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Singular solutions of the L^2 -supercritical biharmonic nonlinear Schrödinger equation

G Baruch and G Fibich

School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel

E-mail: guy.baruch@math.tau.ac.il

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Abstract

We use asymptotic analysis and numerical simulations to study peak-type singular solutions of the supercritical biharmonic nonlinear Schrödinger equation. These solutions have a quartic-root blowup rate, and collapse with a quasi-self-similar universal profile, which is a zero-Hamiltonian solution of a fourth-order nonlinear eigenvalue problem.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The focusing nonlinear Schrödinger equation (NLS)

$$i\psi_t(t, \mathbf{x}) + \Delta\psi + |\psi|^{2\sigma}\psi = 0, \quad \psi(0, \mathbf{x}) = \psi_0(\mathbf{x}) \in H^1(\mathbb{R}^d), \quad (1)$$

where $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $\Delta = \sum_{j=1}^d \partial_j^2$ is the Laplacian, has been the subject of intense study, due to its role in various areas of physics, such as nonlinear optics and Bose–Einstein condensates (BEC). It is well known that the NLS (1) possesses solutions that become singular in a finite time [SS99].

In recent years, there has been a growing interest in extending NLS theory to the focusing biharmonic nonlinear Schrödinger equation (BNLS)

$$i\psi_t(t, \mathbf{x}) - \Delta^2\psi + |\psi|^{2\sigma}\psi = 0, \quad \psi(0, \mathbf{x}) = \psi_0(\mathbf{x}) \in H^2(\mathbb{R}^d), \quad (2)$$

where Δ^2 is the biharmonic operator. The BNLS (2) is called ‘ L^2 -critical’ or simply ‘critical’ if $\sigma d = 4$. In this case, equation (2) can be rewritten as

$$i\psi_t(t, \mathbf{x}) - \Delta^2\psi + |\psi|^{8/d}\psi = 0, \quad \psi(0, \mathbf{x}) = \psi_0(\mathbf{x}) \in H^2(\mathbb{R}^d). \quad (3)$$

Correspondingly, the BNLS with $0 < \sigma d < 4$ is called subcritical, and the BNLS with $\sigma d > 4$ is called supercritical. This is analogous to the NLS, where the critical case is $\sigma d = 2$.

BNLS solutions preserve the *power* (L^2 norm)

$$P(t) \equiv P(0), \quad P = \|\psi\|_2^2,$$

and the Hamiltonian

$$H(t) \equiv H(0), \quad H = \|\Delta\psi\|_2^2 - \frac{1}{\sigma+1} \|\Delta\psi\|_{2\sigma+2}^{2\sigma+2}.$$

In [BAKS00], Ben-Artzi, Koch and Saut obtained sharp dispersive estimates for the biharmonic Schrödinger operator, which imply that when σ is in the H^2 -subcritical regime

$$\begin{cases} 0 < \sigma & d \leq 4, \\ 0 < \sigma < \frac{4}{d-4} & d > 4, \end{cases} \quad (4)$$

the BNLS (2) is locally well-posed in H^2 . Global existence, well-posedness for small data, and scattering of BNLS solutions in the H^2 -critical case $\sigma = 4/(d-4)$ were studied by Miao *et al* [MXZ09] and by Pausader [Pau09b]. The H^2 -critical defocusing BNLS was studied by Miao *et al* [MXZ08] and by Pausader [Pau07, Pau09a].

The above studies focused on non-singular solutions. In this work, we study singular solutions of the BNLS in H^2 , i.e. solutions that exist in $H^2(\mathbb{R}^d)$ over some finite time interval $t \in [0, T_c)$, but for which $\lim_{t \rightarrow T_c} \|\psi\|_{H^2} = \infty$. The first study of singular BNLS solutions was carried out by Fibich *et al* [FIP02], who proved the following results:

Theorem 1. *Let $\psi_0 \in H^2$. Then, the solution of the subcritical BNLS (2) exists globally in H^2 .*

Theorem 2. *Let $\psi_0 \in H^2$, and let $\|\psi_0\|_2^2 < P_{\text{cr}}^B$, where $P_{\text{cr}}^B = \|R_B\|_2^2$, and R_B is the ground state of*

$$-\Delta^2 R_B(x) - R_B + |R_B|^{8/d} R_B = 0. \quad (5)$$

Then, the solution of the critical BNLS (3) exists globally in H^2 .

Pausader and Shao [PS10] proved global well-posedness and scattering of solutions of the critical BNLS (3) for $d \geq 5$, when $\|\psi_0\|_2^2 < P_{\text{cr}}^B$ in the radial case, and when $\|\psi_0\|_2^2 < 4^{-d/8} P_{\text{cr}}^B$ in the non-radial case.

The simulations in [FIP02] suggested that there exist singular solutions for $\sigma d = 4$ and $\sigma d > 4$, and that these singularities are of the blowup type, namely, the solution splits into a collapsing core which becomes infinitely localized, and a non-collapsing component. However, in contradistinction with NLS theory, there is currently no rigorous proof that solutions of the BNLS can become singular in either the critical or the supercritical case.

Most subsequent research of singular BNLS solutions focused on the critical case. Chae *et al* [CHL09] showed that radial singular solutions of (3) have a power-concentration property. In [BFM10b], we showed that radial singular solutions are quasi-self-similar. We also proved, without assuming radial symmetry, that the blowup rate is bounded by a quartic root, the power-concentration property and the existence of the ground state of (5). The two latter properties were also proved by Zhu *et al* [ZZY10]. In [BFM10b], we also provided informal analysis and numerical evidence that peak-type singular solutions of the critical BNLS collapse with a quasi-self-similar R_B profile at a blowup rate which is slightly faster than the quartic-root bound.

In this work, we use asymptotic analysis and numerics to find and characterize peak-type singular solutions of the supercritical BNLS. We find that their properties mirror those of the supercritical NLS. Ring-type singular solutions of the supercritical BNLS were studied in [BFG10, BFM10a].

1.1. Summary of results

We analyse singular solutions of the focusing L^2 -supercritical and H^2 -subcritical BNLS, i.e. when

$$\begin{cases} 4/d < \sigma & d \leq 4, \\ 4/d < \sigma < \frac{4}{d-4} & d > 4. \end{cases} \quad (6)$$

We assume radial symmetry, i.e. that $\psi = \psi(t, r)$, where $r = |\mathbf{x}|$. In this case, equation (2) reduces to

$$i\psi_t(t, r) - \Delta_r^2 \psi + |\psi|^{2\sigma} \psi = 0, \quad \psi(0, r) = \psi_0(r), \quad (7)$$

where

$$\Delta_r^2 = \partial_r^4 + \frac{2(d-1)}{r} \partial_r^3 + \frac{(d-1)(d-3)}{r^2} \partial_r^2 - \frac{(d-1)(d-3)}{r^3} \partial_r$$

is the radial biharmonic operator.

The paper is organized as follows. In section 2, we show that the supercritical BNLS admits the explicit self-similar singular solutions

$$\psi_{S_B}(t, r) = \frac{1}{L^{2/\sigma}(t)} S_B \left(\frac{r}{L(t)} \right) e^{iv \int \frac{1}{L^4(t')} dt'}, \quad (8)$$

where the blowup rate of $L(t)$ is exactly a quartic-root

$$L(t) = \kappa(T_c - t)^{1/4}, \quad \kappa > 0,$$

and the self-similar profile $S_B(\rho)$ is a solution of

$$\begin{aligned} -\nu S_B(\rho) + i \frac{\kappa^4}{4} \left(\frac{2}{\sigma} S_B + \rho S_B' \right) - \Delta_\rho^2 S_B + |S_B|^{2\sigma} S_B &= 0, \\ S_B'(0) = S_B''(0) = 0, \quad S_B(\infty) &= 0. \end{aligned} \quad (9)$$

WKB analysis of the large- ρ behavior of S_B shows that it belongs to $L^{2+2\sigma}$. Since $\lim_{t \rightarrow T_c} \|\psi_{S_B}\|_{2+2\sigma} = \infty$, ψ_{S_B} is a singular solution in $L^{2+2\sigma}$. To the best of our knowledge, this is the first time that explicit singular solutions of the BNLS are presented.

