Self-focusing in the presence of small time dispersion and nonparaxiality

G. Fibich

Department of Mathematics, University of California, Los Angeles, Los Angeles, California 90095-1555

G. C. Papanicolaou

Department of Mathematics, Stanford University, Stanford, California 94305

Received April 24, 1997

We analyze the combined effect of small time dispersion and nonparaxiality on self-focusing and its ability to arrest the blowup of laser pulses by deriving reduced equations that depend on only the propagation distance and time. We calculate the pulse duration for which time dispersion dominates over nonparaxiality, or vice versa. We identify additional terms (shock term, group-velocity nonparaxiality, etc.) that should be retained when time dispersion and nonparaxiality are of comparable magnitude. These additional terms lead to temporal asymmetry, and in the visible spectrum they can dominate over both time dispersion and nonparaxiality. © 1997 Optical Society of America

The simplest model for optical self-focusing is the nonlinear Schrödinger equation (NLS):

$$i\psi_z + \Delta_\perp \psi + |\psi|^2\psi = 0, \qquad \psi(0, r) = \psi_0(r). \tag{1}$$

Here $\psi(z,r)$ is the electric field envelope of a laser beam propagating in a medium with Kerr nonlinearity, z is the distance in the direction of the propagation, $r = (x^2 + y^2)^{1/2}$ is the radial coordinate, and $\Delta_\perp = \partial^2/\partial r^2 + (1/r)(\partial/\partial_r)$ is the Laplacian in the transverse two-dimensional (2D) plane. It is well known that if the initial power is more than a critical value (i.e., $\int |\psi_0|^2 r \mathrm{d}r \geq N_c \cong 1.86$), solutions of Eq. (1) may blow up in a finite distance z. Since physical quantities do not become infinite, it is clear that the validity of Eq. (1) breaks down near the focal point and that additional physical mechanisms, which are initially small, become important there and prevent singularity formation.

In this Letter we focus on the combined effect of two mechanisms that may arrest blowup and that are neglected when one is approximating Maxwell's equations by use of the NLS: small time dispersion and beam nonparaxiality. Previously it was suggested that small nonparaxiality arrests self-focusing and leads to an oscillatory focusing–defocusing behavior.[1,2] In other studies it was shown that small normal time dispersion delays the onset of self-focusing and causes the temporal splitting of the pulse into two peaks that continue to focus.[3,4] However, it is still unknown at present whether the solution will ultimately blow up or not. Recently, pulse splitting was observed experimentally.[5]

An important question that arises when one is modeling physical self-focusing is whether time dispersion and (or) nonparaxiality should be included in the model. In this Letter we answer this question by identifying the regimes in which each mechanism dominates. While we are doing this, additional terms are identified that should be kept in the model when time dispersion and nonparaxiality are of the same order. In fact, these additional terms can even dominate over both time dispersion and nonparaxiality in the visible spectrum. We then derive reduced equations that describe self-focusing when all the above mechanisms are present. We use these reduced equations to analyze the combined effect of normal time dispersion and nonparaxiality (both of which arrest self-focusing), the case of anomalous time dispersion and nonparaxiality, (which have opposite focusing effects), and the influence of the additional terms.

We begin by deriving the NLS with nonparaxiality and time dispersion. If we neglect vectorial effects,[6] the electric field can be assumed to have the form

$$\mathbf{E}(x, y, z, t) = \mathbf{e}A(x, y, z, t)\exp(ik_0 z - i\omega_0 t),$$

where the unit vector $\mathbf{e}$ is perpendicular to the z axis. The equation for the slowly varying envelope A is[7]

$$A_{zz} + 2ik_0 A_z + \Delta_\perp A + \frac{2ik_0}{c_g} A_t - \left(\frac{1}{{c_g}^2} + k_0 k_{\omega\omega}\right) A_{tt} = \frac{2n_2 n_0}{c^2} \exp(i\omega_0 t)\,[|A|^2 A \exp(-i\omega_0 t)]_{tt},$$

