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All-pay auctions with risk-averse players

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Abstract We study independent private-value all-pay auctions with risk-averse players. We show that: (1) Players with low values bid lower and players with high values bid higher than they would bid in the risk neutral case. (2) Players with low values bid lower and players with high values bid higher than they would bid in a first-price auction. (3) Players' expected utilities in an all-pay auction are lower than in a first-price auction. We also use perturbation analysis to calculate explicit approximations of the equilibrium strategies of risk-averse players and the seller's expected revenue. In particular, we show that in all-pay auctions the seller's expected payoff in the risk-averse case may be either higher or lower than in the risk neutral case.

Keywords Private-value auctions · Risk aversion · Perturbation analysis

JEL Classification: D44 · D72 · D82

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1 Introduction

Consider n players who compete for a single item. Every player submits a bid and the player with the highest bid receives the item. All players bear a cost of bidding which is an increasing function of their bids. This setup, which is called an all-pay auction, is commonly used to model applications such as job-promotion competitions, R&D competitions, political campaigns, political lobbying, sport competitions, etc. The literature on contests and particularly on all-pay auctions has dealt mostly with risk-neutral players.[1] In contrast to all-pay auctions, several studies on the classical auction mechanisms (first-price and second-price auctions) with risk-averse players appear in the literature. In independent private-value second-price auctions, risk aversion has no effect on a player's optimal strategy which remains to bid her own valuation for the object. In independent private-value first-price auctions, on the other hand, risk aversion makes players bid more aggressively (Maskin and Riley 1984). Since the (risk-neutral) seller is indifferent to the first-price and second-price auctions when players are risk neutral,[2] she prefers the first-price auction to the second-price auction when players are risk averse. However, the seller's preference relations for auction mechanisms with risk-averse players do not imply anything about the players' preference relations for these auctions, since under risk aversion the combined revenue of the seller and the players is not a constant. Indeed, Matthews (1987) showed that risk averse players with constant absolute risk aversion are indifferent to first and second-price auctions, and that players prefer the first-price auction if they have increasing absolute risk aversion and the second price auction if they have decreasing absolute risk aversion.[3]

In this paper we analyze the role of risk aversion in all-pay auctions by comparing the situation where all players are risk neutral (henceforth referred to as the status quo), with the case where players are risk-averse. In Sect. 2 we show that a risk-averse player with a low valuation bids less aggressively than in the status quo situation. On the other hand, a risk-averse player with a high valuation bids more aggressively than in the status quo. This behavior can be explained as follows. When a player's value is small, she is most likely to lose. Therefore, as she becomes more risk averse, she is willing to pay less, that is, she bids less aggressively. On the other hand, when a player's value is very high, she is afraid of losing the object, therefore, she bids more aggressively. These results are consistent with the experimental studies of Barut et al. (2002) and Noussair and Silver (2005), who observed that players in single-unit and multiple-unit

[1] All-pay auctions with linear cost functions and incomplete information about the players' values include, among others: Weber (1985), Hilman and Riley (1989), Krishna and Morgan (1997), Kaplan et al. (2002). All-pay auctions with complete information about the players' values include, among others: Tullock (1980), Dasgupta (1986), Dixit (1987), Baye et al. (1993, 1996).

[2] This follows from the Revenue Equivalence Theorem (Vickrey 1961; Myerson 1981; Riley and Samuelson 1981).

[3] This result was generalized by Monderer and Tennenholtz (2000) to all k-price auctions.

all pay auctions with low values tend to bid below the risk-neutral equilibrium, and those with large values tend to bid above the risk-neutral equilibrium.

We can learn much about the all-pay auction with risk averse players by comparing it to the first-price auction. Although the first-price auction is a classical auction whereas the all-pay auction is a contest, these models are similar since in both the highest player wins for sure and pays her bid. Intuitively, one can expect that as in the risk-neutral case, the equilibrium bids of risk averse players in all-pay auctions should be lower than in first-price auctions. We show that, indeed, in all-pay auctions, low types bid less aggressively than they bid in first-price auctions. However, high types bid more aggressively in all-pay auctions than they bid in first-price auctions.

In light of the above comparison of the players' bids in first-price auctions and all-pay auctions, it is not clear in which auction the player's expected utility is larger. Nevertheless, we show that, independent of the distribution of the players' valuations and the number of players, the expected utility of a risk-averse player in the first-price auction is always larger than in the all-pay auction. Consequently, a risk-averse player will prefer the first-price auction to the all-pay auction. We note that Eso and White (2004) proved that bidders would prefer the first-price auction to the all-pay auction under symmetric, affiliated values and decreasing absolute risk aversion (DARA). Therefore, the present study shows that this result remains true if one replaces the assumption of affiliation with the stronger assumption of independence, but relaxes the assumption of DARA to any type of risk aversion.

Rigorous analysis of all-pay auctions with risk-averse players is limited since usually explicit expressions for the equilibrium strategies with risk-averse players cannot be obtained. In order to overcome this difficulty, in Sect. 3 we consider the case of weakly risk-averse players. The presence of the small risk-aversion parameter allows us to employ perturbation analysis, one of the most powerful tools in applied mathematics, to calculate explicit approximations of the equilibrium strategies of risk-averse players and the seller's expected revenue.[4] The high accuracy of the explicit approximations of the equilibrium bids is illustrated by an example with two weakly risk-averse players. We show that even when the risk-aversion parameter is not small, the agreement between the explicit approximations obtained by the perturbation analysis and the exact values obtained by numerical analysis is quite remarkable.[5]

The approximate solutions in Sect. 3 show, for example, that risk aversion can lead to an increase, as well as a decrease, in the seller's expected revenue in all pay auctions. In addition, they show that, roughly speaking, weak risk aversion leads to a larger departure from revenue equivalence than weak asymmetry. Altogether, the combination of the quantitative results of risk-aversion in all-pay auctions, together with the qualitative results given in Sect. 2 provide a clear picture of the behavior of risk-averse players in all-pay auctions.