In section 2.1 we show that the zero-Hamiltonian solutions of (9) satisfy the boundary condition

$$\lim_{\rho \rightarrow \infty} \left(\rho S' + \left(\frac{2}{\sigma} + i \frac{4\nu}{\kappa^4} \right) S \right) \rho^\gamma = 0, \quad \frac{2}{3} \left(d - 2 - \frac{2}{\sigma} \right) < \gamma < 4 + \frac{2}{\sigma}.$$

In analogy with the supercritical NLS, we conjecture that for any d, σ and ν , there is a unique *admissible* solution $S_B^{\text{admis.}}(\rho)$, which has a zero Hamiltonian and is monotonically decreasing. This solution is attained for a unique $\kappa = \kappa^{\text{admis.}}(\sigma, d, \nu) > 0$. While a rigorous existence proof for the S_B profile remains open, we provide numerical support for the existence of the admissible solutions.

The WKB analysis suggests that, as in the case of the supercritical NLS, the admissible solutions are not in L^2 . This is also supported by our numerical simulations. Hence, the corresponding ψ_{S_B} solutions are not in H^2 . In section 3 we consider H^2 singular solutions. Using informal asymptotic analysis and the analogy with the supercritical NLS, we conjecture that these solutions undergo a quasi-self-similar collapse with the ψ_{S_B} profile, where S_B is the unique admissible solution $S_B^{\text{admis.}}$. The blowup rate of these solutions is given by $L(t) \sim \kappa^{\text{admis.}}(T_c - t)^{1/4}$. These characteristics are confirmed numerically, in simulations of both the one-dimensional and the two-dimensional BNLS.

The numerical simulations of the BNLS were performed using the IGR/SGR method [RW00, DG09], see [BFM10a] for further details. The numerical solution of the nonlinear fourth-order ODE for S_B is obtained using a modified Petviashvili (SLSR) method, which is described in the appendix. The code is available online at <http://www.math.tau.ac.il/~fibich/publications.html>.

The results of this study are based on asymptotic analysis and numerical simulations. These results show that there is a striking analogy between collapse of peak-type solutions in the supercritical NLS and the supercritical BNLS. We note that the rigorous theory for singular solutions of the supercritical NLS is much less developed than that for the critical NLS. Indeed, a rigorous proof of the blowup rate and blowup profile of the supercritical NLS was obtained very recently, and only in the slightly supercritical regime $0 < \sigma d - 2 \ll 1$ [MRS09]. We hope that this study will motivate a similar rigorous treatment of the supercritical BNLS.

2. Explicit singular solutions

Let us look for explicit self-similar solutions of the supercritical BNLS (2). Since the BNLS is invariant under the dilation symmetry $r \mapsto \frac{r}{L}$, $t \mapsto \frac{t}{L^4}$, $\psi \mapsto \frac{1}{L^{2/\sigma}} \psi$, where L is a constant, this suggests a self-similar solution of the form

$$\psi_{S_B}(t, r) = \frac{1}{L^{2/\sigma}(t)} S_B(\rho) e^{i\tau(t)}, \quad \rho = \frac{r}{L(t)}. \quad (10)$$

Substituting ψ_{S_B} in the BNLS gives

$$-\tau'(t)L^4(t)S_B(\rho) - iL^3(t)L_t \left(\frac{2}{\sigma} S_B + \rho S_B' \right) - \Delta_\rho^2 S_B + |S_B|^{2\sigma} S_B = 0. \quad (11)$$

Since S_B is only a function of ρ , equation (11) must be independent of t . Therefore, there exists a real constant κ such that

$$L^3 L_t \equiv \frac{1}{4} (L^4)_t \equiv -\kappa^4/4.$$

Hence, $L(t)$ is a quartic root, i.e.

$$L(t) = \kappa \sqrt[4]{T_c - t}, \quad \kappa > 0. \quad (12)$$

Likewise, since S_B is only a function of ρ , then $\tau'(t)L^4(t) \equiv \nu$. Hence,

$$\tau(t) = \nu \int_{s=0}^t \frac{1}{L^4(s)} ds = -\frac{\nu}{\kappa^4} \ln \left(1 - \frac{t}{T_c} \right). \quad (13)$$

Substituting (12) and (13) in (11) shows that the equation for S_B is

$$-\nu S_B(\rho) + i \frac{\kappa^4}{4} \left(\frac{2}{\sigma} S_B + \rho S_B' \right) - \Delta_\rho^2 S_B + |S_B|^{2\sigma} S_B = 0. \quad (14a)$$

Since S_B is radially symmetric and decays at infinity, it should satisfy the boundary conditions

$$S_B'(0) = S_B'''(0) = 0, \quad S_B(\infty) = 0. \quad (14b)$$

Equations (14a) and (14b) have the two parameters ν and κ . Note, however, that

$$S_B(\rho; \kappa, \nu) := \nu^{1/2\sigma} \tilde{S}_B(\nu^{1/4} \rho; \tilde{\kappa} = \kappa/\nu^{1/4}), \quad (15)$$

where $\tilde{S}_B(\rho; \tilde{\kappa})$ is the solution of (14a) and (14b) with $\nu = 1$, i.e.

$$-\tilde{S}_B(\rho) + i \frac{\kappa^4}{4} \left(\frac{2}{\sigma} \tilde{S}_B + \rho \tilde{S}_B' \right) - \Delta_\rho^2 \tilde{S}_B + |\tilde{S}_B|^{2\sigma} \tilde{S}_B = 0. \quad (16a)$$

subject to

$$\tilde{S}_B'(0) = \tilde{S}_B'''(0) = 0, \quad \tilde{S}_B(\infty) = 0. \quad (16b)$$

Equations (16a) and (16b) can be viewed as a nonlinear eigenvalue problem with the eigenvalue κ and eigenfunction S_B . By analogy with the supercritical NLS [KL95, Bud01], we make the following conjecture:

Conjecture 3. *Let σ be in the L^2 supercritical and H^2 -subcritical regime (6). Then, there exists a solution $\{\tilde{S}_B(\rho), \tilde{\kappa}\}$ to equations (16a) and (16b), such that $\tilde{S}_B \not\equiv 0$ and $\tilde{\kappa} > 0$.*

Hence, we have the following result:

Lemma 4. *Assume that conjecture 3 holds, and let $S_B(\rho; \kappa, \nu)$ be a nontrivial solution of (14a) and (14b). Then,*

$$\psi_{S_B}(t, r) = \frac{1}{L^{2/\sigma}(t)} S_B\left(\frac{r}{L(t)}\right) e^{i\nu \int^t \frac{1}{L^4(s)} ds}, \quad L(t) = \kappa \sqrt[4]{T_c - t}, \quad (17)$$

is an explicit solution of the BNLS equation (2).

