\}$, with an increasing n, of sets of n players with CDFs $\mathcal{F}^{(n)} = \{F_1^{(n)}, \dots, F_n^{(n)}\}$, such that Conditions* 1–3 *hold. Consider an auction mechanism for which the following* 6 *conditions also hold:*

4. *The object is allocated to the player with the highest bid.*
5. *Let $r_i \geq 0$ be the reservation price of bidder i. Then, any player i with valuation r_i expects a zero surplus.*

6. *The equilibrium bidding strategies* $b_i(v)$ *are strictly monotonically increasing in* $[r_i, 1]$.
7. *The equilibrium bidding strategies* $b_i(v)$ *are three times continuously differentiable near* $v = 1$.
8. *The maximal bid is identical for all bidders; i.e., for a given* n, $b_i(1) = b_j(1)$ *for* $1 \le i, j \le n$.
9. *The asymmetry among the derivatives of the equilibrium bids at the maximal value is at most* $O(1/n)$, *i.e.,* $b_i'(1) - b_j'(1) = O\left(\frac{1}{n}\right)$ *for* $1 \le i, j \le n$ *as* $n \to \infty$.

Then the seller's expected revenue satisfies (15).

Proof. From Conditions 1, 2, 4, and 5 it follows that the expected revenue can be written as [7]

$$R = -\sum_{i=1}^{n} \int_{r_i}^{1} \left(\prod_{j=1}^{n} F_j(b_j^{-1}(b_i(v)) \right) \frac{1 - F_i(v) - v f_i(v)}{F_i(v)} \, dv, \tag{18}$$

where $F_j(b_j^{-1}(b_i)) := F_j(r_j)$ for $b_i < r_j$. Since essentially all the contributions to the revenue come from bidders with nearly maximal valuation ($v \approx 1$), we can expand R asymptotically using the *Laplace method for integrals*. The resulting expressions, however, are long and cumbersome, making the analysis intractable. The key to the proof is, therefore, a series of identities (Lemmas B.2–B.4) that significantly reduce the intermediate resulting expressions. See Appendix B for details. ∎

It is remarkable that we can compute the revenue without any information on the payment rule. As in the case of symmetric auctions [25, 27], this is because expression (18) incorporates the payment rule implicitly though the condition of incentive compatibility. In the symmetric case $b_j^{-1}(b_i(v)) = v$, and so revenue equivalence follows immediately from (18). This is not the case, however, when bidders are asymmetric, which therefore requires considerable more work.

Conditions 1, 2, 4, and 5 of Theorem 3.2 are the same as those of the classical revenue equivalence theorem for symmetric auctions [25, 27], except that in Condition 2 the CDFs are not identical. Condition 6 is the same as in Riley and Samuelson [27] and is required to ensure that the inverse bidding strategies $b_i^{-1}(v)$ in (18) are well defined. Condition 3 is needed in order to make sense of the limit of the CDFs as $n \to \infty$; see section 2.2. Condition 7 is needed in order to expand the bids in a Taylor series around $v = 1$. Condition 8 is satisfied by asymmetric first-price and all-pay auctions (see, e.g., [11]), by asymmetric second-price auctions (since $b_i^{\text{2nd}}(v_i) = v_i$), and by Myerson's optimal mechanism (since $\Psi_i(1) = 1$; see (12)). In addition, it was shown that $\lim_{n\to\infty} b_i(v) = v$ under quite general conditions [1, 29].

The motivation for Condition 9 is as follows. Since the environment (competition) that players i and j face differs by one out of $n-1$ players (i is facing j but not i and vice versa), one could expect the resulting asymmetry among the equilibrium bids to be $O(1/n)$. Indeed, Condition 9 is satisfied for second-price auctions (since $(b_i^{\text{2nd}})'(1) = 1$) and for Myerson's optimal mechanism (since $\Psi_i'(1) = 2$; see (12)). In first-price auctions, however, the equilibrium bids satisfy $\frac{b_i'(1)}{b_k'(1)} = \frac{f_i(1)}{f_k(1)}$ (see [10]), and so Condition 9 is *not* satisfied.[6] Note that in contrast to Theorem 3.1, Theorem 3.2 applies to Myerson's optimal mechanism also in the nonregular case where the virtual valuations may not be strictly monotonically increasing outside a region near $v = 1$.

[6]Indeed, by (20), $v_i'(1) - v_k'(1) = (v_k^{\text{inner}})'(\xi = 0) - (v_i^{\text{inner}})'(\xi = 0) = \sum_{j=1}^{n-1} \lambda_j (a_{kj} - a_{ij}) \neq O(1/n)$.

4. $O(1/n^3)$ asymptotic revenue equivalence. As noted in section 1, the limiting revenue equivalence result (3) suggests that if A and B are two auction mechanisms, then $R^{\mathrm{A}} - R^{\mathrm{B}} \sim c_1/n$. Theorems 3.1 and 3.2, however, lead to the following much stronger result.

COROLLARY 4.1. *Let A and B be a first-price auction, second-price auction, Myerson optimal auction, or any auction mechanism for which Theorem* 3.2 *holds. Consider an increasing sequence of sets of n players with CDFs $\mathcal{F}^{(n)} = \{F_1^{(n)}, \dots, F_n^{(n)}\}$, such that Conditions* 1–3 *hold. Then, the corresponding revenue difference satisfies $R^{\mathrm{A}}[F_1^{(n)}, \dots, F_n^{(n)}] - R^{\mathrm{B}}[F_1^{(n)}, \dots, F_n^{(n)}] = O\left(\frac{1}{n^3}\right)$, where $n \gg 1$.*

A natural question is whether we might get an $O\left(\frac{1}{n^4}\right)$ revenue equivalence if we continue the asymptotic calculations to the next order. The answer to this question is negative.

LEMMA 4.2. *Corollary* 4.1 *is optimal in the sense that there exist cases for which the revenue differences between two large auctions scale as c/n^3, where $c \neq 0$.*

Proof. Consider $n/2$ players whose CDF is $F_1(v) = v$, and $n/2$ players whose CDF is $F_2(v) = v^2$. A direct calculation gives $R^{\mathrm{2nd}} \sim 1 - \frac{4}{3}\frac{1}{n-1} + \frac{52}{27}\frac{1}{(n-1)^2} - \frac{252}{81}\frac{1}{(n-1)^3}$ and $R^{\mathrm{opt}} \sim 1 - \frac{4}{3}\frac{1}{n-1} + \frac{52}{27}\frac{1}{(n-1)^2} - \frac{244}{81}\frac{1}{(n-1)^3}$ (supplementary SM3). Hence, $R^{\mathrm{opt}} - R^{\mathrm{2nd}} \sim \frac{8}{81}\frac{1}{(n-1)^3}$. ∎

Similarly, in Figure 2 we observe numerically for the bidders from Example 1 that

$$R^{\mathrm{2nd}} - R^{\mathrm{1st}} \sim \frac{c_3}{n^3}, \qquad c_3 \approx 0.008. \tag{19}$$

In most other cases that we tested, we also observe numerically that $R^{\mathrm{2nd}} - R^{\mathrm{1st}}$ decays as $1/n^3$. Interestingly, however, in [9] we observed numerically that $R^{\mathrm{1st}} - R^{\mathrm{2nd}} \sim c/n^4$ for the bidders from the proof of Lemma 4.2. We leave as an open problem the characterization of conditions under which $R^{\mathrm{1st}} - R^{\mathrm{2nd}}$ decays faster than $1/n^3$.