where $k = \omega n_0(\omega)/c$, $k_0 = k(\omega_0)$, ${c_g}^{-1} = (\mathrm{d}k/\mathrm{d}\omega)_{\omega_0}$, n_0 is the linear index of refraction, and n_2 is the Kerr coefficient. We change to a nondimensional moving-frame coordinate system with

$$\tilde{r} = \frac{r}{r_0}, \qquad \tilde{z} = \frac{z}{2L_{\mathrm{diff}}}, \qquad \tilde{t} = \frac{t - z/c_g}{T},$$

$$\psi = r_0 k_0 \sqrt{\frac{2n_2}{n_0}}\, A,$$

where r_0 is the initial pulse width, $L_{\mathrm{diff}} = {r_0}^2 k_0$ is the diffraction length, and T is the pulse duration. Dropping the tilde and neglecting the $(|A|^2 A)_{tt}$ term, which is $O(\epsilon_2^2)$, we find that the equation for the nondimensional envelope ψ is

$$i\psi_z + \Delta_\perp \psi + |\psi|^2\psi + \epsilon_1\psi_{zz} + \epsilon_2\left[2i\,\frac{n_0 c_g}{c}\,(|\psi|^2\psi)_t - \psi_{zt}\right] - \epsilon_3\psi_{tt} = 0\,, \quad (2)$$

where

$$\epsilon_1 = \frac{1}{4{r_0}^2{k_0}^2}\,, \qquad \epsilon_2 = \frac{1}{c_g k_0 T} = \frac{1}{\omega_0 T}\,\frac{c}{n_0 c_g}\,, \qquad \epsilon_3 = \frac{L_{\text{diff}} k_{\omega\omega}}{T^2}\,. \quad (3)$$

The dimensionless parameter $\epsilon_1 \sim$ (wavelength/radial pulse width)^2, $\epsilon_2 \sim$(period of one oscillation/pulse duration), and ϵ_3 is a dimensionless measure of group-velocity dispersion. Note that

$${\epsilon_2}^2 = \epsilon_1\epsilon_3 F, \qquad F = \frac{4}{{c_g}^2 k_0 k_{\omega\omega}}\,.$$

The first component of the ϵ_2 term is sometimes called the shock term.[8] The second component can be replaced with

$$-\epsilon_2\psi_{zt} \sim -i\epsilon_2[\Delta_\perp\psi_t + (|\psi|^2\psi)_t]\,, \quad (4)$$

and its linear part $(-i\epsilon_2\Delta_\perp\psi_t)$ was interpreted by Rothenberg as the effect of the variation of the group velocity of a tilted ray projected onto the z axis.[8]

Let us define T_b as the pulse duration for which time dispersion and nonparaxiality are of the same magnitude (i.e., $\epsilon_1 = |\epsilon_3|$):

$$T_b = 2L_{\text{diff}}\sqrt{|k_0 k_{\omega\omega}|} = \frac{4}{\sqrt{|F|}}\,\frac{L_{\text{diff}}}{c_g}\,.$$

If F is $O(1)$, then, when $T \ll T_b$, time dispersion will initially dominate and $\epsilon_1 \ll \epsilon_2 \ll \epsilon_3$, but as the pulse becomes narrower $\epsilon_1 \sim r^{-2}$ increases while $\epsilon_3 \sim r^2$ decreases. When $T \gg T_b$, nonparaxiality dominates and $\epsilon_1 \gg \epsilon_2 \gg \epsilon_3$. Note that it is not possible to include in the model both the ϵ_1 and the ϵ_3 terms without also retaining the ϵ_2 term.