As $\rho \rightarrow \infty$, the nonlinear term in (14a) and (14b) becomes negligible, and (14a) reduces to

$$-\nu S_B^{\text{lin}}(\rho) - \Delta_\rho S_B^{\text{lin}} + i \frac{\kappa^4}{4} \left(\frac{2}{\sigma} S_B^{\text{lin}} + \rho (S_B^{\text{lin}})_\rho \right) = 0, \quad (18)$$

where

$$\Delta_\rho^2 = -\frac{(d-1)(d-3)}{\rho^3} \partial_\rho + \frac{(d-1)(d-3)}{\rho^2} \partial_\rho^2 + \frac{2(d-1)}{\rho} \partial_\rho^3 + \partial_\rho^4.$$

We now use WKB to find the large ρ behavior of (18):

Lemma 5. *Let $S_B^{\text{lin}}(\rho)$ be a solution of (18). Then,*

$$S_B^{\text{lin}} \sim c_1 S_{B,1}(\rho) + c_2 S_{B,2}(\rho) + c_3 S_{B,3}(\rho) + c_4 S_{B,4}(\rho), \quad \rho \rightarrow \infty,$$

where $\{c_i\}_{i=1}^4$ are complex constants,

$$\begin{aligned} S_{B,1}(\rho) &\sim \rho^{-\frac{2}{\sigma} - i\frac{4\nu}{\kappa^4}}, \\ S_{B,2}(\rho) &\sim \frac{1}{\rho^{\frac{2}{3\sigma}(\sigma d - 1)}} \exp\left(-i\frac{3}{4\sqrt[3]{4}}(\kappa\rho)^{4/3} + i\frac{4\nu}{3\kappa^4} \log(\rho)\right), \\ S_{B,3}(\rho) &\sim \frac{\exp\left(+\frac{3\sqrt{3}}{8\sqrt[3]{4}}(\kappa\rho)^{4/3}\right)}{\rho^{\frac{2}{3\sigma}(\sigma d - 1)}} \exp\left(+i\frac{3\sqrt{3}}{8\sqrt[3]{4}}(\kappa\rho)^{4/3} + i\frac{4\nu}{3\kappa^4} \log(\rho)\right), \\ S_{B,4}(\rho) &\sim \frac{\exp\left(-\frac{3\sqrt{3}}{8\sqrt[3]{4}}(\kappa\rho)^{4/3}\right)}{\rho^{\frac{2}{3\sigma}(\sigma d - 1)}} \exp\left(+i\frac{3\sqrt{3}}{8\sqrt[3]{4}}(\kappa\rho)^{4/3} + i\frac{4\nu}{3\kappa^4} \log(\rho)\right). \end{aligned}$$

Proof. In order to apply the WKB method, we substitute $S_B^{\text{lin}}(\rho) = \exp(w(\rho))$, and expand

$$w(\rho) \sim w_0(\rho) + w_1(\rho) + \dots$$

Substituting $w_0(\rho) = \alpha\rho^p$ and balancing terms shows that $p = 4/3$, and that the equation for the leading-order, the $\mathcal{O}(\rho^{4/3})$ terms, is

$$(w_0')^3 = \left(\frac{4\alpha}{3}\right)^3 \rho = i\frac{\kappa^4}{4}\rho.$$

Therefore,

$$\alpha = \frac{3}{4} \sqrt[3]{i \frac{\kappa^4}{4}} = \frac{3\kappa^{4/3}}{4\sqrt[3]{4}} \cdot \left\{ -i, \frac{\sqrt{3} + i}{2}, \frac{-\sqrt{3} + i}{2} \right\}.$$

The equation for the next order, the $\mathcal{O}(1)$ terms, is

$$v + i \frac{\kappa^4}{2} d = i \frac{\kappa^4}{4} \left(\frac{2}{\sigma} + 3\rho w'_1 \right),$$

implying that

$$w_1 = \frac{1}{3} \left(\frac{2}{\sigma} (1 - \sigma d) + \frac{4v}{\kappa^4} i \right) \log \rho.$$

The next-order terms are $\mathcal{O}(\rho^{-4/3}) = o(1)$ and can be neglected. We therefore obtain the three solutions $S_{B,2}$, $S_{B,3}$ and $S_{B,4}$.

Since (18) is a fourth order ODE, another solution is required. To obtain the fourth solution, we substitute $w_0 \sim \beta \log(\rho)$ in (18) and obtain that the equation for the leading-order, the $\mathcal{O}(1)$ terms, is

$$-v + i \frac{\kappa^4}{4} \left(\frac{2}{\sigma} + \beta \right) = 0,$$

and that the next-order terms are $\mathcal{O}(\rho^{-4}) = o(1)$ and can be neglected. The fourth solution is therefore $S_{B,1}$. □

Equation (18) thus has the two algebraically decaying solutions, $S_{B,1}$ and $S_{B,2}$, the exponentially increasing solution $S_{B,3}$, and the exponentially decreasing solution $S_{B,4}$. The fact that $S_{B,3}$ increases exponentially as $\rho \rightarrow \infty$ is inconsistent with the boundary condition (14b). Therefore,

$$S_B(\rho) \sim c_1 S_{B,1}(\rho) + c_2 S_{B,2}(\rho) + c_4 S_{B,4}(\rho). \tag{19}$$

Since $\sigma d > 4$, the exponent $\frac{2}{3\sigma}(\sigma d - 1)$ of $S_{B,2}$ is larger than the exponent $\frac{2}{\sigma}$ of $S_{B,1}$, hence

$$S_{B,1} \gg S_{B,2}, \quad \rho \rightarrow \infty. \tag{20}$$

Direct calculations give that

$$\begin{aligned} S_{B,1} &\notin L^2(\mathbb{R}^d), & \Delta S_{B,1} &\in L^2(\mathbb{R}^d), & S_{B,1} &\in L^{2+2\sigma}(\mathbb{R}^d), \\ S_{B,2} &\in L^2(\mathbb{R}^d), & \Delta S_{B,2} &\notin L^2(\mathbb{R}^d), & S_{B,2} &\in L^{2+2\sigma}(\mathbb{R}^d), \\ S_{B,4} &\in L^2(\mathbb{R}^d), & \Delta S_{B,4} &\in L^2(\mathbb{R}^d), & S_{B,4} &\in L^{2+2\sigma}(\mathbb{R}^d). \end{aligned} \tag{21}$$

Therefore, S_B is in $L^{2+2\sigma}$. Unless $c_1 = c_2 = 0$, however, S_B is not in H^2 . Furthermore, since

$$\|\psi_{S_B}\|_{2+2\sigma}^{2+2\sigma} = \frac{1}{L^{4/\sigma - (d-4)}(t)} \|S_B\|_{2+2\sigma}^{2+2\sigma},$$

then $\psi_{S_B} \in L^{2+2\sigma}$ for $0 \leq t < T_c$. In the H^2 -subcritical regime $4/\sigma - (d - 4) > 0$. Therefore,

$$\lim_{t \rightarrow T_c} \|\psi_{S_B}\|_{2+2\sigma} = \infty.$$

Hence,

Lemma 6. *Assume that conjecture 3 holds. Then, ψ_{S_B} is an explicit solution of the BNLS equation (2) that becomes singular in $L^{2+2\sigma}$ as $t \rightarrow T_c$.*

2.1. Zero-Hamiltonian solutions

As is the case of peak-type solutions of the supercritical NLS, a key role is played by the zero-Hamiltonian solutions.