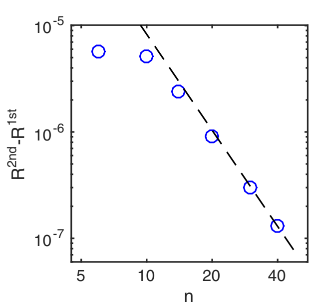

FIG. 2. *$R^{\mathrm{2nd}} - R^{\mathrm{1st}}$ (○) as a function of n for $n = 6, 10, 14, 20, 30$, and 40 players from Example* 1. *The dashed curve is $0.0082/n^3$. Both axes are on a log scale.*

4.1. Effect of asymmetry on the revenue. The first question in homogenization theory is to find the correct average. Our results show that in large auctions, it is given by the geometrical average. The second question in homogenization theory is whether the homogeneous solution lies above or below the inhomogeneous one. This problem was first considered by Cantillon [2], who showed that in the second-price auction and in certain cases also in the first-price auction, the revenue in the asymmetric case is smaller than in the symmetric case with $F = F_G$. We now extend this result and quantify its extension to a wide class of large auctions. Indeed, since $\mathrm{Var}[f_1(1), \ldots, f_n(1)]$ is equal to zero in the symmetric case and is positive in the asymmetric case, Theorems 3.1 and 3.2 show that asymmetry always hurts the revenue in large auctions.

Corollary 4.3. *Let* $\mathrm{Var}[f_1(1), \ldots, f_n(1)] > 0$. *Then asymmetry decreases the revenue in large second-price auctions, regular optimal auctions, first-price auctions, and auctions that satisfy the conditions of Theorem* 3.2. *Moreover, the leading-order effect of asymmetry on the revenue is given by*

$$R[F_1, \ldots, F_n] - R[F_G, \ldots, F_G] \sim -\frac{1}{n^2} \frac{\mathrm{Var}[f_1(1), \ldots, f_n(1)]}{f_G^3(1)}, \qquad n \gg 1.$$

Proof. The revenue in the symmetric case can be expanded as [5]

$$R[F, \ldots, F] = 1 - \frac{1}{n-1} \frac{2}{f(1)} - \frac{1}{(n-1)^2} \left[\frac{3f'(1)}{f^3(1)} - \frac{4}{f(1)} \right] + O\left(\frac{1}{n^3}\right), \qquad n \gg 1.$$

Hence, the result follows from Theorems 3.1 and 3.2. ∎

5. $O(\epsilon^2/n^3)$ asymptotic revenue equivalence. The fitted constant c_3 in (19) is very small. Thus, while we expected the revenue difference with, e.g., $n = 10$ bidders to be in the third digit, it turns out to be in the fifth digit. We observed a similar behavior, namely, that $R^{\text{2nd}} - R^{\text{1st}} \sim c_3/n^3$ with $c_3 \ll 1$ in numerous other simulations (data not shown). To understand why $c_3 \ll 1$, we recall that there are two mechanisms which act to decrease the revenue difference $R^{\text{2nd}} - R^{\text{1st}}$. The first one is the asymptotic revenue equivalence of large auctions (Corollary 4.1). The second one is the asymptotic revenue equivalence of a weakly asymmetric auction (section 2.3). Indeed, combining Corollary 4.1 and relation (10) yields our main result.

Theorem 5.1 (asymptotic revenue equivalence). *Assume the conditions of Corollary* 4.1, *and let* $F_i^{(n)} = F_i^{(n)}(v; \epsilon)$, *such that* (9) *holds. Then*

$$R^{\mathrm{A}}[F_1^{(n)}, \ldots, F_n^{(n)}] - R^{\mathrm{B}}[F_1^{(n)}, \ldots, F_n^{(n)}] = O\left(\frac{\epsilon^2}{n^3}\right), \qquad n \gg 1, \quad \epsilon \ll 1.$$

Example 2. We reinterpret relation (19) in light of Theorem 5.1. Since "there is no ϵ" in Example 1, we "introduce" it into the problem as follows. Let

$$\bar{F} := \frac{F_1 + F_2}{2} = \frac{1}{2}\left(v^2 + \frac{1 - e^{v/2}}{1 - e^{1/2}}\right), \qquad \epsilon_0 := \max_{i=1,2} \max_{0 \le v \le 1} |F_i - \bar{F}| \approx 0.094.$$

Then we can rewrite (17) as

$$F_1 = \bar{F} + \epsilon_0 H, \qquad F_2 = \bar{F} - \epsilon_0 H, \qquad H := \frac{F_1 - F_2}{2\epsilon_0} = \frac{1}{2\epsilon_0}\left(v^2 + \frac{1 - e^{v/2}}{1 - e^{1/2}}\right).$$

Therefore, if we define $F_{1,2}(v; \epsilon) = \bar{F}(v) \pm \epsilon H(v)$, the CDFs (17) correspond to $F_{1,2}(v; \epsilon_0)$. Figure 3A shows that for a fixed ϵ, $R^{\text{2nd}}(n, \epsilon) - R^{\text{1st}}(n, \epsilon) \sim c_3(\epsilon)/n^3$. To show that $c_3(\epsilon) \sim \tilde{c}_3 \cdot \epsilon^2$, in Figure 3B we plot $(R^{\text{2nd}} - R^{\text{1st}})/\epsilon^2$ as a function of n and observe that for all ϵ, it scales as $\tilde{c}_3/n^3$, where $\tilde{c}_3 \approx 2$. Therefore, we conclude that, as predicted in Theorem 5.1,

$$R^{\text{2nd}} - R^{\text{1st}} \sim \tilde{c}_3 \frac{\epsilon^2}{n^3},$$

where $\tilde{c}_3 \approx 2$. The fact that $\tilde{c}_3$ is of order 1 provides strong support that this is, indeed, the "correct" scaling. In particular, for the CDFs (17) we have that $\tilde{c}_3\epsilon_0^2 \approx 2 \cdot 0.15^2 = 0.045$, explaining the smallness of c_3 in (19).

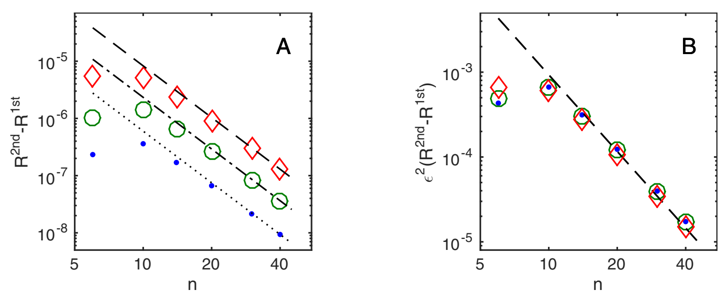

FIG. 3. A: $R^{\text{2nd}} - R^{\text{1st}}$ *as a function of* n *for* $n = 4, 6, 10, 20,$ *and* 40 *players from Example* 2. *Here* $\epsilon = \epsilon_0 \approx 0.094$ ($\diamond$), $\epsilon = \epsilon_0/2$ ($\circ$), *and* $\epsilon = \epsilon_0/4$ ($\cdot$). *The dashed curve is* $0.0082/n^3$, *the dash-dotted curve is* $0.0023/n^3$, *and the dotted curve is* $0.0006/n^3$. B: *The normalized difference* $(R^{\text{2nd}} - R^{\text{1st}})/\epsilon^2$ *as a function of* n *for the same data as in* A. *Both subplots are on a loglog scale.*

Theorem 5.1 implies, in particular, that large asymmetric auctions are $O(\epsilon^2/n^3)$ asymptotically optimal.