[4] Fibich and Gavious (2003) and Fibich et al. (2004) employed perturbation analysis to study asymmetric auctions.

[5] This is, more often than not, the case in perturbation analysis (see, e.g., Bender and Orszag 1978).

Remark For clarity, the proofs are delegated to the Appendices and presented in the order in which they are proved. The results in Sect. 2 are presented in a different order in which they are proved in the Appendices, in order to better present the economic results.

2 All-pay auctions with risk-averse players

Consider n players that compete to acquire a single object in an all-pay auction. The valuation of each player for the object v is independently distributed according to a distribution function $F(v)$ on the interval $[\underline{v}, \overline{v}]$, where $\underline{v} \geq 0$. Each player submits a bid b and pays her bid regardless of whether she wins or not, but only the highest player wins the object. Each player's utility is given by a function $U(v-b)$ which is twice continuously differentiable, monotonically increasing, normalized such that $U(0) = 0$, and satisfies $U'' < 0$ (i.e., risk-averse players). Given that the equilibrium bid function $b(v)$ is monotonically increasing, we can define the equilibrium inverse bid function $v = v(b)$. The maximization problem of player i with valuation v is given by

$$\max_b V(v,b) = F^{n-1}(v(b))U(v-b) + (1 - F^{n-1}(v(b)))U(-b).$$

Differentiating with respect to b gives the first-order condition

$$0 = \frac{\partial V}{\partial b} = (n-1)F^{n-2}(v(b))f(v(b))v'(b)[U(v-b) - U(-b)] \\ -F^{n-1}(v(b))[U'(v-b) - U'(-b)] - U'(-b).$$

Therefore, the inverse bid function satisfies the ordinary differential equation

$$v'(b) = \frac{F(v(b))[U'(v-b) - U'(-b)]}{(n-1)f(v(b))[U(v-b) - U(-b)]} \\ + \frac{U'(-b)}{(n-1)F^{n-2}(v(b))f(v(b))[U(v-b) - U(-b)]}, \quad (1)$$

subject to the initial condition $v(0) = \underline{v}$.

Equation (1) is exact in the risk-neutral case $U(x) = c \cdot x$ where c is a constant. In that case, its solution is given by

$$b_{\mathrm{rn}}^{\mathrm{all}}(v) = vF^{n-1}(v) - \int_{\underline{v}}^{v} F^{n-1}(s)\mathrm{d}s. \quad (2)$$

For comparison, the equilibrium bid in a first price auction with risk-neutral bidders is given by

$$b_{\rm rn}^{\rm 1st}(v) = \frac{1}{F^{n-1}(v)} b_{\rm rn}^{\rm all}(v). \tag{3}$$

Therefore, it immediately follows that

$$b_{\rm rn}^{\rm all}(v) < b_{\rm rn}^{\rm 1st}(v), \quad \underline{v} < v < \bar{v}. \tag{4}$$

Although there are no explicit solutions of Eq. (1) for a general utility function U, we can derive some qualitative results by comparing the equilibrium bids in the risk-averse and the risk-neutral cases. These results are in the spirit of the ones obtained by Maskin and Riley (1984), who showed that in a first-price auction the equilibrium bid of a risk-averse player is higher than the equilibrium bid of a risk-neutral player with the same type, that is,

$$b_{\rm rn}^{1st}(v) < b^{\rm 1st}(v), \quad \underline{v} < v \leq \bar{v}. \tag{5}$$

We also show how relation (4) is affected by risk aversion, by comparing the bids of risk-averse bidders in all-pay auctions with the ones in first-price auctions, denoted by $b^{\rm 1st}(v)$.

We first show that risk aversion affects low type players to bid less aggressively:

Proposition 1 *In an all-pay auction the equilibrium bid of a risk-averse player with low type v is smaller than the equilibrium bid of a risk-neutral player with the same type, i.e.,*

$$b^{\rm all}(v) < b_{\rm rn}^{\rm all}(v), \quad 0 < v - \underline{v} \ll 1. \tag{6}$$

Proof See Appendix B.

The following result shows that risk-aversion affects high type players and low type players quite differently:

Proposition 2 *In an all-pay auction, the equilibrium bid of a risk-averse player with high type v is higher than the equilibrium bid of a risk-neutral player with the same type, i.e.,*

$$b^{\rm all}(v) > b_{\rm rn}^{\rm all}(v), \quad 0 \leq \bar{v} - v \ll 1. \tag{7}$$