The ϵ_2 term is usually assumed to be small compared with either time dispersion or nonparaxiality. However, we now show that in the visible spectrum it can dominate both. The index of refraction of optical materials such as water[9] or silica[10] in the range of transparency is almost constant, and $|\omega n_\omega| \ll 1$.[11] For example, by use of data digitized from Ref. 9 it was estimated that for water in the visible spectrum $|\omega n_\omega| \sim 0.03$.[12] Therefore, $c_g \sim c/n_0$, $\epsilon_2 > 0$ and

$$|F| \sim \frac{2n_0}{|\omega n_\omega|} \gg 1\,,$$

with $F \sim 100$ for water, for example. This implies that in the visible regime and with $T = O(T_b)$ both ϵ_1 and ϵ_3 are small $[O(1/\sqrt{|F|})]$ compared with ϵ_2. When $T = T_b\sqrt{|F|}$ (or $T = T_b/\sqrt{|F|}$), $\epsilon_1 = \epsilon_2$ (or $\epsilon_3 = \epsilon_2$) and $\epsilon_3/\epsilon_2 = O(1/F)$ [or $\epsilon_1/\epsilon_2 = O(1/F)$]. Only when $T \gg T_b\sqrt{|F|}$ (or $T \ll T_b/\sqrt{|F|}$) do we have that $\epsilon_3 \ll \epsilon_2 \ll \epsilon_1$ (or $\epsilon_1 \ll \epsilon_2 \ll \epsilon_3$). Moreover, using relation (4) and $c_g \sim c/n_0$, we find that Eq. (2) reduces to

$$i\psi_z + \Delta_\perp\psi + |\psi|^2\psi + \epsilon_1\psi_{zz} + i\epsilon_2[(|\psi|^2\psi)_t - \Delta_\perp\psi_t] - \epsilon_3\psi_{tt} = 0\,. \quad (5)$$

The separate effects of small time dispersion and nonparaxiality were analyzed before[2,4] by use of a perturbation method that permits the derivation of simplified equations.[13] Briefly, near the focal point the solutions of Eq. (2) or (5) have the form

$$\psi(z,t,r) \sim \frac{1}{L(z,t)}\,R\!\left(\frac{r}{L}\right)\exp\!\left[i\zeta(z,t) + i\,\frac{L_z r^2}{4L}\right],$$

where $R(r) > 0$, the radial profile (Townes soliton), satisfies $\Delta_\perp R - R + R^3 = 0$ and $\int R^2 r\mathrm{d}r = N_c$. By averaging over the transverse coordinates we find that the modulation functions L and ζ must satisfy the reduced equations

$$\zeta_z(z,t) = \frac{1}{L^2}\,, \qquad L_{zz}(z,t) = -\frac{\beta(z,t)}{L^3}\,, \quad (6)$$

$$\beta_z(z,t) = -\gamma_1\left(\frac{1}{L^2}\right)_z - \gamma_2\left(\frac{1}{L^2}\right)_t + \gamma_3\zeta_{tt}\,, \quad (7)$$

where $\gamma_1 = 2\epsilon_1 N_c/M$, $\gamma_2 = \epsilon_2(6c_g n_0/c - 2)N_c/M$ for Eq. (2) and $\gamma_2 = 4\epsilon_2 N_c/M$ for Eq. (5), $\gamma_3 = 2\epsilon_3 N_c/M$ and $M = 1/4\int R^2 r^3 r\mathrm{d}r \cong 0.55$. The modulation functions have the following meaning[13,14]: β is proportional to the excess cross-sectional power above critical, L is the nondimensional radial pulse width and is also inversely proportional to the on-axis intensity $|\psi(z,t,r=0)|$ so that blowup occurs when $L = 0$, and ζ is the rescaled axial distance. The system of equations (6) and (7) is much easier than Eq. (2) for both analysis and simulations, since the radial dependence has been eliminated.