COROLLARY 5.2 ($O(\epsilon^2/n^3)$ asymptotic optimality). *Let A be an auction mechanism that satisfies the conditions of Theorem* 5.1. *Then*

$$R^{\text{opt}}[F_1^{(n)}, \ldots, F_n^{(n)}] - R^{\text{A}}[F_1^{(n)}, \ldots, F_n^{(n)}] = O\left(\frac{\epsilon^2}{n^3}\right), \qquad n \gg 1, \quad \epsilon \ll 1.$$

Example 3. Consider $n/2$ players with $F_1(v) = v + \epsilon v(1 - v)$ and $n/2$ players with $F_2(v) = v - \epsilon v(1 - v)$. Figure 4 shows that $R^{\text{opt}} - R^{\text{1st}} \sim 4.61\epsilon^2/n^3$ and $R^{\text{opt}} - R^{\text{2nd}} \sim 5.13\epsilon^2/n^3$, in agreement with Corollary 5.2. In particular, for $n \geq 20$ the revenue "losses" $R^{\text{opt}} - R^{\text{1st}}$ and $R^{\text{opt}} - R^{\text{2nd}}$ are below 0.01%.

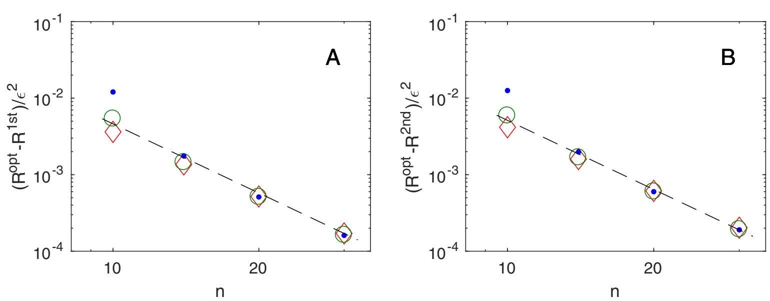

FIG. 4. A: *The normalized difference* $(R^{\text{opt}} - R)/\epsilon^2$ *for* $n = 10, 14, 20,$ *and* 30 *players from Example* 3 *with* $\epsilon = 0.4$ ($\diamond$), $\epsilon = 0.2$ ($\circ$), *and* $\epsilon = 0.1$ ($\cdot$). A: $R = R^{\text{1st}}$. *The dashed curve is* $4.61/n^3$. B: $R = R^{\text{2nd}}$. *The dashed curve is* $5.13/n^3$. *Both subplots are on a loglog scale.*

6. How small can a large auction be? The asymptotic approximation $R = R_{\text{asymp}} + O(1/n^3)$ was derived for $n \gg 1$. To test the regime for which this approximation is valid, in Table 1 we compute the accuracy of R_{asymp} for several examples with few ($5 \leq n_0 \leq 10$) asymmetric bidders. Since in each example we are given a single set of payers $\mathcal{F}^{(n_0)}$, in order to relate it to the asymptotic analysis, we can think of it as belonging to a sequence $\{\mathcal{F}^{(n)}\}$ whose geometrical mean is constant. Therefore, we can approximate the corresponding revenue $R[\mathcal{F}^{(n_0)}]$ using the asymptotic expansions for $R[\mathcal{F}^{(n)}]$, which was derived for $n \to \infty$. For example, the CDFs in the top line can be embedded in the sequence $\left\{v, v^3, v^5\right\}, \left\{v, v^2, v^3, v^4, v^5\right\}, \dots$.

TABLE 1

The asymptotic approximation R_{asymp} and its relative accuracy $\frac{R_{\text{asymp}}-R}{R}$ for various auctions and bidders.

CDFs	R_{asymp}	$\frac{R_{\text{asymp}}-R^{\text{1st}}}{R^{\text{1st}}}$	$\frac{R_{\text{asymp}}-R^{\text{2nd}}}{R^{\text{2nd}}}$	$\frac{R_{\text{asymp}}-R^{\text{opt}}}{R^{\text{opt}}}$
$F_i = v^i, \quad i = 1, \dots, 5$	0.8703	1.09%	1.11%	0.91%
$F_{1,2} = \frac{v+v^2}{2}$, $F_{3,4} = \frac{1-e^v}{1-e}$, $F_{5,6} = v^3$	0.8449	1.05%	1.06%	0.93%
$F_{1,2} = \frac{1-e^v}{1-e}$, $F_{3,4} = \frac{1-e^{v/2}}{1-e^{1/2}}$, $F_{5,6} = v^2$	0.81345	1.50%	1.50%	1.38%
$F_{1,2} = \frac{1-e^v}{1-e}$, $F_{3,4} = \frac{1-e^{v/2}}{1-e^{1/2}}$, $F_{5,6} = v$	0.7733	2.12%	2.12%	1.86%
$F_{1,2} = \frac{v+v^2}{2}$, $F_{3,4} = \frac{1-e^v}{1-e}$, $F_{5,\dots,10} = v^3$	0.91856	0.14%	0.14%	0.13%
$F_i = v^i, \quad i = 1, \dots, 10$	0.9624	0.041%	0.041%	-0.002%

Table 1 shows that already for as few as 5 bidders, the accuracy of R_{asymp} is 1%–2%. In contrast, the accuracy of the limiting result $\lim_{n\to\infty} R = 1$ is only 10%–20%. With 10 bidders, the accuracy of R_{asymp} improves to 0.002%–0.14%, whereas that of the limiting result $\lim_{n\to\infty} R = 1$ is only 4%–8%. This data, together with Figures 3 and 4 and additional numerical examples (data not shown), suggests that our asymptotic expressions provide a good estimate for the revenue, even with relatively few bidders and not-so-small asymmetry. This suggests, therefore, that the revenue difference between asymmetric auctions is generically small, even in small auctions.

To illustrate the last point, consider a small auction with 2 players with $F_{1,2}(v) = v \pm \epsilon v(1-v)$. Previously, Fibich, Gavious, and Sela [7] showed that $R^{\text{1st}} - R^{\text{2nd}} \sim c\epsilon^2$ for $0 \le \epsilon \le 0.4$, where $c \approx 0.06$, thus confirming the $O(\epsilon^2)$ revenue equivalence. At that time, however, we did not pay attention to the smallness of c. Motivated by Theorem 5.1, we now observe that $R^{\text{1st}} - R^{\text{2nd}} \sim \tilde{c}_3\epsilon^2/n^3$, where $\tilde{c}_3 \approx 1/2$; see Table 2. Since $\tilde{c}_3$ is $O(1)$, this appears to be the "correct" scaling. In particular, we observe that the scaling $R^{\text{1st}} - R^{\text{2nd}} \sim \tilde{c}_3\epsilon^2/n^3$ is valid for n as small as 2 and ϵ as large as 0.4.

TABLE 2

The revenue difference for $n = 2$ players with $F_{1,2}(v) = v \pm \epsilon v(1-v)$ is well approximated by $\frac{1}{2}\frac{\epsilon^2}{n^3}$. Data in first row are taken from [7].

ϵ	0	0.1	0.2	0.3	0.4
$R^{\text{1st}} - R^{\text{2nd}}$	0	0.006	0.0025	0.0057	0.01
$\frac{1}{2}\frac{\epsilon^2}{n^3}$	0	0.006	0.0025	0.0055	0.01

7. Final remarks. The key result of this study is Theorem 3.2—the $O(1/n^3)$ asymptotic computation of the revenue of asymmetric auctions, without any information on the payment rules. Theorem 3.2 presents a significant improvement over the current $o(1)$ limiting revenue result for asymmetric auctions; see (2). Moreover, this theorem shows that the revenue of large auctions is asymptotically independent of the reserve price. This is intuitive, since the revenue depends "only" on the high-valuation players. Unfortunately, Theorem 3.2 is far from optimal, predominantly because of Condition 9 on the derivatives of the unknown bidding strategies. This condition, however, is probably not essential. Indeed, the result of Theorem 3.2 holds for first-price auctions which satisfy all conditions except Condition 9. Thus, it seems that Theorem 3.2 should be applicable to a larger class of auctions. We leave this open question for a future study.