Proof From Eq. (3) it follows that

$$b_{\rm rn}^{\rm all}(\bar{v}) = b_{\rm rn}^{\rm 1st}(\bar{v}). \tag{8}$$

Similarly, from Eq. (5) it follows that

$$b_{\rm rn}^{\rm 1st}(\bar{v}) < b^{\rm 1st}(\bar{v}). \tag{9}$$

By Proposition 4, the equilibrium bid of a risk-averse player with type $\bar{v}$ in an all-pay auction is larger than in a first-price auction, that is,

$$b^{\rm 1st}(\bar{v}) < b^{\rm all}(\bar{v}). \tag{10}$$

The combination of the three inequalities (8)+(9)+(10) completes the proof. □

Since in an all-pay auction a player pays her bid regardless of whether she wins, whereas in a first-price auction she pays only if she wins, it seems natural that players will bid more carefully (i.e., have lower bids) in all-pay auctions than in first-price auctions. Indeed, the bid of a risk-neutral player in an all-pay auction is smaller than her bid in a first-price auction, see Eq. (5), and we can expect this relation to be even stronger for risk-averse players. However, as Propositions 3 and 4 show, the relation of bids in first-price and all-pay auctions with risk-averse players is more complex:

Proposition 3 *The equilibrium bid of a risk-averse player with sufficiently low type v in an all-pay auction is smaller than her bid in a first-price auction.*

Proof From Proposition 1, Eq. (3) and (9) we have that

$$b^{\rm all}(v) < b_{\rm rn}^{\rm all}(v) < b_{\rm rn}^{\rm 1st}(v) < b^{\rm 1st}(v). \tag{11}$$

□

Proposition 4 *The equilibrium bid of a risk-averse player with sufficiently high type v in an all-pay auction is larger than in a first-price auction.*

Proof See Appendix C.

Example 1 *Consider two players where each player's valuation is distributed on* $[0,1]$ *according to the uniform distribution function* $F(v) = v$. *Assume that each player's utility function is* $U(x) = x - \varepsilon x^2$. *In Fig.* 1 *we show the equilibrium bids of risk neutral and risk averse bidders in first-price and all-pay auctions. The results illustrate our finding that* (1) *Players in the all pay auction with low values bid lower and players with high values bid higher than they would bid in the risk neutral case.* (2) *Players in the all pay auction with low values bid lower and players with high values bid higher than they would bid in a first-price auction.*

Propositions 3 and 4 show that there is no dominance relation among the bids in first-price and all-pay auctions. Nevertheless, first-price auctions dominate all-pay auctions from the player's point of view:

Proposition 5 *The expected utility of a risk-averse player with type* $\underline{v} < v \leq \bar{v}$ *in the first-price auction is larger than her expected payoff in the all-pay auction.*

Proof See Appendix A.

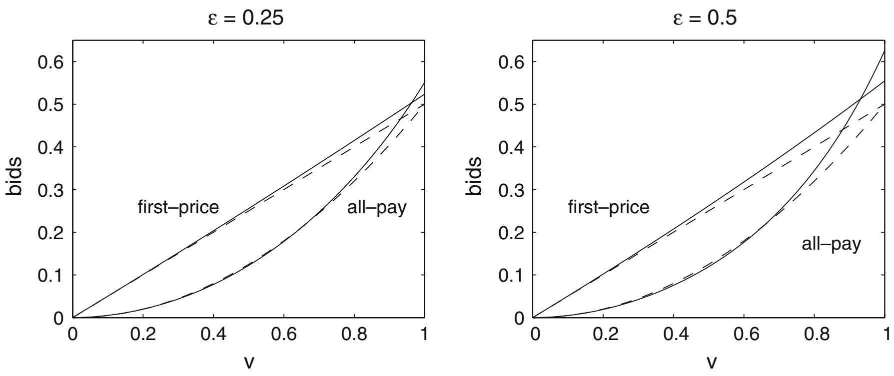

Fig. 1 Bids of risk-averse players (*solid lines*) and of risk-neutral players (*dashed lines*) in all-pay auctions and in first-price auctions

3 All-pay auctions with weakly risk-averse players

The results of the previous section leave many open questions. For example, because of the complex way that risk aversion affects the equilibrium bids, it is not clear whether, overall, risk aversion leads to an increase or a decrease in the seller's expected revenue. In addition, the tools that we used in the previous section, which are standard in auction theory, typically provide qualitative results (e.g., which of two possibilities is larger), but do not give a quantitative estimate (e.g., by how much).

In order to address such questions, we consider the case of weak risk aversion,[6] i.e., $U \approx x$. This is the case, for example, for players with a constant absolute risk aversion (CARA) utility function $U(x) = [1 - \exp(-\epsilon x)]/\epsilon$, or for players with constant relative risk aversion (CRRA) utility function $U(x) = x^{1-\epsilon}$, if $0 < \epsilon \ll 1$. Therefore, in general, the utility function of weakly risk-averse players can be written as

$$U(x) = x + \varepsilon u(x) + O(\varepsilon^2), \quad \varepsilon \ll 1. \tag{12}$$

Thus, ε is the risk aversion parameter and $\varepsilon \ll 1$ implies weak risk aversion. Note that $u(0) = 0$ and $u'' < 0$. On the other hand, u' can be either positive or negative (given that $u'(x) > -\frac{1}{\varepsilon}$) since in either case $U = x + \epsilon u$ is monotonically increasing.

The existence of a small risk aversion parameter enables us to use perturbation methods to calculate explicit approximations to the bidding strategies:

[6] The assumption of weak risk aversion is quite reasonable. Indeed, while most people would prefer to receive \$500 dollar with probability 1 rather than \$1,000 with probability 1/2, much fewer would prefer receiving \$300 dollar with probability 1 rather than \$1,000 with probability 1/2.