Theorem 7. *Let σ be in the L^2 supercritical and H^2 -subcritical regime (6). Let S_B be a solution of (14a) and (14b). If $H[S_B] < \infty$, then $H[S_B] = 0$.*

Proof. The Hamiltonian of ψ_{S_B} , see (17), is equal to

$$H[\psi_{S_B}] = \frac{1}{L^{4/\sigma-(d-4)}} H[S_B].$$

From H^2 -subcriticality, it follows that $L^{-4/\sigma-4+d}(t) \neq \text{const}$. Therefore, Hamiltonian conservation ($H[\psi_{S_B}] \equiv \text{const}$) implies that $H[S_B] = 0$. \square

Lemma 8. *Let σ be in the L^2 supercritical and H^2 -subcritical regime (6). Let $S_B(\rho)$ be a zero-Hamiltonian solution of (14a) and (14b). Then,*

- (i) $c_2 = c_3 = 0$.
(ii) If $c_1 \neq 0$ then

$$S_B(\rho) \sim c_1 S_{B,1}(\rho), \quad \rho \rightarrow \infty, \quad (22)$$

or, equivalently,

$$\lim_{\rho \rightarrow \infty} \left(\rho S' + \left(\frac{2}{\sigma} + i \frac{4\nu}{\kappa^4} \right) S \right) \rho^\gamma = 0, \quad (23a)$$

where

$$\gamma_0 < \gamma < \gamma_1, \quad \gamma_0 = \frac{2}{3} \left(d - 2 - \frac{2}{\sigma} \right), \quad \gamma_1 = 4 + \frac{2}{\sigma}. \quad (23b)$$

Moreover, $S_B \in L^{2\sigma+2}$ and $S_B \notin L^2$.

Proof. The exponentially increasing solution $c_3 S_{B,3}$ must vanish, as explained above. Convergence of the Hamiltonian requires that $\Delta S_B \in L^2$. Since $\Delta S_{B,2} \notin L^2$, see (21), it follows that $c_2 = 0$. Since $S_{B,1} \notin L^2$, then $S_B \notin L^2$.

To show that (23a) and (23b) are equivalent to demanding that $c_2 = 0$, we first note that in the H^2 -subcritical regime $d - 4 < \frac{4}{\sigma}$ and so $\gamma_0 < \frac{2}{3} \left(2 + \frac{2}{\sigma} \right) < \gamma_1$. Next, direct calculation gives that

$$\left(\rho \frac{d}{d\rho} + \frac{2}{\sigma} + i \frac{4\nu}{\kappa^4} \right) S_{B,2} \sim \mathcal{O}(\rho^{4/3 - \frac{2}{3\sigma}(\sigma d - 1)}), \quad \rho \rightarrow \infty,$$

and that

$$\left(\rho \frac{d}{d\rho} + \frac{2}{\sigma} + i \frac{4\nu}{\kappa^4} \right) S_{B,1} \sim \mathcal{O}(\rho^{-\frac{2}{\sigma}-4}), \quad \rho \rightarrow \infty,$$

where the LHS is the result of the next term in the WKB approximation of $S_{B,1}$. Therefore,

$$\rho^\gamma \left(\rho \frac{d}{d\rho} + \frac{2}{\sigma} + i \frac{4\nu}{\kappa^4} \right) S_B \sim \mathcal{O}(c_1 \cdot \rho^{\gamma - \frac{2}{\sigma} - 4}) + \mathcal{O}(c_2 \cdot \rho^{\gamma + 4/3 - \frac{2}{3\sigma}(\sigma d - 1)}), \quad \rho \rightarrow \infty.$$

Since, for $\gamma_0 < \gamma < \gamma_1$,

$$\lim_{\rho \rightarrow \infty} \rho^{\gamma - \frac{2}{\sigma} - 4} = 0, \quad \lim_{\rho \rightarrow \infty} \rho^{\gamma + \frac{4}{3} - \frac{2}{3\sigma}(\sigma d - 1)} = \infty,$$

it follows that $c_2 = 0$ if and only if the limit (23a) is satisfied. \square

The fourth-order nonlinear ODE (14a) requires four boundary conditions. Three boundary conditions are given by (14b), and the fourth condition will be the zero-Hamiltonian condition (23a). Generically, one can expect that for a given ν , this nonlinear eigenvalue problem has an enumerable number of eigenvalues $\kappa^{(n)}$ with corresponding eigenfunctions $S_B^{(n)}$. As in the case of the supercritical NLS [KL95, Bud01], we conjecture that for any (σ, d, ν) there is a unique *admissible solution*, which is monotonically decreasing.

Conjecture 9. *Let σ be in the L^2 supercritical and H^2 -subcritical regime (6), and let $\nu > 0$. Then, the nonlinear eigenvalue problem posed by equation (14a), subject to the boundary conditions*

$$S'_B(0) = S'''_B(0) = S_B(\infty) = 0, \quad \lim_{\rho \rightarrow \infty} \left(\rho S' + \left(\frac{2}{\sigma} + i \frac{4\nu}{\kappa^4} \right) S \right) \rho^\gamma = 0, \tag{24}$$

where γ satisfies (23b), admits a unique eigenpair $(S_B^{\text{admis.}}(\rho), \kappa^{\text{admis.}})$, such that

$$\kappa^{\text{admis.}} = \kappa^{\text{admis.}}(\sigma, d, \nu) > 0,$$

and $|S_B^{\text{admis.}}(\rho)|$ is monotonically decreasing. Furthermore,

$$S_B^{\text{admis.}}(\rho) \sim c \rho^{-2/\sigma - 4\nu i/\kappa^4}, \quad \rho \rightarrow \infty,$$

and

$$\kappa^{\text{admis.}}(\sigma, d, \nu) = \nu^{1/4} \tilde{\kappa}^{\text{admis.}}(\sigma, d), \quad \tilde{\kappa}^{\text{admis.}} := \kappa^{\text{admis.}}(\sigma, d, \nu = 1).$$

3. Peak-type H^2 -singular solutions

3.1. Informal analysis

As in the supercritical NLS [LPSS88, SKL93], we expect that singular peak-type solutions of the supercritical BNLS undergo a quasi-self-similar collapse, so that

$$\psi(t, r) \sim \begin{cases} \psi_{S_B}(t, r) & 0 \leq r \leq r_c, \\ \psi_{\text{non-singular}}(t, r) & r \geq r_c, \end{cases} \tag{25}$$

where ψ_{S_B} is the self-similar profile (10), and r_c is a number, whose value depends on the initial condition. The singular region $r \in [0, r_c]$ is constant in the coordinate r . Therefore, in the rescaled variable $\rho = r/L(t)$, the singular region $\rho \in [0, r_c/L(t)]$ becomes infinite as $L(t) \rightarrow 0$. This is in contradistinction with the critical-BNLS case, where the singular region $\rho \in [0, \rho_c]$ is constant in the rescaled variable ρ , but shrinks to a point in the original coordinate r [BFM10b].

Lemma 10. *Let $\sigma d > 4$, and let ψ be a peak-type singular solution of the BNLS that collapses with the ψ_{S_B} profile (10). If $L(t) \sim \kappa(T_c - t)^p$, then $p \geq \frac{1}{4}$. Furthermore,*

- If $p = 1/4$ if then the self-similar profile $S_B(\rho)$ satisfies the equation (14a).
- If $p > 1/4$, then the profile satisfies the equation

$$-\Delta^2 R_B(x) - R_B + |R_B|^{2\sigma} R_B = 0. \tag{26}$$

Proof. If $\psi \sim \psi_{S_B}$, then the equation for S_B is

$$-\nu S_B - i \left(\lim_{t \rightarrow T_c} L_t L^3 \right) \left(\frac{2}{\sigma} S_B + \rho S'_B \right) - \Delta_\rho^2 S_B + |S_B|^{2\sigma} S_B = 0, \tag{27}$$

implying that $L_t L^3$ should be bounded as $t \rightarrow T_c$. Since $L^3 L_t \sim -p \kappa^4 (T_c - t)^{4p-1}$, it follows that $p \geq \frac{1}{4}$. If $p = 1/4$, then equation (27) reduces to (14a), see section 2. \square

From Hamiltonian conservation it follows that $H[\psi_{S_B}]$ is bounded, because otherwise the non-singular region would also have an infinite Hamiltonian. Therefore, from theorem 7 it follows that $H[S_B] = 0$.