The assumption that the supports of $\{F_i\}$ are common to all players can be relaxed. For example, if the number of players for which the right boundary point is maximal is unbounded as $n \to \infty$, then it is possible to neglect the revenue contributions of all other players (whose right boundary point is less than maximal).

An immediate consequence of the asymptotic revenue equivalence is *asymptotic optimality.* Indeed, Corollary 5.2 shows that large auctions such as first-price or second-price auctions are nearly optimal. Hence, the additional revenue that may be generated by "modifying" these suboptimal auction mechanisms (e.g., by imposing a reserve price) is at most $O(\frac{\epsilon^2}{n^3})$ and so may not be worth the effort.

Symmetric auctions are significantly easier to study than asymmetric auctions.[7] Therefore, it is convenient to assume that bidders are symmetric, as is indeed done in almost all studies in auction theory. Until now, however, the validity of this assumption was not analyzed. Theorem 3.2 quantifies the error introduced by this approximation and shows that it is justified even for auctions with as little as 5–6 bidders.

From an economics perspective, this study provides a quantitative meaning to the terminology "large auction." Until now, the limiting revenue equivalence result, equation (3), implicitly implied that asymmetric auctions are essentially revenue equivalent only when there are hundreds of players. Our surprising result that large asymmetric auctions are $O(\epsilon^2/n^3)$ revenue equivalent shows that auctions with several bidders are already essentially revenue equivalent. This insight is important for experimental economists who study large auctions. In these experiments, knowing that the auction is already large with 5 instead of 50 bidders means that with the same budget, one can conduct 10 times the number of experiments.

Appendix A. Asymptotic expansion of $R^{\mathbf{1st}}$. To evaluate (13) for $n \gg 1$, we use the asymptotic expansion for $\{v_i(b)\}$ and $\bar{b}$ (see [10]):

(20a)
$$v_i(b) = v_i^{\text{outer}}(b) + \frac{v_i^{\text{inner}}(\xi(b))}{(n-1)^2} + O\left(\frac{1}{(n-1)^3}\right), \qquad \xi(b) = (n-1)^2\left(\bar{b} - b\right), \qquad n \to \infty.$$

Here the outer solution and its derivatives are $O(1)$ for $0 \le b \le \bar{b}$:

(20b)
$$v_i^{\text{outer}}(b) = b + \frac{u(b)}{n-1} + \frac{w_i(b)}{(n-1)^2}, \quad u(b) = \frac{F_G(b)}{f_G(b)}, \quad w_i(b) = \left(\frac{F_G^2(b)}{f_G^2(b)}\frac{f_i(b)}{F_i(b)} - \frac{F_G(b)}{f_G(b)}\right).$$

The $O(1/n^2)$ inner solution is given by

$$v_i^{\text{inner}}(\xi) = \frac{1}{(n-1)^2}\sum_{j=1}^{n-1} a_{ij} e^{-\lambda_j \xi}, \tag{20c}$$

[7]For example, in an asymmetric first-price auction, the equilibrium bidding strategies satisfy a nonstandard, free-boundary, nonlinear boundary value problem of n coupled equations (see (14)), while in the symmetric case this system reduces to a single first-order initial value problem that can be solved explicitly.

where $\bar{b}$ has the expansion

$$\text{(20d)}\quad \bar{b} = 1-\frac{1}{n-1}\frac{1}{f_G(1)}+\frac{\bar{b}_2}{(n-1)^2}+O\left(\frac{1}{n^3}\right), \qquad \bar{b}_2 = \frac{2}{f_G(1)}-\frac{1}{f_G^3(1)}\frac{1}{n}\sum_{i=1}^{n} f_i'(1).$$

Unlike in second-price auctions (see (11)), the right boundary point $\bar{b}$ of the integral in (13), where the maximum of the integrand is attained, varies with n. Therefore, we expand $R^{1\text{st}}$ using the *Laplace method for integrals with a moving maximum point*, as follows. By (13), $R^{1\text{st}} = \bar{b} - I^{1\text{st}}$, where

$$\text{(21)}\qquad I^{1\text{st}} := \int_0^{\bar{b}} \prod_{i=1}^{n} F_i(v_i(b))db = \int_0^{\bar{b}} \exp\left[\sum_{i=1}^{n} \log F_i(v_i(b))\right] db.$$

To expand $I^{1\text{st}}$ in $\frac{1}{n-1}$, we first substitute $v_i(b)$ (see (20)) into $F_i(v_i(b))$. This yields

$$F_i(v_i(b)) = F_i(b)+\frac{f_i(b)u(b)}{n-1}+\frac{1}{(n-1)^2}\left[\left(w_i(b)+v_i^{\text{inner}}(\xi)\right)f_i(b)+\frac{f_i'(b)}{2}u^2\right]+O\left(\frac{1}{n^3}\right).$$

Hence, $\log F_i(v_i(b)) = \log F_i(b) + \frac{u(b)}{n-1}\frac{f_i(b)}{F_i(b)} + \frac{g_i(b)}{(n-1)^2} + O\left(\frac{1}{n^3}\right)$, where

$$g_i(b) = \frac{1}{F_i(b)}\left[\left(w_i(b)+v_i^{\text{inner}}(\xi)\right)f_i(b)+\frac{f_i'(b)}{2}u^2-\frac{f_i^2(b)u^2(b)}{2F_i(b)}\right].$$

Thus, using (6),

$$\text{(22)}\quad \begin{aligned} \sum_{i=1}^{n}\log F_i(v_i(b)) &= \sum_{i=1}^{n}\log F_i(b)+\frac{u(b)}{n-1}\sum_{i=1}^{n}\frac{f_i(b)}{F_i(b)}+\frac{1}{(n-1)^2}\sum_{i=1}^{n} g_i(b)+O\left(\frac{1}{n^2}\right)\\ &= \log\prod_{i=1}^{n}F_i(b)+\frac{n}{n-1}+\frac{1}{(n-1)^2}\sum_{i=1}^{n} g_i(b)+O\left(\frac{1}{n^2}\right)\\ &= \log F_G^n(b)+1+\frac{1}{n-1}g(b)+O\left(\frac{1}{n^2}\right), \end{aligned}$$

where $g(b) = 1+\frac{1}{n-1}\sum_{i=1}^{n} g_i(b)$. Hence

$$\text{(23)}\quad \begin{aligned} \exp\left[\sum_{i=1}^{n}\log F_i(v_i(b))\right] &= \exp\left[\log F_G^n(b)+1+\frac{1}{n-1}g(b)+O\left(\frac{1}{n^2}\right)\right]\\ &= e\cdot F_G^n(b)\cdot\exp\left[\frac{g(b)}{n-1}+O\left(\frac{1}{n^2}\right)\right] = e\cdot F_G^n(b)\left[1+\frac{g(b)}{n-1}+O\left(\frac{1}{n^2}\right)\right], \end{aligned}$$

where in the last equality we used that $e^x = 1+x+O(x^2)$ for $x \ll 1$.

Substituting (23) into (21) yields

$$\text{(24)}\qquad I^{1\text{st}} = e\int_0^{\bar{b}} F_G^n(b)\left[1+\frac{g(b)}{n-1}+O\left(\frac{1}{n^2}\right)\right]db.$$

To expand this integral asymptotically using the Laplace method for integrals, let $t = -\log h(b)$ where $h(b) := F_G(b)/F_G(\bar{b})$.[8] Then $F_G^n(b) = e^{n\log F_G(b)} = e^{-n\left(t-\log F_G(\bar{b})\right)} =$

[8]The division by $F_G(\bar{b})$ maps the moving maximum point $\bar{b} = \bar{b}(n)$ to a stationary point $t = 0$.