Proposition 6 *The symmetric equilibrium bid function in an all-pay auction with weakly risk-averse players is given by*

$$b^{\text{all}}(v) = b^{\text{all}}_{\text{rn}}(v) + \varepsilon b^{\text{all}}_1(v) + O(\varepsilon^2),$$

where $b^{\text{all}}_{\text{rn}}(v)$ *is the equilibrium bid in the risk-neutral case* (2), *and*

$$b^{\text{all}}_1(v) = u\left(-b^{\text{all}}_{\text{rn}}(v)\right) + F^{n-1}(v)\left[u\left(v - b^{\text{all}}_{\text{rn}}(v)\right) - u\left(-b^{\text{all}}_{\text{rn}}(v)\right)\right] - \int_{\underline{v}}^{v} F^{n-1}(s)u'\left(s - b^{\text{all}}_{rn}(s)\right)\,\mathrm{d}s. \tag{13}$$

Proof See Appendix D.

We thus found an explicit expression for $\varepsilon b^{\text{all}}_1(v)$, i.e., the leading-order effect of risk-aversion on the equilibrium strategy. Roughly speaking, for a 10% level risk aversion, we calculated the corresponding 10% change in the equilibrium strategy with 1% accuracy.

Example 2 *The results of our perturbation analysis can be illustrated by the following example. Consider two players where each player's valuation is distributed on* $[0, 1]$ *according to the distribution function* $F(v) = v^{\alpha}$. *Assume that each player's utility function is* $U(x) = x - \varepsilon x^2$. *From Proposition 6 the equilibrium bid function in the all-pay auction is given by*

$$b^{\text{all}}(v) = \frac{\alpha}{1+\alpha}v^{1+\alpha} + \varepsilon\left(-\frac{\alpha}{2+\alpha}v^{2+\alpha} + \frac{\alpha}{1+\alpha}v^{2+2\alpha}\right) + O\left(\varepsilon^2\right). \tag{14}$$

In Fig. 2 *we compare the approximation* (14) *with the exact bid functions* (*i.e., the numerical solutions of equation* (1)), *for the case* $\alpha = 1$. *At* $\epsilon = 0.25$, *the approximations are almost indistinguishable from the exact bids. Although when* $\epsilon = 0.5$ *the risk-aversion parameter is not small,*[7] *the agreement between the explicit approximations and the exact values is quite remarkable.*[8]

In addition for providing quantitative predictions for the equilibrium bids, the explicit approximations obtained in Proposition 6 can be used to approximate the seller's expected revenue under risk aversion:

[7] In fact, $\epsilon = 0.5$ is the *largest* possible value of ϵ for which $U = x - \epsilon x^2$ is monotonically increasing.

[8] Such good agreement was also observed in numerous other comparisons that we made with different distribution functions and utility functions.

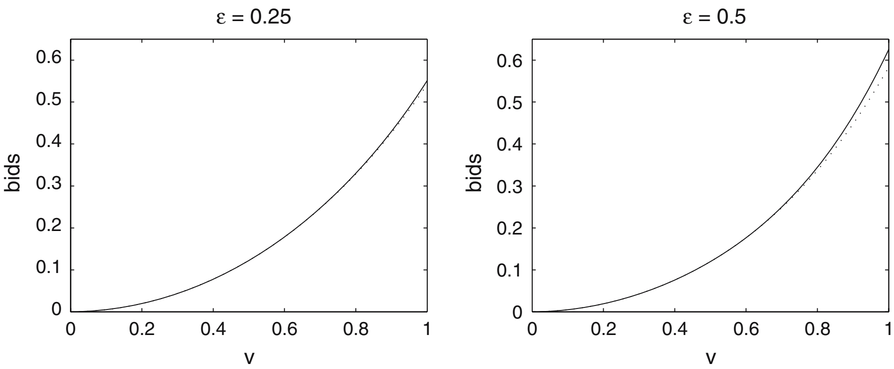

Fig. 2 Bids of risk-averse buyers (*solid lines*) and their explicit approximation [Eq. (14), *dotted lines*] in all-pay auctions

Proposition 7 *In an all-pay auction with weakly risk-averse players, the seller's expected revenue is given by*

$$R^{\text{all}} = R_{\text{rn}} + \varepsilon n \left\{ \int_{\underline{v}}^{\bar{v}} \left[F^{n-1}(v) u(v - b_{\text{rn}}^{\text{all}}(v)) + \left(1 - F^{n-1}(v)\right) u(-b_{\text{rn}}^{\text{all}}(v)) \right] f(v)\, \mathrm{d}v - \int_{\underline{v}}^{\bar{v}} F^{n-1}(v)(1 - F(v)) u'(v - b_{\text{rn}}^{\text{all}}(v))\, \mathrm{d}v \right\} + O(\varepsilon^2), \tag{15}$$

where R_{rn} is the expected revenue in the risk-neutral case.

Proof See Appendix E.

As we have said, unlike first price auctions, the effect of risk aversion on the seller's revenue in all pay auctions is not obvious, since it lowers the bids for low values but increases the bids for large values. Indeed, the result of Proposition 7 shows that risk-aversion can lead to an increase, as well as to a decrease, of the seller's expected revenue in all-pay auctions:

Example 3 *Consider $n = 2$ risk averse players with distribution functions $F(v) = v^\alpha$ in $[0, 1]$, such that $U(x) = x - \epsilon x^2$. Substituting $u(x) = -x^2$ in (15) and integrating gives*

$$R^{\text{all}} = R_{\text{rn}} + \epsilon \Delta R + O(\epsilon^2), \quad \Delta R = \frac{(2 - \alpha)\, \alpha^2}{\left(2 + 5\alpha + 3\alpha^2\right)(\alpha + 2)}.$$