In the pulse-splitting experiment[5] the values of the nondimensional parameters are $\epsilon_1 = 1.3 \times 10^{-6}$, $\epsilon_2 = 5 \times 10^{-3}$, $\epsilon_3 = 1.5 \times 10^{-1}$. Using these values and the initial conditions $L(0,t) \equiv 1$, $\beta(0,t) = N_c[1.05\exp(-t^2) - 1]/M$, we integrated Eqs. (6) and (7). These initial conditions may not be close to those of the experiment in the focusing regime, which are unknown, but they do give an idea of how the pulse evolves. We observe (Fig. 1) pulse splitting (owing to normal time dispersion), accompanied by a temporal

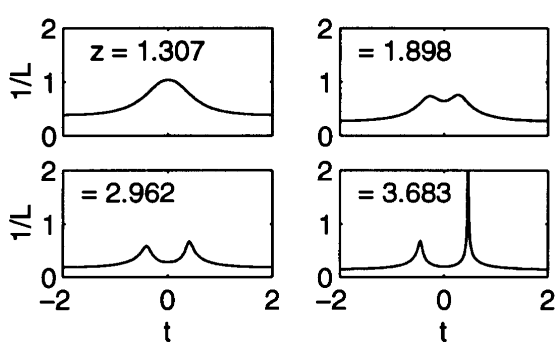

Fig. 1. Evolution of the on-axis intensity $(1/L)$ versus time according to Eqs. (6) and (7) at the propagation distances indicated.

shift of the focus toward later times and enhanced focusing of the second peak (owing to the ϵ_2 term).

Following Ref. 4, we can analyze the initial effect of the three terms in Eq. (2) by looking at special solutions of Eqs. (6) and (7). Away from the focal point, the three perturbing terms in Eq. (2) are small and each t cross section of the pulse [i.e., the 2D plane $t =$ const in the (x, y, t) space] focuses independently with

$$L(z,t) = L(Z_c(t) - z), \qquad \beta(z,t) = \beta(Z_c(t) - z), \qquad \zeta(z,t) = \zeta(Z_c(t) - z). \tag{8}$$

Here $Z_c(t)$ is the location of the focus in the (z, t) plane when $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$.[14] Therefore, Eq. (7) becomes

$$\beta_z = -\gamma_1\left(\frac{1}{L^2}\right)_z + \gamma_2\dot{Z}_c\left(\frac{1}{L^2}\right)_z + \gamma_3(-\ddot{Z}_c\zeta_z + \dot{Z}_c{}^2\zeta_{zz}), \quad \text{where } \dot{} = \frac{\mathrm{d}}{\mathrm{d}t}\cdot \tag{9}$$

Equation (9) can be transformed into a nonlinear Airy equation[4]

$$g_{ss} = sg + \kappa g^3, \quad \text{with } g = L^{-1} > 0\,. \tag{10}$$

Here

$$s = (\beta_0 - \gamma_3\ddot{Z}_c\zeta)(\gamma_3\ddot{Z}_c)^{-2/3}, \qquad \beta_0 \sim \beta(0,t)\,,$$

$$\kappa = -(\gamma_1 - \gamma_2\dot{Z}_c - \gamma_3\dot{Z}_c{}^2)(\gamma_3\ddot{Z}_c)^{-2/3}.$$

The initial conditions for Eq. (10) are given at

$$s_0(t) := s(z = 0, t) \sim \beta(0,t)(\gamma_3\ddot{Z}_c)^{-2/3}.$$

At time t_0 of the initial peak power of the pulse, $Z_c(t)$ attains its minimum, $\dot{Z}_c(t_0) = 0$, and the evolution is given by Eq. (10) with $\kappa = -\gamma_1(\gamma_3\ddot{Z}_c)^{-2/3} < 0$. Because $\ddot{Z}_c(t_0) > 0$, as $z \to Z_c$ and $\zeta \to +\infty$, $s \to -\infty$ for normal time dispersion ($\epsilon_3 > 0$), and both time dispersion and nonparaxiality [the first and second terms on the right-hand side of Eq. (10), respectively] contribute to the arrest of the blowup by preventing g from becoming infinite. When time dispersion is anomalous ($\epsilon_3 < 0$), it enhances blowup ($s \to +\infty$), whereas nonparaxiality opposes it. Eventually, as $s \to +\infty$ nonparaxiality prevails and the solution of Eq. (10) will decay (no blowup).