$F_G^n(\bar b)e^{-nt}$. In addition, since $e^{-t}=h(b)$ and $b(t)=h^{-1}(e^{-t})$, then $dt=-\frac{f_G(b)}{F_G(b)}db=-\frac{f_G(b(t))}{F_G(\bar b)e^{-t}}db$. Therefore,

$$I^{1\mathrm{st}}=eF_G^{n+1}(\bar b)\int_0^\infty \frac{e^{-(n+1)t}}{f_G(b(t))}\left[1+\frac{g(b(t))}{n-1}+O\left(\frac{1}{n^2}\right)\right]dt.$$

To expand the integrand about $t=0$, we first note that

$$f_G(b(t))=f_G(b(0))+tf_G'(b(0))b'(0)+O(t^2)=f_G(\bar b)-t\frac{f_G'(\bar b)}{f_G(1)}F_G(\bar b)+O(t^2), \tag{25}$$

where in the last equality we used the fact that $b(0)=\bar b$ and

$$b'(0)=\left.\frac{db}{dt}\right|_{t=0}=-\frac{F_G(\bar b)}{f_G(\bar b)}. \tag{26}$$

Thus, using (20d) and (25) gives

$$\begin{aligned}\frac{1}{f_G(b(t))}&=\frac{1}{f_G(\bar b)}+\frac{f_G'(\bar b)F_G(\bar b)}{f_G(1)f_G^2(\bar b)}t+O(t^2)\\ &=\frac{1}{f_G(1)}+\frac{1}{n-1}\frac{f_G'(1)}{f_G^3(1)}+\frac{f_G'(1)F_G(1)}{f_G^3(1)}t+O\left(\frac{t}{n},t^2,\frac{1}{n^2}\right),\end{aligned} \tag{27}$$

where in the last equality we used that $\bar b-b(t)=O(t)$, since by (26), $b'(0)$ is bounded. Similarly,

$$1+\frac{g(b(t))}{n-1}=1+\frac{g(\bar b)}{n-1}+O\left(\frac{t}{n}\right)=1+\frac{g(1)}{n-1}+O\left(\frac{t}{n},\frac{1}{n^2}\right).$$

Overall,

$$I^{1\mathrm{st}}=e\cdot F_G^{n+1}(\bar b)\int_0^\infty\left[A_0+\frac{A_1}{n-1}+B_0t+O\left(\frac{t}{n},t^2,\frac{1}{n^2}\right)\right]e^{-(n+1)t}dt,$$

where $A_0=\frac{1}{f_G(1)}$, $A_1=\frac{g(1)}{f_G(1)}+\frac{f_G'(1)}{f_G^3(1)}$, and $B_0=\frac{f_G'(1)}{f_G^3(1)}$. Therefore, using the relation

$$\int_0^\infty n^\ell t^k e^{-(n+1)t}\,dt=\frac{n^\ell k!}{(n+1)^{k+1}},$$

we have

$$I^{1\mathrm{st}}=e\cdot F_G^{n+1}(\bar b)\left[\frac{A_0}{n-1}+\frac{A_1+B_0-2A_0}{(n-1)^2}+O\left(\frac{1}{n^3}\right)\right].$$

Substituting $\bar b$ (see (20d)) into $F_G(\bar b)$ yields $F_G(\bar b)=1-\frac{1}{n-1}+\frac{a_0}{(n-1)^2}+O\left(\frac{1}{n^3}\right)$, where $a_0=\frac{2f_G^3\bar b_2+f_G'(1)}{2f_G^2}$. Since $\ln(1+x)=x-x^2/2+O(x^3)$ for $x\ll 1$, one obtains $\ln F_G(\bar b)=-\frac{1}{n-1}+\frac{a_0}{(n-1)^2}-\frac{1}{2(n-1)^2}+O\left(\frac{1}{n^3}\right)$, and so $(n+1)\ln F_G(\bar b)=-1+\frac{a_0-\frac52}{n-1}+O\left(\frac{1}{n^2}\right)$ and $F_G^{n+1}(\bar b)=e^{(n+1)\ln F_G(\bar b)}=\frac1e(1+\frac{a_0-\frac52}{n-1})+O\left(\frac{1}{n^2}\right)$. Therefore,

$$\begin{aligned}I^{1\mathrm{st}}&=\frac{A_0}{n-1}+\frac{1}{(n-1)^2}\left[\left(a_0-\frac52\right)A_0+A_1+B_0-2A_0\right]+O\left(\frac{1}{n^3}\right)\\ &=\frac{1}{n-1}\frac{1}{f_G(1)}+\frac{1}{(n-1)^2}\left[\frac{a_0}{f_G(1)}+\frac{g(1)}{f_G(1)}+2\frac{f_G'(1)}{f_G^3(1)}-\frac{9}{2f_G(1)}\right]+O\left(\frac{1}{n^3}\right)\\ &=\frac{1}{n-1}\frac{1}{f_G(1)}+\frac{1}{(n-1)^2}\left[\bar b_2+\frac{2g(1)-9}{2f_G(1)}+\frac52\frac{f_G'(1)}{f_G^3(1)}\right]+O\left(\frac{1}{n^3}\right).\end{aligned} \tag{28}$$

Finally, to evaluate $g(1) = 1 + \frac{1}{n-1}\sum_{i=1}^n g_i(1)$, we recall that $w_i(1) + v_i^{\text{inner}}(0) = \frac{1}{f_G(1)} - \frac{f'_G(1)}{f_G^3(1)} - \bar{b}_2$; see [10]. Therefore, by (20d),

$$g_i(1) = g_i(\bar{b}) + O\left(\frac{1}{n}\right) = \left(w_i(\bar{b}) + v_i^{\text{inner}}(\xi(\bar{b}))\right) f_i(\bar{b}) + \frac{f'_i(\bar{b}) - f_i^2(\bar{b})}{2}u^2(\bar{b}) + O\left(\frac{1}{n}\right)$$
$$= \left[\frac{1}{f_G^3(1)}\frac{1}{n}\sum_{i=1}^n f_i^2(1) - \frac{1}{f_G(1)}\right] f_i + \frac{f'_i(1) - f_i^2(1)}{2}u^2(1) + O\left(\frac{1}{n}\right),$$

where we used that $\xi(\bar{b}) = 0$. Consequently,

$$\frac{1}{n}\sum_{i=1}^n g_i(1) = \frac{1}{f_G^2}\frac{1}{n}\sum_{i=1}^n f_i^2(1) - 1 + \frac{1}{2nf_G^2}\sum_{i=1}^n [f'_i(1) - f_i^2(1)] + O\left(\frac{1}{n}\right)$$
$$= \frac{1}{f_G^2}\frac{1}{n}\sum_{i=1}^n f_i^2(1) - 1 + \frac{1}{2}\left(\frac{f'_G(1)}{f_G^2(1)} - 1\right) + O\left(\frac{1}{n}\right),$$

where in the last equality we used (7). Hence,

$$g(1) = \frac{1}{f_G^2}\frac{1}{n}\sum_{i=1}^n f_i^2(1) + \frac{f'_G(1)}{2f_G^2(1)} - \frac{1}{2} + O\left(\frac{1}{n}\right).$$

Substituting $g(1)$ into (28) yields

$$I^{\text{1st}} = \frac{1}{n-1}\frac{1}{f_G(1)} + \frac{1}{(n-1)^2}\left[\bar{b}_2 + \frac{\frac{1}{n}\sum f_i^2(1)}{f_G^3(1)} + 3\frac{f'_G(1)}{f_G^3(1)} - \frac{5}{f_G(1)}\right].$$

Substituting $\bar{b}$ (see (20d)) and I^{1st} into $R^{\text{1st}} = \bar{b} - I^{\text{1st}}$ and using relation (8) yields the result.