We thus see that depending on the value of α*,* ΔR *can be either positive or negative. Hence, we conclude that risk-aversion can lead to an increase, as well as to a decrease, of the seller's expected revenue in all-pay auctions.*

An immediate, yet important consequence from Proposition 7 is as follows:

Proposition 8 *An* $O(\epsilon)$ *risk aversion leads to an* $O(\epsilon)$ *difference in the seller's revenue among different auction mechanisms.*

Proof Since risk-aversion does not affect the revenue in a second price auction, the result follows from Proposition 7. □

In Fibich et al. (2004) we showed that if ϵ is the level of asymmetry among the distribution functions of the players' valuations, then weak asymmetry only leads to an $O(\epsilon^2)$ difference in the seller's revenue among different auction mechanisms. Hence, Proposition 8 shows that, roughly speaking, weak risk aversion leads to a larger revenue differences among different auction mechanisms than weak asymmetry.

We can also use the explicit expression obtained in Proposition 6 to analyze the effect of weak risk aversion on the players' expected utility.

Proposition 9 *The expected utility of a weakly risk averse player with type v in an all-pay auction is given by*

$$V^{\text{all}}(v) = V_{\text{rn}}(v) + \varepsilon \int_{\underline{v}}^{v} F^{n-1}(s) u' \left(s - b_{\text{rn}}^{\text{all}}(s) \right) \, \mathrm{d}s + O(\varepsilon^2),$$

where $V_{\text{rn}}(v) = \int_{\underline{v}}^{v} F^{n-1}(v) \, \mathrm{dv}$ *is the expected utility in the risk-neutral case.*

Proof See Appendix F.

Note that the difference between the expected payoffs of a weakly risk-averse player and a risk-neutral player does not depend on the value of u, but depends on the value of u'. That is, if the utility function of a risk-averse player $U(x)$ always larger or equal than the utility function of a risk-neutral player $U_{\text{rn}}(x) = x$, it does not necessarily imply that the expected utility of the risk-averse player is larger than the expected utility of the risk-neutral player.

A natural question that arises is whether in the case of weak-risk aversion one cannot simply approximate the bidding functions using the risk-neutral expressions. In other words, when ϵ is small, is there an advantage for the approximation $b^{\text{all}}(v;\epsilon) \approx b_{\text{rn}}^{\text{all}}(v) + \epsilon b_1^{\text{all}}(v)$ over the continuous approximation $b^{\text{all}}(v;\epsilon) \approx b^{\text{all}}(v;\epsilon = 0) = b_{\text{rn}}^{\text{all}}(v)$? The answer is that the accuracy of the first approximation is $O(\epsilon^2)$, whereas that of the second approximation is only $O(\epsilon)$. Therefore, the first approximation is significantly more accurate when ϵ is moderately small (but not negligible). Indeed, comparison of Figs. 1 and 2 shows that the (exact) bids in the risk-averse case are well-approximated with the explicit approximation that we derived, but are not well-approximated with the bids in the risk-neutral case.

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A Proof of proposition 5

The proof here is similar to the one in Milgrom and Weber (1982) and Matthews (1987), who used it to obtain similar results. When all players follow their equilibrium bidding strategies, a player's expected utility given that his type is v and that he plays as if his type is t is

$$\begin{aligned} V^{\text{all}}(t|v) &= F^{n-1}(t)U\left(v - b^{\text{all}}(t)\right) + (1 - F^{n-1}(t))U\left(-b^{\text{all}}(t)\right), \\ V^{\text{1st}}(t|v) &= F^{n-1}(t)U\left(v - b^{\text{1st}}(t)\right), \end{aligned} \tag{16}$$

for all-pay and first-price auctions, respectively. Let $V^{\text{all}}(v) = V^{\text{all}}(v|v)$ and $V^{\text{1st}}(v) = V^{\text{1st}}(v|v)$. By a standard argument, in equilibrium

$$\left.\frac{\partial V^j(t|v)}{\partial t}\right|_{t=v} = 0, \quad j = \text{all}, \text{1st}. \tag{17}$$

Therefore,

$$\left(V^j(v)\right)' = F^{n-1}(v)U'(v - b^j(v)), \quad j = \text{all}, \text{1st}. \tag{18}$$

In addition, $V^{\text{all}}(\underline{v}) = V^{\text{1st}}(\underline{v})$, since in both auctions the lowest type expects a zero utility.

We prove by negation. Assume that for some type v, $\underline{v} < v < \bar{v}$, we have $V^{\text{all}}(v) \geq V^{\text{1st}}(v)$. Then, by (16) it follows that $b^{\text{1st}}(v) > b^{\text{all}}(v)$. From the concavity of U it follows that $U'(v - b^{\text{all}}(v)) < U'(v - b^{\text{1st}}(v))$. Thus $\left(V^{\text{all}}(v)\right)' < \left(V^{\text{1st}}(v)\right)'$.

Let $y = V^{\text{all}} - V^{\text{1st}}$. Then, $y(\underline{v}) = 0$, and for $\underline{v} < v < \bar{v}$, $y(v) \geq 0$ implies that $y'(v) < 0$. Therefore, it follows that $y < 0$ for $\underline{v} < v < \bar{v}$.