In the case of normal time dispersion and $\epsilon_1 = \epsilon_2 = 0$, blowup is arrested only in an exponentially small neighborhood of t_0, where pulse splitting occurs.[4] To assess the added effects of nonparaxiality and the mixed term, we note that the condition for blowup[4] in Eq. (10) as $s \to -\infty$ is $\kappa > 2L^2(0,t)Ai^2(s_0)$ or

$$\gamma_3\dot{Z}_c{}^2 > \gamma_1 - \gamma\dot{Z}_c + 2L^2(0,t)Ai^2(s_0)(\gamma_3\ddot{Z}_c)^{2/3},$$

where $Ai(s)$ is the Airy function. Therefore, if nonparaxiality dominates, arrest of blowup occurs over a much larger region (possibly everywhere). If the ϵ_2 term dominates, blowup will occur when $\epsilon_3 > -\epsilon_2/\dot{Z}_c$, i.e., only for $t > t_0$. Note that as the solution starts to deviate from that of the unperturbed NLS, the 2D self-similar structure [Eqs. (8)] will gradually break down. Therefore, for later z this 2D self-similar argument becomes invalid, and the full three-dimensional nature of Eq. (7) has to be considered.

From Eq. (9) we see that the effect of the ϵ_2 term on a self-focusing pulse is a temporal power transfer toward later times (recall that β is proportional to the excess power above critical). This will result in an asymmetric temporal development of the pulse, with a greatly enhanced trailing portion and a suppressed leading part, in agreement with previous results on the effect of the shock term[15] and of the linear component of the ϵ_2 terms.[8]

We thank R. Medina for supplying us the digitized data from Ref. 9. The research of G. Fibich was supported in part by National Science Foundation (NSF) grant DMS-9623087. The research of G. Papanicolaou was supported in part by NSF grant DMS-9622854 and by Air Force Office of Scientific Research grant F49620-95-1-0315.

References

1. M. Feit and J. Fleck, J. Opt. Soc. Am. B **5,** 633 (1988); J. Soto-Crespo and N. Akhmediev, Opt. Commun. **101,** 223 (1993); N. Akhmediev, A. Ankiewicz, and J. Soto-Crespo, Opt. Lett. **18,** 411 (1993).
2. G. Fibich, Phys. Rev. Lett. **76,** 4356 (1996).
3. N. Zharova, A. G. Litvak, T. A. Petrova, A. M. Sergeev, and A. D. Yunakovsky, JETP Lett. **44,** 13 (1986); J. E. Rothenberg, Opt. Lett. **17,** 583 (1992); P. Chernev and V. Petrov, Opt. Lett. **17,** 172 (1992); G. Luther, A. Newell, and J. Moloney, Phys. D **74,** 59 (1994).
4. G. Fibich, V. Malkin, and G. Papanicolaou, Phys. Rev. A **52,** 4218 (1995).
5. J. Ranka, R. Schirmer, and A. Gaeta, Phys. Rev. Lett. **77,** 3783 (1996).
6. S. Chi and Q. Guo, Opt. Lett. **20,** 1598 (1995).
7. N. Tzoar and M. Jain, Phys. Rev. A **23,** 1266 (1981).
8. J. Rothenberg, Opt. Lett. **17,** 1340 (1992).
9. M. Querry, D. Wieliczka, and D. Segelstein, in *Handbook of Optical Constants of Solids II*, E. D. Palik, ed. (Academic, San Diego, Calif., 1991), pp. 1059–1077.
10. I. Malitson, J. Opt. Soc. Am. **55,** 1205 (1965).
11. Z. Bor and B. Rácz, Appl. Opt. **24,** 3440 (1985).
12. G. Fibich, "Self-focusing in the nonlinear Schrödinger equation for ultrashort laser–tissue interactions," Ph.D. dissertation (Courant Institute, New York University, New York, New York, 1994).
13. G. Fibich and G. Papanicolaou, Computational and Applied Mathematics report 97-21 (University of California, Los Angeles, Los Angeles, Calif., 1997).
14. G. Fibich, Opt. Lett. **21,** 1735 (1996).
15. D. Anderson and M. Lisak, Opt. Lett. **7,** 394 (1982).