Appendix B. Proof of Theorem 3.2. Let

$$G_i(v) = \left(\prod_{j=1}^n F_j(b_j^{-1}(b_i(v))\right)^{1/n}, \qquad g_i(v) = \frac{d}{dv}G_i(v). \tag{29}$$

By Condition 6, $b_i(v)$, and hence $G_i(v)$, is strictly monotonically increasing. Since

$$G_i(1) = 1 \tag{30}$$

(see Condition 8), $G_i(v) < 1$ for $v < 1$. Hence,

$$R = -\sum_{i=1}^n \int_{v=r_i}^1 e^{n\ln G_i(v)}\left(\frac{1 - F_i(v) - vf_i(v)}{F_i(v)}\right) dv$$
$$= -\sum_{i=1}^n \int_{s=0}^{1-r_i} e^{n\ln G_i(1-s)}\left(\frac{1 - F_i(1-s) - (1-s)f_i(1-s)}{F_i(1-s)}\right) ds, \tag{31}$$
$$= -\sum_{i=1}^n \int_{x=0}^{-\ln G_i(r_i)} e^{-nx_i}\left(\frac{1 - F_i(1-s(x_i)) - (1-s(x_i))f_i(1-s(x_i))}{F_i(1-s(x_i))}\right)\frac{ds}{dx_i}\,dx_i,$$

where we made the changes of variables $s = 1 - v$ and $x_i(s) = -\ln G_i(1-s)$. To approximate R for large n using the Laplace method for integrals, we replace the integrand with its Taylor expansion near $x_i = 0$ and also extend the domain of integration from $[0, -\ln G_i(r_i)]$ to $[0, \infty)$. Throughout the proof, existence of the various Taylor expansions is ensured by Conditions 2 and 7.

LEMMA B.1. *Let $x_i(s) = -\ln G_i(1-s)$. Then for $0 \le x_i \ll 1$,*

$$s = \frac{1}{g_i(1)}x_i - \frac{g_i^2(1)-g_i'(1)}{2g_i^3(1)}x_i^2 + \frac{g_i^4(1)-3g_i^2(1)g_i'(1)-g_i''(1)g_i(1)+3(g_i'(1))^2}{6g_i^5(1)}x_i^3 + O\left(x_i^4\right)$$

and

$$\frac{ds}{dx_i} = \frac{1}{g_i(1)} - \frac{g_i^2(1)-g_i'(1)}{g_i^3(1)}x_i + \frac{g_i^4(1)-3g_i^2(1)g_i'(1)-g_i''(1)g_i(1)+3(g_i'(1))^2}{2g_i^5(1)}x_i^2 + O\left(x_i^3\right).$$

Proof. See Appendix B.1. ∎

In addition, by Taylor expansion and Lemma B.1,

$$\frac{1-F_i(1-s)-(1-s)f_i(1-s)}{F_i(1-s)} = -f_i(1)+\alpha_1 s+\alpha_2 s^2+O(s^3) = -f_i(1)+\beta_1 x_i+\beta_2 x_i^2+O(x_i^3),$$

where

$$\alpha_1 = 2f_i(1)+f_i'(1)-f_i^2(1), \quad \alpha_2 = \frac{3}{2}f_i'(1)f_i(1) - \frac{3}{2}f_i'(1) - \frac{1}{2}f_i''(1) + 2f_i^2(1) - f_i^3(1),$$

$$\beta_1 = \frac{\alpha_1}{g_i(1)}, \qquad \beta_2 = \frac{\alpha_2}{g_i^2(1)} - \frac{(g_i^2(1)-g_i'(1))\alpha_1}{2g_i^3(1)}.$$

Since x_i is a dummy variable, we can rename $x_i = x$. Substituting the above in (31) gives $R = \int_0^\infty \sum_{i=1}^n e^{-nx}\left[A_{i,0}+A_{i,1}x+A_{i,2}x^2+O(x^3)\right]dx$, where

$$A_{i,0} = \frac{f_i(1)}{g_i(1)}, \qquad A_{i,1} = \left[\frac{f_i^2(1)-f_i'(1)-2f_i(1)}{g_i^2(1)} - \frac{(g_i^2(1)-g_i'(1))}{g_i^3(1)}f_i(1)\right],$$

$$A_{i,2} = \frac{f_i(1)[g_i^4(1)-3g_i^2(1)g_i'(1)-g_i(1)g_i''(1)+3(g_i'(1))^2]}{2g_i^5(1)} + \frac{3(g_i^2(1)-g_i'(1))(2f_i(1)+f_i'(1)-f_i^2(1))}{2g_i^4(1)}$$
$$- \frac{3f_i'(1)f_i(1)-3f_i'(1)-f_i''(1)+4f_i^2(1)-2f_i^3(1)}{2g_i^3(1)}.$$

Since $\int_0^\infty e^{-nx}x^k\,dx = \frac{k!}{n^{k+1}}$ for $k = 0, 1, \ldots$, we have that

$$R = \frac{1}{n}\sum_{i=1}^n A_{i,0} + \frac{1}{n^2}\sum_{i=1}^n A_{i,1} + \frac{2}{n^3}\sum_{i=1}^n A_{i,2} + O\left(\frac{1}{n^3}\right). \tag{32}$$

We first compute the leading-order term $\frac{1}{n}\sum_{i=1}^n A_{i,0} = \frac{1}{n}\sum_{i=1}^n \frac{f_i(1)}{g_i(1)}$.

LEMMA B.2. *Let $G_i(v)$ and $g_i(v)$ be defined by (29). Then*

$$\frac{1}{n}\sum_{i=1}^n \frac{f_i(1)}{g_i(1)} = 1. \tag{33}$$

Proof. See Appendix B.2. ∎

Therefore, $\frac{1}{n}\sum_{i=1}^n A_{i,0} = 1$. The first-order correction term reads

$$\frac{1}{n^2}\sum_{i=1}^n A_{i,1} = \frac{1}{n^2}\sum_{i=1}^n\left(\frac{f_i(1)g_i'(1)}{g_i^3(1)} + \frac{f_i^2(1)-f_i'(1)}{g_i^2(1)}\right) - \frac{1}{n^2}\sum_{i=1}^n\frac{f_i(1)}{g_i(1)} - \frac{2}{n^2}\sum_{i=1}^n\frac{f_i(1)}{g_i^2(1)}.$$

To evaluate this term, we make use of the following lemma.

LEMMA B.3. *The following relations hold:*

$$g_i(1) = f_G(1) + O\left(\frac{1}{n}\right), \tag{34}$$

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{f_i(1)g_i'(1)}{g_i^3(1)} + \frac{f_i^2(1) - f_i'(1)}{g_i^2(1)}\right) = 1, \tag{35}$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{f_i(1)}{g_i^2(1)} = \frac{1}{f_G(1)} + O\left(\frac{1}{n^2}\right). \tag{36}$$

Proof. See Appendix B.3. ∎

Remark 1. The only place where we make use of Condition 9 of Theorem 3.2 is in the proof of relations (34) and (36).

Therefore, $\frac{1}{n^2}\sum_{i=1}^{n} A_{i,1} = -\frac{2}{nf_G(1)} + O\left(\frac{1}{n^3}\right)$. Since the approximation error is $O(1/n^3)$, it does not contribute to the next-order term in (32), which is $O(1/n^2)$.