In lemma 8 we saw that the zero-Hamiltonian solutions of (14a) are in $L^{2+2\sigma}$, but not in L^2 . Hence, $\psi_{S_B} \notin L^2$. From power conservation, however, it follows that if $\psi_0 \in H^2$, then $\psi \in L^2$. As in the NLS case, see [BP92], this ‘contradiction’ can be resolved as follows.

Corollary 11. *Let $S_B(\rho)$ be a zero-Hamiltonian solution of (14a) and (14b). Then, $\|S_B\|_2 = \infty$. Nevertheless, $\lim_{t \rightarrow T_c} \|\psi_{S_B}\|_{L^2(r < r_c)} < \infty$.*

Proof. Since $S_B(\rho) \sim c_1 S_{B,1}(\rho)$,

$$\|S_{B,1}\|_2^2 \sim C \int_{\rho=0}^{\infty} \rho^{-4/\sigma+d-1} d\rho \sim C \rho^{d-4/\sigma} \Big|_{\rho=0}^{\infty} = \infty.$$

The profile ψ_{S_B} satisfies

$$\begin{aligned} \|\psi_{S_B}\|_{L^2(r < r_c)}^2 &= L^{d-4/\sigma}(t) \cdot \int_{\rho=0}^{r_c/L(t)} |S_B(\rho)|^2 \rho^{d-1} d\rho \\ &\sim L^{d-4/\sigma}(t) \cdot (C \rho^{d-4/\sigma} \Big|_{\rho=0}^{r_c/L(t)}) = \mathcal{O}(1). \end{aligned} \quad \square$$

In summary, we conjecture the following:

Conjecture 12. *Let ψ be peak-type singular solution of the supercritical BNLS. Then,*

(i) *The collapsing core approaches the self-similar profile ψ_{S_B} , i.e.*

$$\psi(t, r) \sim \psi_{S_B}(t, r), \quad 0 \leq r \leq r_c, \quad (28a)$$

where

$$\psi_{S_B}(t, r) = \frac{1}{L^{2/\sigma}(t)} S_B(\rho) e^{i\nu\tau(t)}, \quad \rho = \frac{r}{L}, \quad \tau(t) = \int_{s=0}^t \frac{1}{L^4(s)} ds, \quad (28b)$$

and r_c is a number, whose value depends on the initial condition.

(ii) *The self-similar profile $S_B(\rho) = S_B^{\text{admis.}}(\rho)$ is the unique admissible solution of*

$$\begin{aligned} -\nu S_B(\rho) + i \frac{\kappa^4}{4} \left(\frac{2}{\sigma} S_B + \rho S_B' \right) - \Delta_\rho^2 S_B + |S_B|^{2\sigma} S_B &= 0, \\ S_B'(0) = S_B''(0) = 0, \quad S_B(\infty) = 0, \quad H[S_B] &= 0, \end{aligned} \quad (28c)$$

where $\kappa^{\text{admis.}}(\sigma, d, \nu) = \nu^{1/4} \kappa^{\text{admis.}}(\sigma, d) > 0$.

(iii) *In particular, $S_B(\rho) \neq R_B(\rho)$.*

(iv) *The blowup rate of singular peak-type solutions is exactly a quartic root, i.e.*

$$L(t) \sim \kappa \sqrt[4]{T_c - t}, \quad \kappa > 0. \quad (28d)$$

(v) *The coefficient κ of the blowup rate of $L(t)$ is equal to the value of κ of the admissible solution S_B , i.e.*

$$\kappa := \lim_{t \rightarrow T_c} \frac{L(t)}{\sqrt[4]{T_c - t}} = \kappa^{\text{admis.}}(\sigma, d, \nu).$$

In particular, κ is universal (i.e. it does not depend on the initial condition).

In section 3.2 we provide numerical evidence in support of conjecture 12.

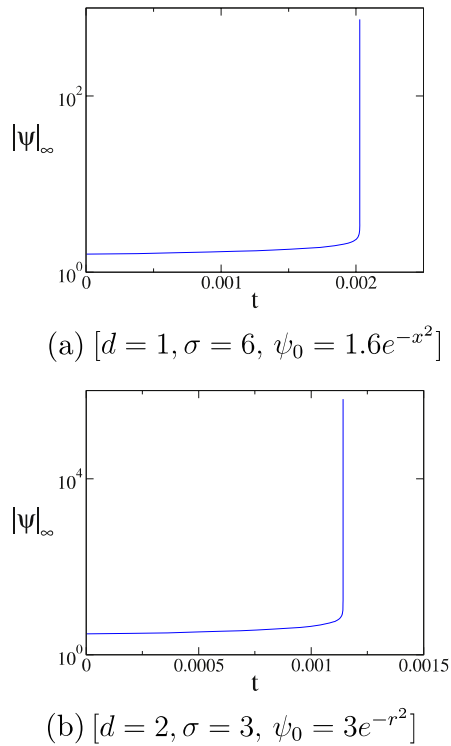


Figure 1. Maximal amplitude of singular solutions of the supercritical BNLS.

3.2. Simulations

The radially symmetric BNLS (7) was solved in the supercritical cases:

- (i) $d = 1, \sigma = 6$ with the initial condition $\psi_0(x) = 1.6e^{-x^2}$.
- (ii) $d = 2, \sigma = 3$ with the initial condition $\psi_0(r) = 3e^{-r^2}$.

In both cases, the solutions blowup at a finite time, see figure 1.

To check whether the solutions collapse with the self-similar profile (28a)–(28d), the solution was rescaled according to

$$\psi_{\text{rescaled}}(t, \rho) = L^{2/\sigma}(t)\psi(t, r = \rho \cdot L), \quad L(t) = \|\psi\|_{\infty}^{-2/\sigma}. \quad (29)$$

Comparing this rescaling with (28b) shows that it implies that $|S_B(0)| = \|S_B\|_{\infty} = 1$. This requirement can always be satisfied with a proper choice of ν , see (15). Figure 2(a) shows the rescaled solutions at the focusing levels $L = 10^{-4}$ and $L = 10^{-8}$, and the rescaled solution $S_B(\rho)$ of (28a)–(28d).¹ The three curves are indistinguishable, showing that the solution is self-similar with the S_B profile, and not with the R_B profile. As additional evidence, figure 3 shows that as $\rho \rightarrow \infty$, the self-similar profile of ψ decays as $\rho^{-2/\sigma}$, which is in agreement with the decay rate of $S_{B,1}(\rho)$.

We next verify that the solution converges to the asymptotic profile for $r \in [0, r_c]$, i.e. for $\rho \in [0, r_c/L(t)]$. To do this, we plot in figure 4 the rescaled solution at focusing levels of $1/L = 10, 100, 1000, 10000$, as a function of $\log(r/L)$. The curves are indistinguishable

¹ In the calculation of S_B , see the appendix, the values of ν and κ were extracted from the BNLS simulation as discussed below, see equations (30) and (32).