To complete the proof, we now prove that $y(\bar{v}) = V^{\text{all}}(\bar{v}) - V^{\text{1st}}(\bar{v}) < 0$. Since $y(v) < 0$ for $\underline{v} < v < \bar{v}$, we only need to prove that it is not possible to have $y(\bar{v}) = 0$. Assume, therefore, by negation that $y(\bar{v}) = 0$. We will show that this implies that $y'(\bar{v}) = 0$ and $y''(\bar{v}) > 0$, which is in contradiction with the fact that $y < 0$ for $\underline{v} < v < \bar{v}$. Indeed,

$$\begin{aligned} y(\bar{v}) = 0 &\Longrightarrow U(\bar{v} - b^{\text{all}}(\bar{v})) = U\left(\bar{v} - b^{\text{1st}}(\bar{v})\right) \Longrightarrow b^{\text{1st}}(\bar{v}) = b^{\text{all}}(\bar{v}) \\ &\Longrightarrow \left(V^{\text{all}}\right)'(\bar{v}) = \left(V^{\text{1st}}\right)'(\bar{v}) \Longrightarrow y'(\bar{v}) = 0. \end{aligned}$$

In addition, substituting $t = v = \bar{v}$ in (17) gives that

$$\left(b^{\text{all}}\right)'(\bar{v}) - \left(b^{\text{1st}}\right)'(\bar{v}) = -\frac{(n-1)f(\bar{v})}{U'(\bar{v} - \bar{b})} U\left(-b^{\text{all}}(\bar{v})\right) > 0.$$

Therefore, by (18),

$$y''(\bar{v}) = \left(V^{\text{all}}\right)''(\bar{v}) - \left(V^{1\text{st}}\right)''$$
$$(\bar{v}) = -U''\left(\bar{v} - b^{\text{all}}(\bar{v})\right)\left[\left(b^{\text{all}}\right)'(\bar{v}) - \left(b^{1\text{st}}\right)'(\bar{v})\right] > 0.$$

B Proof of Proposition 1

Since $b^{\text{all}}(\underline{v}) = b^{\text{all}}_{\text{rn}}(\underline{v}) = 0$, we can prove the result by showing that

$$(b^{\text{all}})'(v) < (b^{\text{all}}_{\text{rn}})'(v), \quad 0 < v - \underline{v} \ll 1. \tag{19}$$

Let us first note that (17) implies that

$$\left(b^{\text{all}}(v)\right)'$$
$$= (n-1)F^{n-2}(v)f(v)\frac{U\left(v - b^{\text{all}}(v)\right) - U\left(-b^{\text{all}}(v)\right)}{F^{n-1}(v)U'\left(v - b^{\text{all}}(v)\right) + (1 - F^{n-1}(v))U'\left(-b^{\text{all}}(v)\right)}.$$

In particular, in the case of risk neutrality

$$\left(b^{\text{all}}_{\text{rn}}\right)'(v) = (n-1)F^{n-1}(v)f(v)v.$$

Therefore,

$$\frac{\left(b^{\text{all}}\right)'(v) - \left(b^{\text{all}}_{\text{rn}}\right)'(v)}{(n-1)F^{n-1}(v)f(v)}$$
$$= \frac{U\left(v - b^{\text{all}}(v)\right) - U\left(-b^{\text{all}}(v)\right)}{F^{n-1}(v)U'\left(v - b^{\text{all}}(v)\right) + (1 - F^{n-1}(v))U'\left(-b^{\text{all}}(v)\right)} - v. \tag{20}$$

Let us begin with the case when $\underline{v} > 0$. Since $b^{\text{all}}(\underline{v}) = 0$, then

$$\left.\frac{\left(b^{\text{all}}\right)'(v) - \left(b^{\text{all}}_{\text{rn}}\right)'(v)}{(n-1)F^{n-1}(v)f(v)}\right|_{v=\underline{v}} = \frac{U(\underline{v}) - U(0)}{U'(0)} - \underline{v} = \left[\frac{U'(x)}{U'(0)} - 1\right]\underline{v},$$

where $0 < x < \underline{v}$. By the concavity of U, $\frac{U'(x)}{U'(0)} < 1$. Therefore, we proved (19) for $\underline{v} > 0$.

To prove (19) when $\underline{v} = 0$, we first expand,

$$U\left(v - b^{\text{all}}(v)\right) = U\left(-b^{\text{all}}(v)\right) + vU'\left(-b^{\text{all}}(v)\right) + \frac{v^2}{2}U''\left(-b^{\text{all}}(v)\right) + O(v^3),$$
$$U'\left(v - b^{\text{all}}(v)\right) = U'(-b^{\text{all}}(v)) + vU''\left(-b^{\text{all}}(v)\right) + O(v^2).$$

Therefore,

$$
\begin{aligned}
&\frac{U(v-b^{\text{all}}(v))-U(-b^{\text{all}}(v))}{F^{n-1}(v)U'(v-b^{\text{all}}(v))+(1-F^{n-1}(v))U'(-b^{\text{all}}(v))}\\
&\quad=\frac{vU'(-b^{\text{all}}(v))+\frac{v^2}{2}U''(-b^{\text{all}}(v))+O(v^3)}{U'(-b^{\text{all}}(v))+F^{n-1}(v)\left[vU''(-b^{\text{all}}(v))+O(v^2)\right]}\\
&\quad=\frac{v+\frac{v^2}{2}\frac{U''(-b^{\text{all}}(v))}{U'(-b^{\text{all}}(v))}+O(v^3)}{1+F^{n-1}(v)v\frac{U''(-b^{\text{all}}(v))}{U'(-b^{\text{all}}(v))}+O(v^2)}\\
&\quad=v\left[1+\frac{v}{2}\frac{U''(-b^{\text{all}}(v))}{U'(-b^{\text{all}}(v))}+O(v^2)\right]\left[1-F^{n-1}(v)v\frac{U''(-b^{\text{all}}(v))}{U'(-b^{\text{all}}(v))}+O(v^2)\right]\\
&\quad=v+v^2\frac{U''(-b^{\text{all}}(v))}{U'(-b^{\text{all}}(v))}\left[\frac{1}{2}-F^{n-1}(v)\right]+O(v^3).
\end{aligned}
$$