Using relations (33) and (35),

$$\frac{2}{n}\sum_{i=1}^{n} A_{i,2} = -\frac{1}{n}\sum_{i=1}^{n}\frac{f_i(1)g_i''(1)}{g_i^4(1)} - 2 + \frac{1}{n}\sum_{j=1}^{n}\frac{f_j''(1) - 3f_j'(1)f_j(1) + 2f_j^3}{g_j^3(1)} - \frac{3}{n}\sum_{j=1}^{n}\frac{[f_j'(1) - f_j^2(1)]g_i'(1)}{g_j^4(1)}$$
$$+ 3\frac{1}{n}\sum_{j=1}^{n}\frac{f_j(1)[g_i'(1)]^2}{g_j^5(1)} + \frac{6}{n}\sum_{i=1}^{n}\frac{f_i(1)}{g_i^2(1)} - \frac{6}{n}\sum_{i=1}^{n}\frac{f_i(1)g_i'(1)}{g_i^4(1)} + \frac{1}{n}\sum_{i=1}^{n}\frac{-4f_i^2(1) + 3f_i'(1)}{g_i^3(1)}.$$

LEMMA B.4. *The following relation holds:*

$$\frac{1}{n}\sum_{i=1}^{n}\frac{f_i(1)g_i''(1)}{g_i^4(1)} = -2 + \frac{1}{n}\sum_{j=1}^{n}\frac{f_j''(1) - 3f_j'(1)f_j(1) + 2f_j^3}{g_j^3(1)} - \frac{3}{n}\sum_{j=1}^{n}\frac{[f_j'(1) - f_j^2(1)]g_i'(1)}{g_j^4(1)}$$
$$+ 3\frac{1}{n}\sum_{j=1}^{n}\frac{f_j(1)[g_i'(1)]^2}{g_j^5(1)}.$$

Proof. See Appendix B.4. ∎

Therefore,

$$\frac{2}{n}\sum_{i=1}^{n} A_{i,2} = \frac{6}{n}\sum_{i=1}^{n}\frac{f_i(1)}{g_i^2(1)} - \frac{6}{n}\sum_{i=1}^{n}\frac{f_i(1)g_i'(1)}{g_i^4(1)} + \frac{1}{n}\sum_{i=1}^{n}\frac{-4f_i^2(1) + 3f_i'(1)}{g_i^3(1)}.$$

Using relations (34) and (35), we obtain

$$\frac{1}{n}\sum_{i=1}^{n}\frac{f_i(1)g_i'(1)}{g_i^4(1)} = \frac{1}{f_G(1)}\frac{1}{n}\sum_{i=1}^{n}\frac{f_i(1)g_i'(1)}{g_i^3(1)} + O(1/n) = \frac{1}{f_G(1)}\left[1 + \frac{1}{n}\sum_{i=1}^{n}\frac{f_i'(1) - f_i^2(1)}{g_i^2(1)}\right]$$
$$= \frac{1}{f_G^3(1)}\left[f_G^2(1) + \frac{1}{n}\sum_{i=1}^{n}(f_i'(1) - f_i^2(1))\right] + +O\left(\frac{1}{n}\right).$$

Therefore, using again relation (34),

$$\frac{2}{n}\sum_{i=1}^{n} A_{i,2} = \frac{1}{f_G^3(1)}\left[\frac{3}{n}\sum_{i=1}^{n}\left(f_i^2(1) - f_i'(1)\right) - \frac{1}{n}\sum_{i=1}^{n} f_i^2(1) + f_G^2(1)\right] + O\left(\frac{1}{n}\right)$$
$$= \frac{2f_G^2(1) - 3f_G'(1) - \mathrm{Var}[f_1, \ldots, f_n]}{f_G^3(1)} + O\left(\frac{1}{n}\right),$$

where in the last equality we used (7) and (8). Therefore,

$$\begin{aligned} R &= 1 - \frac{2}{f_G(1)}\frac{1}{n} + \frac{2f_G^2(1) - 3f_G'(1) - \mathrm{Var}[f_1,\dots,f_n]}{f_G^3(1)}\frac{1}{n^2} + O\left(\frac{1}{n^3}\right) \\ &= 1 - \frac{2}{f_G(1)}\frac{1}{n-1} + \frac{4f_G^2(1) - 3f_G'(1) - \mathrm{Var}[f_1,\dots,f_n]}{f_G^3(1)}\frac{1}{(n-1)^2} + O\left(\frac{1}{n^3}\right). \end{aligned}$$

B.1. Proof of Lemma B.1. Let $x_i(s) = -\ln G_i(1-s)$. Since G_i is continuous and since $G_i(1) = 1$ (see (30)), $0 \le x_i \ll 1$ implies that $0 \le s \ll 1$. Taylor expansion of $x_i(s)$ about $s = 0$ gives

$$x_i = sg_i(1) + \frac{s^2}{2}\left(g_i^2(1) - g_i'(1)\right) + \frac{s^3}{6}\left(g_i''(1) - 3g_i(1)g_i'(1) + 2g_i^3(1)\right) + O\left(s^4\right). \tag{37}$$

To invert this relation, we substitute $s = a_0 + a_1x_i + a_2x_i^2 + a_3x_i^3 + O\left(x_i^4\right)$ into (37) and compute the coefficients by equating powers of x_i. This yields

$$a_0 = 0, \quad a_1 = \frac{1}{g_i(1)}, \quad a_2 = -\frac{\left(g_i^2(1) - g_i'(1)\right)}{2g_i^3(1)}, \quad a_3 = \frac{g_i^4(1) - 3g_i^2(1)g_i'(1) - g_i''(1)g_i(1) + 3(g_i'(1))^2}{6g_i^5(1)}.$$

Differentiation of s with respect to x_i gives $\frac{ds}{dx_i}$.

B.2. Proof of Lemma B.2. By (29),

$$\begin{aligned} g_i(v) &= \frac{d}{dv}G_i(v) = \frac{d}{dv}\left(\prod_{j=1}^n F_j(b_j^{-1}(b_i(v))\right)^{1/n} \\ &= \frac{1}{n}\left(\prod_{k=1}^n F_k(b_k^{-1}(b_i(v))\right)^{1/n-1}\sum_{j=1}^n\left(\prod_{k\neq j}^n F_k(b_k^{-1}(b_i(v))\right) f_j(b_j^{-1}(b_i(v))\frac{b_i'(v)}{b_j'(b_j^{-1}(b_i(v))} \\ &= G_i(v)\frac{1}{n}b_i'(v)\sum_{j=1}^n \frac{f_j(b_j^{-1}(b_i(v))}{F_j(b_j^{-1}(b_i(v))}\frac{1}{b_j'(b_j^{-1}(b_i(v))}. \end{aligned} \tag{38}$$

Since all bidders have the same maximal bid, $b_j^{-1}(b_i(1)) = 1$. Therefore, by (30),

$$g_i(1) = \frac{b_i'(1)}{n}\sum_{j=1}^n \frac{f_j(1)}{b_j'(1)}. \tag{39}$$

From this, it follows that

$$g_i(1) = \alpha b_i'(1), \tag{40}$$

where α is independent of i. Substituting (40) into (39) yields (33).