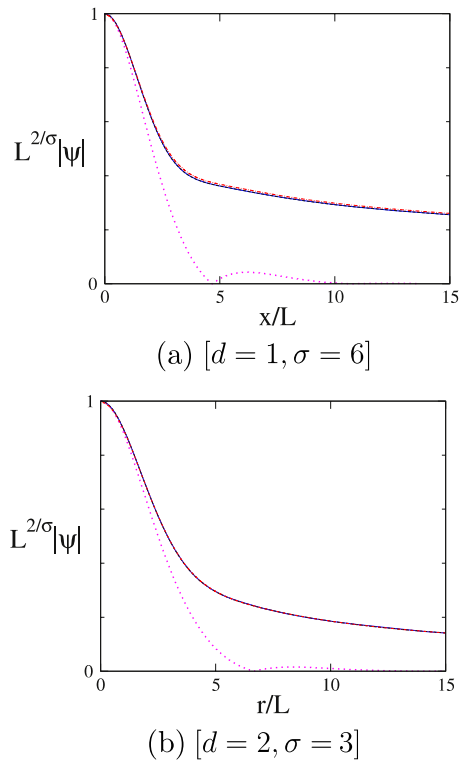


Figure 2. The solutions of figure 1, rescaled according to (29), at the focusing levels $1/L = 10^4$ (blue solid line) and $1/L = 10^8$ (black dashed line). The red dashed–dotted line is the rescaled solution S_B of (28a)–(28d). The (magenta) dotted line is the rescaled ground state R .

at $r/L = \mathcal{O}(1)$, but bifurcate at increasing values of r/L . These ‘bifurcation positions’ are marked by circles in figure 4, and their r/L values are listed in table 1. The ‘bifurcation positions’ are linear in $1/L$, indicating that the region where $\psi \sim \psi_{S_B}$ is indeed $\rho \in [0, r_c/L(t)]$, which corresponds to $r \in [0, r_c]$.

In order to compute the blowup rate p , we performed a least-squares fit of $\log(L)$ with $\log(T_c - t)$, see figure 5. The resulting values are $p \approx 0.2502$ in the $d = 1, \sigma = 6$ case, and $p \approx 0.2504$ in the $d = 2, \sigma = 3$ case. Next, we provide two indications that the blowup rate is exactly $1/4$, i.e. that

$$L(t) \sim \kappa \sqrt[4]{T_c - t}, \quad \kappa > 0.$$

- (i) If the blowup rate is exactly a quartic root, then $L^3 L_t \rightarrow -\frac{\kappa^4}{4} < 0$. Indeed, figure 6(a) shows that in the case $d = 1, \sigma = 6$, $L^3 L_t \rightarrow -0.289$, implying that

$$\kappa^{\text{admis.}}(d = 1, \sigma = 6) \approx \sqrt[4]{4 \cdot 0.289} \approx 1.037. \quad (30a)$$

In the case $d = 2, \sigma = 3$, $L^3 L_t \rightarrow -0.171$, implying that

$$\kappa^{\text{admis.}}(d = 2, \sigma = 3) \approx \sqrt[4]{4 \cdot 0.171} \approx 0.909. \quad (30b)$$

Since $L^3 L_t$ converges to a finite, negative constant, this shows that the blowup rate is exactly $1/4$.

- (ii) According to lemma 10, if $\lim_{t \rightarrow T_c} L^3 L_t < 0$, the self-similar profile $S_B(\rho)$ does not satisfy the standing-wave equation (26), but rather is a solution to the problem (28c), as is clearly demonstrated in figure 2.

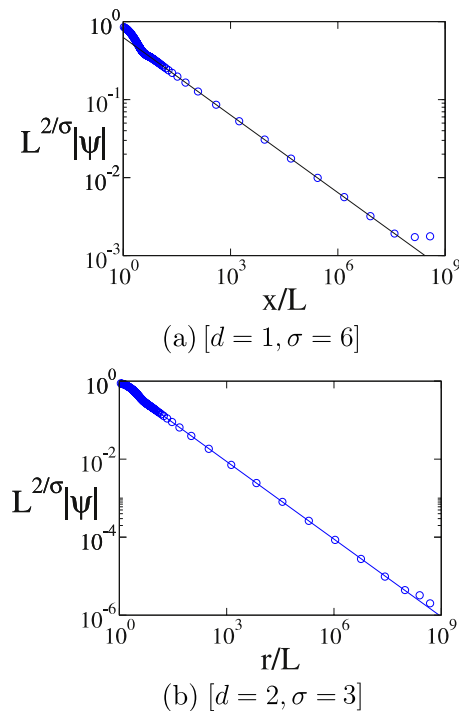


Figure 3. The solutions of figure 1, rescaled according to (29), at the focusing level $1/L = 10^8$ (circles). Solid lines are the fitted curves $y = 0.63 \cdot (x/L)^{-0.33}$ (left) and $y = 0.85 \cdot (r/L)^{-0.66}$ (right).

We recall that calculation of the profile S_B requires knowing the numerical values for ν and κ , see the appendix. The value of κ was obtained previously from the limit $\lim_{t \rightarrow T_c} (-4L^3 L_t)^{1/4}$. We approximate the value of the coefficient ν from

$$\nu_{\text{numeric}} = \lim_{t \rightarrow T_c} L^4(t) \frac{d\tau}{dt}, \quad \tau = \arg \psi(t, r = 0). \tag{31}$$

Indeed, figure 6(b) shows that in both cases $L^4(t) \frac{d\tau}{dt}$ quickly converge to

$$\nu_{\text{numeric}}(d = 1, \sigma = 6) = 0.36187, \quad \nu_{\text{numeric}}(d = 2, \sigma = 3) = 0.22826. \tag{32}$$

As a further verification, using the above values for ν , we seek a value of κ such that the solution of (28c) will satisfy $|S_B(0)| = 1$, and obtain

$$\kappa(d = 1, \sigma = 6, \nu = 0.36187) = 1.007, \quad \kappa(d = 2, \sigma = 3, \nu = 0.22826) = 0.894.$$

These values are within 1–3% from the values of 1.037 and 0.909 we obtained directly from the BNLS simulations, see (30a) and (30b).

Finally, we verified that the value of κ in the blowup rate (28d) is universal. We solve the BNLS in the case $d = 1, \sigma = 6$ with the initial condition $\psi_0(x) = 2e^{-x^4}$. In this case, the calculated value of $\kappa(d = 1, \sigma = 6)$ is $\kappa = \lim_{t \rightarrow T_c} \sqrt[4]{-4L_t L^3} \approx 1.037$, which is equal, to first 3 significant digits, to the previously obtained value, see (30a), for the initial condition $\psi_0(x) = 1.6e^{-x^2}$. Similarly, in the case $d = 2, \sigma = 3$, we solve the equation with the initial condition $\psi_0(x) = 3e^{-x^4}$. The calculated value of $\kappa(d = 2, \sigma = 3)$

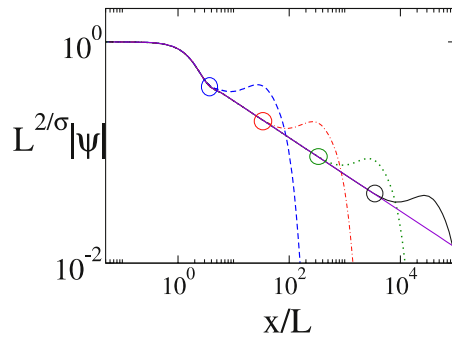
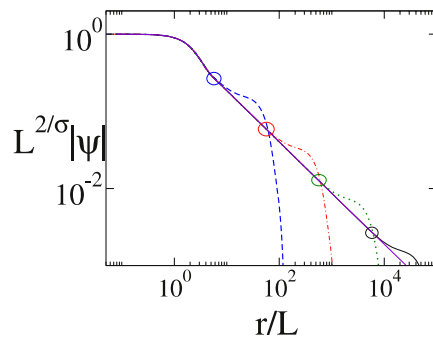
(a) [$d = 1, \sigma = 6$](b) [$d = 2, \sigma = 3$]

Figure 4. Convergence to a self-similar profile. The solutions of figure 1, rescaled according to (29), as a function of $\log(r/L)$, at the focusing levels $L = 10^{-1}$ (dashed blue line), $L = 10^{-2}$ (dashed-dotted red line), $L = 10^{-3}$ (dotted green line), $L = 10^{-4}$ (solid black line) and $L = 10^{-8}$ (solid magenta line). The circles mark the approximate position where each curve bifurcates from the limiting profile, see also table 1.