Therefore, by (20),

$$
\frac{(b^{\text{all}})'(v)-(b^{\text{all}}_{\text{rn}})'(v)}{(n-1)F^{n-1}(v)f(v)}=v^2\frac{U''(-b^{\text{all}}(v))}{U'(-b^{\text{all}}(v))}\left(\frac{1}{2}-F^{n-1}(v)\right)+O(v^3)<0.
$$

C Proof of Proposition 4

By Proposition 5, the expected utility of a risk-averse player with type $\bar{v}$ in the first-price auction is larger than her expected payoff in the all-pay auction ($V^{\text{all}}(\bar{v})<V^{1\text{st}}(\bar{v})$). Since $V^j(\bar{v})=U(v-b^j(\bar{v}))$ for $j=\text{all},1st$, see Eq. (16), and since U is monotonically increasing, the result follows.

D Proof of Proposition 6

We can write the equilibrium bid as $v(b)=v_{\text{rn}}(b)+\varepsilon v_1(b)+O(\varepsilon^2)$, where $v_{\text{rn}}(b)$ is the inverse function of the risk-neural equilibrium strategy in all-pay auctions (2). For clarity, we drop the superscript *all*. We first note that when $\varepsilon\ll 1$,

$$
\begin{aligned}
F(v(b))&=F(v_{\text{rn}})+\varepsilon v_1F'(v_{\text{rn}})+O(\varepsilon^2),\\
f(v(b))&=f(v_{\text{rn}})+\varepsilon v_1f'(v_{\text{rn}})+O(\varepsilon^2),\\
U(v(b)-b)-U(-b)&=v(b)+\varepsilon[u(v(b)-b)-u(-b)]\\
&=v_{\text{rn}}(b)+\varepsilon[v_1(b)+u(v_{\text{rn}}(b)-b)-u(-b)]+O(\varepsilon^2),\\
U'(v(b)-b)-U'(-b)&=\varepsilon[u'(v(b)-b)-u'(-b)]\\
&=\varepsilon[u'(v_{\text{rn}}(b)-b)-u'(-b)]+O(\varepsilon^2).
\end{aligned}
$$

Substitution in (1) and expanding in a power series in ε, the equation for the $O(1)$ term is identical to the one in the risk-neutral case and therefore is automatically satisfied. The equation for the $O(\varepsilon)$ terms is

$$\begin{aligned} v_1'(b) = & \frac{F(v_{\rm rn}(b))[u'(v_{\rm rn}(b)-b)-u'(-b)]}{(n-1)f(v_{\rm rn}(b))v_{\rm rn}(b)} \\ & + \frac{u'(-b)}{(n-1)F^{n-2}(v_{\rm rn}(b))f(v_{\rm rn}(b))v_{\rm rn}(b)} \\ & - \frac{(n-2)v_1(b)}{(n-1)F^{n-1}(v_{\rm rn}(b))v_{\rm rn}(b)} - \frac{v_1 f'(v_{\rm rn}(b))}{(n-1)F^{n-2}(v_{\rm rn}(b))f^2(v_{\rm rn}(b))v_{\rm rn}(b)} \\ & - \frac{[v_1(b)+u(v_{\rm rn}(b)-b)-u(-b)]}{(n-1)F^{n-2}(v_{\rm rn}(b))f(v_{\rm rn}(b))v_{\rm rn}^2(b)}, \end{aligned}$$

subject to $v_1(0)=0$. Since, by (1),

$$v_{\rm rn}'(b) = \frac{1}{(n-1)F^{n-2}(v_{\rm rn}(b))f(v_{\rm rn}(b))v_{\rm rn}(b)}, \tag{21}$$

the equation for $v_1'(b)$ can be rewritten as

$$v_1'(b) + v_1(b)B(b) = G(b) \tag{22}$$

where

$$B(b) = \left[\frac{v_{\rm rn}'(b)}{v_{\rm rn}(b)} + \frac{f'(v_{\rm rn}(b))}{f(v_{\rm rn}(b))}v_{\rm rn}'(b) + (n-2)\frac{f(v_{\rm rn}(b))}{F(v_{\rm rn}(b))}v_{\rm rn}'(b)\right],$$

and

$$\begin{aligned} G(b) = v_{\rm rn}'(b)\Big\{ & -\Big[u(v_{\rm rn}(b)-b)-u(-b)\Big](n-1)F^{n-2}(v_{\rm rn}(b))f(v_{\rm rn}(b))v_{\rm rn}'(b) \\ & + F^{n-1}(v_{\rm rn}(b))\Big(u'(v_{\rm rn}(b)-b)-u'(-b)\Big) + u'(-b)\Big\}. \end{aligned} \tag{23}$$

The solution of (22) is given by

$$v_1(b) = \mathrm{e}^{\int_b^{\bar b_{\rm rn}} B}\left(C_1 - \int_b^{\bar b_{\rm rn}} G(x)\mathrm{e}^{-\int_x^{\bar b_{\rm rn}} B}\,\mathrm{d}x\right),$$

where $\bar{b}_{\rm rn} = b_{\rm rn}(\bar{v})$. It is easy to verify that (see (21))

$$\mathrm{e}^{\int\limits_{b}^{\bar{b}_{\rm rn}} B} = \frac{v'_{\rm rn}(b)}{v'_{\rm rn}(\bar{b}_{\rm rn})}.$$