B.3. Proof of Lemma B.3. Substituting Condition 9 into (39) and using (7) gives relation (34). To prove relation (35), we differentiate (38) and use $b_j^{-1}(b_i(1)) = 1$

to obtain

$$g_i'(1) = \frac{1}{n^2}\left(b_i'(1)\right)^2 \left(\sum_{j=1}^{n} \frac{f_j(1)}{b_j'(1)}\right)^2 + \frac{1}{n} b_i''(1) \sum_{j=1}^{n} \frac{f_j(1)}{b_j'(1)}$$
$$+\frac{1}{n} b_i'(1) \sum_{j=1}^{n} \left(\frac{f_j'(1)}{b_j'(1)} - \frac{f_j^2(1)}{b_j'(1)} - \frac{f_j(1) b_j''(1)}{(b_j'(1))^2}\right) \frac{b_i'(1)}{b_j'(1)}$$
$$= g_i^2(1) + \frac{b_i''(1)}{b_i'(1)} g_i(1) + \frac{1}{n} b_i'(1) \sum_{j=1}^{n} \left(\frac{f_j'(1)}{b_j'(1)} - \frac{f_j^2(1)}{b_j'(1)} - \frac{f_j(1) b_j''(1)}{(b_j'(1))^2}\right) \frac{b_i'(1)}{b_j'(1)}. \tag{41}$$

Multiplying by $f_i(1)$ and using (40) gives

$$f_i(1) g_i'(1) = f_i(1) g_i^2(1) + \alpha b_i''(1) f_i(1) + \frac{1}{n} f_i(1) g_i^2(1) \sum_{j=1}^{n} \left(\frac{f_j'(1)}{g_j^2(1)} - \frac{f_j^2(1)}{g_j^2(1)} - \frac{\alpha f_j(1) b_j''(1)}{g_j^3(1)}\right).$$

Dividing by $g_i^3(1)$, summing, and using (33) gives

$$\sum_{i=1}^{n} \left(\frac{f_i(1) g_i'(1)}{g_i^3(1)}\right) = \underbrace{\sum_{i=1}^{n} \frac{f_i(1)}{g_i(1)}}_{=n} + \cancel{\alpha \sum_{i=1}^{n} \frac{b_i''(1) f_i(1)}{g_i^3(1)}} + \underbrace{\left(\frac{1}{n} \sum_{i=1}^{n} \frac{f_i(1)}{g_i(1)}\right)}_{=1} \sum_{j=1}^{n} \left(\frac{f_j'(1)}{g_j^2(1)} - \frac{f_j^2(1)}{g_j^2(1)} - \cancel{\frac{\alpha f_j(1) b_j''(1)}{g_j^3(1)}}\right).$$

Therefore, relation (35) follows.

To prove relation (36), let

$$h_i := \frac{1}{g_i(1)} - \frac{1}{f_G(1)}. \tag{42}$$

By (33) and (7),

$$\frac{1}{n} \sum_{i=1}^{n} h_i f_i(1) = \frac{1}{n} \sum_{i=1}^{n} \frac{f_i(1)}{g_i(1)} - \frac{1}{f_G(1)} \frac{1}{n} \sum_{i=1}^{n} f_i(1) = 0.$$

Therefore, the left-hand side of (36) can be written as

$$\frac{1}{n} \sum_{i=1}^{n} \frac{f_i(1)}{g_i^2(1)} = \frac{1}{n} \sum_{i=1}^{n} f_i(1) \left(\frac{1}{f_G(1)} + h_i\right)^2 = \frac{1}{n} \sum_{i=1}^{n} f_i(1) \left(\frac{1}{f_G^2(1)} + 2 h_i f_G(1) + h_i^2\right)$$
$$= \frac{1}{f_G(1)} + 0 + \frac{1}{n} \sum_{i=1}^{n} f_i(1) h_i^2.$$

Finally, by relation (34), $h_i = O\left(\frac{1}{n}\right)$. Therefore, relation (36) follows.

B.4. Proof of Lemma B.4. Differentiating (38) twice and using $b_j^{-1}(b_i(1)) = 1$ gives

$$\begin{aligned}
g_i''(1) &= [g_i'(1)b_i'(1) + 2g_i(1)b_i''(1) + b_i'''(1)]\frac{1}{n}\sum_{j=1}^{n}\frac{f_j(1)}{b_j'(1)} \\
&\quad + [g_i(1)b_i'(1) + b_i''(1)]b_i'(1)\frac{1}{n}\sum_{j=1}^{n}\frac{f_j'(1) - f_j^2(1)}{[b_j'(1)]^2} - \frac{f_j(1)b_j''(1)}{[b_j'(1)]^3} \\
&\quad + \left[g_i(1)[b_i'(1)]^2 + 2b_i''(1)b_i'(1)\right]\frac{1}{n}\sum_{j=1}^{n}\frac{f_j'(1) - f_j^2(1)}{[b_j'(1)]^2} \\
&\quad + \frac{1}{n}[b_i'(1)]^3\sum_{j=1}^{n}\frac{f_j''(1) - 3f_j'(1)f_j(1) + 2f_j^3(1)}{[b_j'(1)]^3} - 2[b_i'(1)]^3\frac{1}{n}\sum_{j=1}^{n}\frac{f_j'(1) - f_j^2(1)}{[b_j'(1)]^4}b_j''(1) \\
&\quad - \left[g_i(1)[b_i'(1)]^2 + 2b_i''(1)b_i'(1)\right]\frac{1}{n}\sum_{j=1}^{n}\frac{f_j(1)b_j''(1)}{[b_j'(1)]^3} - \frac{1}{n}[b_i'(1)]^3\sum_{j=1}^{n}\frac{[f_j'(1) - f_j^2(1)]b_j''(1)}{[b_j'(1)]^4} \\
&\quad - \frac{1}{n}[b_i'(1)]^3\sum_{j=1}^{n}\frac{f_j(1)(b_j'''(1)b_j'(1) + 3[b_j''(1)]^2)}{[b_j'(1)]^5}.
\end{aligned}$$

By (39) and (41)

$$g_i(1) = \alpha b_i'(1), \qquad g_i'(1) = \alpha b_i''(1) + \beta g_i^2(1), \tag{43a}$$

where

$$\alpha = \frac{1}{n}\sum\frac{f_j(1)}{b_j'(1)}, \quad \beta = 1 + \frac{1}{n}\sum_{j=1}^{n}\left[\frac{f_j'(1) - f_j^2(1)}{g_j^2(1)} - \alpha\frac{f_j(1)b_j''(1)}{g_j^3(1)}\right]. \tag{43b}$$

Using relations (43), as well as relation (35), yields

$$\begin{aligned}
g_i''(1) =\ & (2+\beta)g_i'(1)g_i(1) - (\beta^2+1)g_i^3 + \alpha b_i'''(1) + \left[(1-2\beta)g_i^3(1) + 2g_i'(1)g_i(1)\right]\frac{1}{n}\sum_{j=1}^{n}\frac{f_j'(1) - f_j^2(1)}{g_j^2(1)} \\
& + \frac{1}{n}g_i^3(1)\sum_{j=1}^{n}\frac{f_j''(1) - 3f_j'(1)f_j(1) + 2f_j^3}{g_j^3(1)} + 2\beta g_i^3(1)\frac{1}{n}\sum_{j=1}^{n}\frac{f_j'(1) - f_j^2(1)}{g_j^2(1)} \\
& - 2g_i^3(1)\frac{1}{n}\sum_{j=1}^{n}\frac{f_j'(1) - f_j^2(1)}{g_j^4(1)}g_i'(1) \\
& - \left[(1+3\beta)g_i^3(1) + 2g_i'(1)g_i(1)\right]\frac{1}{n}\sum_{j=1}^{n}\frac{f_j(1)g_i'(1)}{g_i^3(1)} - \beta\left[\beta g_i^3(1) + 2g_i'(1)g_i(1)\right] \\
& - g_i^3(1)\frac{1}{n}\sum_{j=1}^{n}\frac{[f_j'(1) - f_j^2(1)]g_i'(1)}{g_j^4(1)} + \frac{\alpha}{n}g_i^3(1)\sum_{j=1}^{n}\frac{f_j(1)b_i'''(1)}{g_j^4(1)} + \frac{3}{n}g_i^3(1)\sum_{j=1}^{n}\frac{f_j(1)[g_i'(1)]^2}{g_j^5(1)}.
\end{aligned}$$

Multiplying both sides by $\frac{f_i(1)}{ng_i^4(1)}$, summing, and using (33) and (35) yields the result.

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