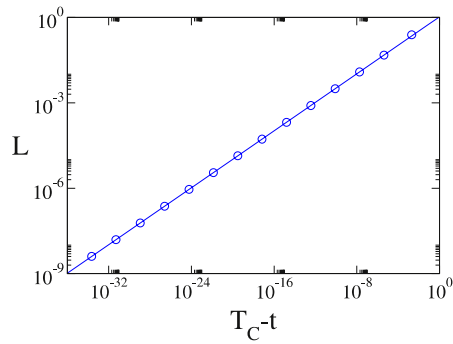
Table 1. Position of circles in figure 3.

	1/L				r_c
	10	100	1000	10000	
x/L ($d = 1$)	3.6	36	360	3600	0.36
r/L ($d = 2$)	6	60	600	6000	0.6

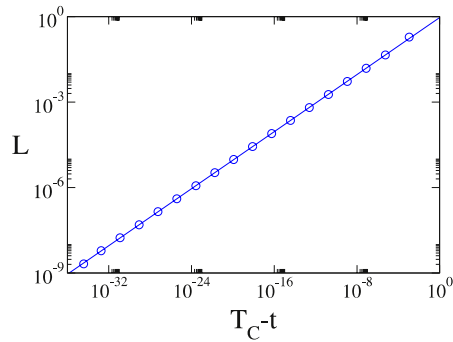
is $\kappa = \lim_{t \rightarrow T_c} \sqrt[4]{-4L_t L^3} \approx 0.913$, which is equal, to first 2 significant digits, to the previously obtained value, see (30b), for the initial condition $\psi_0(x) = 3e^{-x^2}$.

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(a) $[d = 1, \sigma = 6]$



(b) $[d = 2, \sigma = 3]$

Figure 5. $L(t)$ as a function of $(T_c - t)$, on a logarithmic scale, for the solutions of figure 1 (circles). Solid lines are the fitted curves $L = 1.048 \cdot (T_c - t)^{0.2502}$ (a) and $L = 0.931 \cdot (T_c - t)^{0.2504}$ (b).

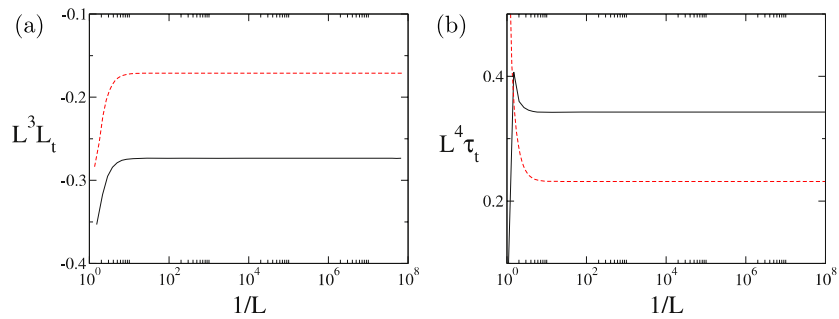


Figure 6. (a) $L^3 L_t$ as a function of $1/L$, for the solution of figure 1(a) (black solid line) and of figure 1(b) (red dashed line). (b) same as (a) for $L^4 \tau_t$, where $\tau = \arg \psi(t, r = 0)$.

Appendix. Numerical calculation of the S_B profile

In order to solve equation (28c), we first define its linear part, which is the fourth-order linear differential operator $L[S]$

$$L[S(\rho)] = -\nu S(\rho) + i \frac{\kappa^4}{4} \left(\frac{2}{\sigma} S + \rho S' \right) - \Delta_\rho^2 S, \tag{A.1a}$$

under the BCs, see equation (24),

$$\begin{aligned} S'(0) = S'''(0) = S(\infty) = 0, \quad \lim_{\rho \rightarrow \infty} \rho^\gamma \left(\rho S' + \left(\frac{2}{\sigma} + i \frac{4\nu}{\kappa^4} \right) S \right) = 0 \\ \gamma_0 < \gamma < \gamma_1, \quad \gamma_0 = \frac{2}{3} \left(d - 2 - \frac{2}{\sigma} \right), \quad \gamma_1 = 4 + \frac{2}{\sigma}. \end{aligned} \quad (\text{A.1b})$$

The nonlinear ODE (28c) is therefore rewritten as

$$L[S(\rho)] + |S|^{2\sigma} S = 0. \quad (\text{A.2})$$

For given numerical values of ν and κ , we wish to calculate the ground state of the nonlinear boundary-value problem (A.2). In order to do so, we modify the SLSR method for the calculation of the ground state of the NLS [Pet76, PS01, AM05] and BNLS [BFM10b] as follows. We consider the fixed-point iterative scheme

$$S^{(k+1)}(\rho) = -L^{-1}[|S^{(k)}|^{2\sigma} S^{(k)}], \quad k = 0, 1, \dots \quad (\text{A.3})$$

for the solution of (A.2). In the standard application of the SLSR method, L is a differential operator of constant coefficients, and its inversion is easily performed using the Fourier transform. In our case, L is a variable-coefficient operator, and the Fourier transform cannot be used. Therefore, we discretize the operator L using finite differences, see appendix A.1, and invert it using the LU decomposition.

We observe numerically that generically, the iterations (A.3) converge to zero for a small initial guess and diverge to infinity for a large initial guess. To avoid this divergence, we rescale the approximate solutions at each iteration, so that they satisfy the integral relation:

$$\int |S|^2 \rho^{d-1} d\rho = \langle S, S \rangle = -\text{Re} \langle S, L^{-1} |S|^{2\sigma} S \rangle,$$

which follows from multiplication of (A.3) by S . Here, $\langle \cdot, \cdot \rangle$ denotes the standard inner product $\langle f, g \rangle = \int f^* g \rho^{d-1} d\rho$. Following a similar argumentation as in [BFM10b], we obtain that the iterations are

$$S^{(k+1)} = - \left(\frac{-\langle S^{(k)}, S^{(k)} \rangle}{\text{Re} \langle S^{(k)}, L^{-1} |S^{(k)}|^{2\sigma} S^{(k)} \rangle} \right)^{1 + \frac{1}{2\sigma}} L^{-1} [|S^{(k)}|^{2\sigma} S^{(k)}]. \quad (\text{A.4})$$

In our simulation, this method converged for every value of ν and κ that we tried. The numerical values of ν was obtained from the on-axis phase of the BNLS simulation solutions, as explained in section 3.2. In order to obtain a prediction of κ , we recall that the specific choice (29) of the blowup rate $L(t)$ implies that $|S_B(0)| = 1$. We therefore use the SLSR solver to search for the value of κ for which $|S_B(0) = 1|$.

A.1. Discretization of L

Using half-integer grid

$$\rho_n = \left(n + \frac{1}{2} \right) h, \quad n = 0, \dots, N-1, \quad h = \frac{R_{\max}}{N},$$

the $\mathcal{O}(h^2)$ centred-difference discretizations of the radial biharmonic operator D_ρ^2 and of the first-derivative, the approximation at the interior nodes is

$$\left(-\nu + \frac{i\kappa^4}{2\sigma} \right) S_n + \frac{i\kappa^4}{4} \rho_n \frac{S_{n+1} - S_{n-1}}{2h} - D_\rho^2 S_n + |S_n|^{2\sigma} S_n = \mathcal{O}(h^2).$$

The stencil is five-nodes wide, so two ghost-nodes are needed at each boundary. In order to enfold the ghost-nodes at $\rho = 0$, we relate them to the interior nodes, using the symmetry of the solution $S(\rho) = S(-\rho)$, so that

$$\begin{bmatrix} S_{-2} \\ S_{-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \end{bmatrix}.$$

This relation is substituted in the discretization of the equation at ρ_0 and ρ_1 .