Thus, as $b \to 0$, $v_{\rm rn}(b) \to \underline{v}$ and $\mathrm{e}^{\int_b^{\bar{b}_{\rm rn}} B} \to \infty$. Therefore it follows that $C_1 = \int_0^{\bar{b}_{\rm rn}} G(x)\mathrm{e}^{-\int_x^{\bar{b}_{\rm rn}} B}\,\mathrm{d}x$ and that

$$v_1(b) = v'_{\rm rn}(b) \int\limits_0^b G(x)/v'_{\rm rn}(x)\,\mathrm{d}x.$$

In addition, we note that if we differentiate the identity $v = v(b(v;\varepsilon);\varepsilon)$ with respect to ε and set $\varepsilon = 0$, we get that $v_1(b_{\rm rn}(v)) + v'_{\rm rn}(b_{\rm rn}(v))b_1(v) = 0$ or $b_1(v) = -v_1/v'_{\rm rn}(b)$. Thus, we get that

$$b_1(v) = -\int\limits_0^{b_{\rm rn}(v)} G(x)/v'_{\rm rn}(x)\,\mathrm{d}x.$$

Substitution of G from (23) gives

$$b_1(v) = \int\limits_0^{b_{\rm rn}(v)} \left\{\Big[u(v_{\rm rn}(b) - b) - u(-b)\Big](F^{n-1}(v_{\rm rn}(b)))' - F^{n-1}(v_{\rm rn}(b))\Big(u'(v_{\rm rn}(b) - b) - u'(-b)\Big) - u'(-b)\right\}\mathrm{d}b.$$

A few more technical calculations complete the proof.

E Proof of Proposition 7

The seller's revenue is given by $R^{\rm all} = n\int_{\underline{v}}^{\bar{v}} b(s)f(s)\,\mathrm{d}s$. Substituting $b = b_{\rm rn} + \varepsilon b_1 + O(\varepsilon^2)$, we have

$$R^{\text{all}} = n\int_{\underline{v}}^{\bar{v}} (b_{\text{rn}} + \varepsilon b_1) f(s) \mathrm{d}s + O(\varepsilon^2) = n\int_{\underline{v}}^{\bar{v}} b_{\text{rn}} f(s)\mathrm{d}s + \varepsilon n \int_{\underline{v}}^{\bar{v}} b_1 f(s)\mathrm{d}s + O(\varepsilon^2)$$

$$= R_{\text{rn}} + \varepsilon n \int_{\underline{v}}^{\bar{v}} b_1 f(s)\, \mathrm{d}s + O(\varepsilon^2).$$

Substituting b_1 from (13) yields

$$\int_{\underline{v}}^{\bar{v}} b_1 f(s)\, \mathrm{d}s = \int_{\underline{v}}^{\bar{v}} (1 - F^{n-1}(v)) u(-b_{\text{rn}}(v)) f(v)\, \mathrm{d}v$$
$$+ \int_{\underline{v}}^{\bar{v}} F^{n-1}(v) u(v - b_{\text{rn}}(v)) f(v)\, \mathrm{d}v$$
$$- \int_{\underline{v}}^{\bar{v}} \left[\int_{\underline{v}}^{v} F^{n-1}(s) u'(s - b_{\text{rn}}(s))\, \mathrm{d}s \right] f(v)\, \mathrm{d}v.$$

Integrating by parts the double integral gives

$$\int_{\underline{v}}^{\bar{v}} \left[\int_{\underline{v}}^{v} F^{n-1}(s) u'(s - b_{\text{rn}}(s))\, \mathrm{d}s \right] f(v)\, \mathrm{d}v$$
$$= \int_{\underline{v}}^{\bar{v}} F^{n-1}(v)(1 - F(v)) u'(v - b_{\text{rn}}(v))\, \mathrm{d}v.$$

Therefore, the result follows.

F Proof of Proposition 9

The expected utility for a player with type v in all-pay auctions in equilibrium is given by

$$V^{\text{all}}(v) = F^{n-1}(v) U(v - b(v)) + [1 - F^{n-1}(v)] U(-b(v)).$$

In the case of weak risk aversion (12),

$$V^{\text{all}}(v) = F^{n-1}(v) v - b(v)$$
$$+ \varepsilon \left[F^{n-1}(v) \Big(u(v - b(v)) - u(-b(v)) \Big) + u(-b(v)) \right] + O(\varepsilon^2).$$

Using the relation $b(v) = b_{\rm rn}(v) + \varepsilon b_1(v) + O(\varepsilon^2)$, we have

$$V^{\rm all}(v) = V^{\rm all}_{\rm rn}(v) - \varepsilon \left\{ b_1(v) - \left[F^{n-1}(v) \Big(u(v - b_{\rm rn}(v)) - u(-b_{\rm rn}(v)) \Big) + u(-b_{\rm rn}(v)) \right] \right\} + O(\varepsilon^2).$$

By the revenue equivalence theorem, $V^{\rm all}_{\rm rn}(v) = V_{\rm rn}(v) = \int_{\underline{v}}^{v} F^{n-1}(s)\,{\rm d}s$ is independent of the auction mechanism. Substituting (13) in the last equation yields the result.

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