At the other boundary $\rho = R_{\max}$ we use the approximate form of the solution obtained from the WKB approximation, i.e. we require

$$S_n = c_1 S_{B,1}(\rho_n) + c_4 S_{B,4}(\rho_n), \quad n = N - 2, N - 1, N, N + 1, \dots$$

In matrix form, this becomes

$$\begin{bmatrix} S_{B,1}(\rho_{N-1}) & S_{B,4}(\rho_{N-1}) \\ S_{B,1}(\rho_N) & S_{B,4}(\rho_N) \\ S_{B,1}(\rho_{N+1}) & S_{B,4}(\rho_{N+1}) \\ S_{B,1}(\rho_{N+2}) & S_{B,4}(\rho_{N+2}) \end{bmatrix} \begin{bmatrix} c_1 \\ c_4 \end{bmatrix} = \begin{bmatrix} S_{N-1} \\ S_N \\ S_{N+1} \\ S_{N+2} \end{bmatrix},$$

which is then solved to obtain

$$\begin{bmatrix} S_{N+1} \\ S_{N+2} \end{bmatrix} = \begin{bmatrix} S_{B,1}(\rho_{N+1}) & S_{B,4}(\rho_{N+1}) \\ S_{B,1}(\rho_{N+2}) & S_{B,4}(\rho_{N+2}) \end{bmatrix} \begin{bmatrix} S_{B,1}(\rho_{N-1}) & S_{B,4}(\rho_{N-1}) \\ S_{B,1}(\rho_N) & S_{B,4}(\rho_N) \end{bmatrix}^{-1} \begin{bmatrix} S_{N-1} \\ S_N \end{bmatrix}.$$

Some care should be taken when choosing the parameters R_{\max} and N . On the one hand, we use the closed-form approximations for $S_{B,1}$ and $S_{B,4}$ that become more accurate for $R_{\max} \gg 1$. On the other hand, since $S_{B,4}$ has a super-exponentially decreasing term $e^{-\rho^{4/3}}$, choosing too-large a value of R_{\max} leads to numerical instabilities. Finally, in order to resolve the rapid-oscillations $e^{i\rho^{4/3}}$ of $S_{B,4}$, the grid-size N must be chosen such that $\rho^{1/3}h_\rho = \mathcal{O}(R_{\max}^{4/3}/N) \ll 1$, hence that $N \geq \mathcal{O}(R_{\max}^{4/3})$. The grid-size N , however, cannot be arbitrarily large, since the condition number of L is $\mathcal{O}(N^4)$.

In the simulations presented in this study, we used an extension of above approach to a fourth-order approximation, and set $R_{\max} = 160$ and $N = 32000$.

References

- [AM05] Ablowitz M J and Musslimani Z H 2005 Spectral renormalization method for computing self-localized solutions to nonlinear systems *Opt. Lett.* **30** 2140–2
- [BAKS00] Ben-Artzi M, Koch H and Saut J-C 2000 Dispersion estimates for fourth order Schrödinger equations *C. R. Acad. Sci. Paris Sér. I Math.* **330** 87–92
- [BFG10] Baruch G, Fibich G and Gavish N 2010 Singular standing ring solutions of nonlinear partial differential equations *Physica D* **239** 1968–83
- [BFM10a] Baruch G, Fibich G and Mandelbaum E 2010 Ring-type singular solutions of the biharmonic nonlinear Schrödinger equation *Nonlinearity* **23** 2867–87
- [BFM10b] Baruch G, Fibich G and Mandelbaum E 2010 Singular solutions of the L^2 -critical biharmonic nonlinear Schrödinger equation *SIAM J. Appl. Math.* **70** 3319–41
- [BP92] Bergé L and Pesme D 1992 Time dependent solutions of wave collapse *Phys. Lett. A* **166** 116–22
- [Bud01] Budd C J 2001 Asymptotics of multibump blow-up self-similar solutions of the nonlinear Schrödinger equation *SIAM J. Appl. Math.* **62** 801–30
- [CHL09] Chae M, Hong S and Lee S 2011 Mass concentration for the L^2 -critical Nonlinear Schrodinger equations of higher orders *Discrete Contin. Dyn. Syst.* **29** 909–28
- [DG09] Ditzkowsky A and Gavish N 2009 A grid redistribution method for singular problems *J. Comput. Phys.* **228** 2354–65
- [FIP02] Fibich G, Ilan B and Papanicolaou G 2002 Self-focusing with fourth-order dispersion *SIAM J. Appl. Math.* **62** 1437–62

- [KL95] Koppel N and Landman M 1995 Spatial structure of the focusing singularity of the nonlinear Schrödinger equation: a geometrical analysis *SIAM J. Appl. Math.* **55** 1297–323
- [LPSS88] LeMesurier B J, Papanicolaou G, Sulem C and Sulem P L 1988 Focusing and multi-focusing solutions of the nonlinear Schrödinger equation *Physica D* **31** 78–102
- [MRS09] Merle F, Raphael P and Szeftel J 2010 Stable self similar blow up dynamics for slightly L^2 supercritical NLS equations *Geom. Funct. Anal.* **20** 1028–71
- [MXZ08] Miao C, Xu G and Zhao L 2008 Global wellposedness and scattering for the defocusing energy-critical nonlinear Schrödinger equations of fourth order in dimensions $d \geq 9$ arXiv:0807.0692
- [MXZ09] Miao C, Xu G and Zhao L 2009 Global well-posedness and scattering for the focusing energy-critical nonlinear Schrödinger equations of fourth order in the radial case *J. Diff. Eqns* **246** 3715–49
- [Pau07] Pausader B 2007 Global well-posedness for energy critical fourth-order Schrödinger equations in the radial case *Dyn. Partial Diff. Eqns* **4** 197–225
- [Pau09a] Pausader B 2009 The cubic fourth-order Schrödinger equation *J. Funct. Anal.* **256** 2473–517
- [Pau09b] Pausader B 2009 The focusing energy-critical fourth-order Schrödinger equation with radial data *Discrete Contin. Dyn. Syst.* **24** 1275–92
- [PS10] Pausader B and Shao S 2010 The mass-critical fourth-order Schrödinger equation in high dimensions *J. Hyp. Diff. Eqns* **7** 651–705
- [Pet76] Petviashvili V I 1976 Equation of an extraordinary soliton *Sov. J. Plasma Phys.* **2** 469–72
- [PS01] Pelinovsky D B and Stepanyants Y A 2004 Convergence of petviashvili's iteration method for numerical approximation of stationary solutions of nonlinear wave equations *SIAM J. Numer. Anal.* **42** 1110–27
- [RW00] Ren W and Wang X P 2000 An iterative grid redistribution method for singular problems in multiple dimensions *J. Comput. Phys.* **159** 246–73
- [SKL93] Shvets V F, Kosmatov N E and LeMesurier B J 1993 On collapsing solutions of the nonlinear Schrödinger equation in supercritical case *Singularities in Fluids, Plasmas and Optics* ed R E Caflisch and G C Papanicolaou (Dordrecht: Kluwer) pp 317–21
- [SS99] Sulem C and Sulem P L 1999 *The Nonlinear Schrödinger Equation* (New York: Springer)
- [ZZY10] Zhu S, Zhang J and Yang H 2010 Limiting profile of the blow-up solutions of the fourth-order Nonlinear Schrödinger equation *Dyn. Partial Diff. Eqns* **7** 187–205