Critical power of collapsing vortices

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We calculate the critical power for collapse of linearly polarized phase vortices, and show that this expression is more accurate than previous results. Unlike the nonvortex case, deviations from radial symmetry do not increase the critical power for collapse, but rather lead to disintegration into collapsing non vortex filaments. The cases of circular, radial, and azimuthal polarizations are also considered.

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The nonlinear optical process of self-focusing sets an upper limit on the amount of laser power that can be propagated through a medium with an intensity dependent refractive index (i.e., $n=n_0+n_2I$, where n_0 is the linear refractive index, n_2 is the nonlinear refractive index, and I is the intensity). For powers above this threshold the beam will undergo collapse, with the peak intensity becoming sufficiently high that damage to the material can occur. Ultimately, collapse will be arrested by some physical mechanism, such as plasma formation, normal dispersion, or damping.

Let us briefly review the situation in the nonvortex case. The value of the critical power is given by [1]

$$P_{cr}=\frac{\lambda^2}{4\pi n_0 n_2}p_{cr},$$

where p_{cr} is the nondimensional critical power for collapse in the dimensionless nonlinear Schrödinger (NLS)

$$i\psi_z(z,x,y)+\Delta\psi+|\psi|^2\psi=0, \quad \psi(0,x,y)=\psi_0(x,y). \tag{1}$$

In the NLS model, there is no mechanism for arrest of collapse, hence collapse is defined as the maximal amplitude becoming infinite. Weinstein [2] proved that the lower bound for the critical power is equal to $p_{\mathrm{cr}}=\int|R|^2rdr\approx 1.86$, i.e., the power of the *Townes profile*, which is the ground state solution of

$$R''+\frac{1}{r}R'-R+R^3=0, \quad R'(0)=0, \quad R(\infty)=0.$$

While the Townesian input beams $\psi_0=\lambda R(\lambda r)$, where $\lambda>0$, can collapse with exactly the input power p_{cr}, all other input profiles require power strictly above p_{cr} for collapse [3,4]. In practice, however, the critical power of peak-type (i.e., nonring-type) radially symmetric input beams is only a few percent above p_{cr} [1,5]. For example, the critical power of Gaussian and super-Gaussian ($\psi_0=ce^{-r^4}$) input beams is $\approx 2\%$ and $\approx 8\%$ above p_{cr}, respectively.

We now consider the critical power of vortex input beams. In [6], Kruglov *et al.* derived an expression for the critical power of vortex beams, and showed that it increases with the winding number (or topological charge) m. In this study, we show that this expression is inaccurate, and derive the correct expression for the critical power. Unlike the vortex-free case, deviations from radial symmetry do not increase the critical power, but rather lead to disintegration into collapsing nonvortex filaments.

We first consider radially symmetric vortex input beams of the form $\psi_0=A_0(r)e^{im\theta}$. In this case, the solution remains a vortex with winding number m, i.e., it is of the form $\psi(z,r,\theta)=A(z,r)e^{im\theta}$ [7]. Following a similar derivation to [2], it can be rigorously shown that the lower bound for the critical power of radially symmetric vortex input beams $\psi_0=A_0(r)e^{im\theta}$ is

$$p_{\mathrm{cr}}(m)=\int|R_m|^2rdr,$$

where R_m is the ground state solution of

$$R_m''(r)+\frac{1}{r}R_m'-\left(1+\frac{m^2}{r^2}\right)R_m+R_m^3=0, \quad R_m'(0)=0,$$
$$R_m(\infty)=0.$$

The values of $p_{\mathrm{cr}}(m)$ for $m=1,\ldots,6$ are listed in Table I. Using the approximation [8]

$$R_m(r)\approx\sqrt{3}\,\mathrm{sech}\left(\frac{r-\sqrt{2}m}{\sqrt{2/3}}\right), \tag{2}$$

we can derive the analytic approximation $p_{\mathrm{cr}}(m)\approx 4\sqrt{3}m$. Figure 1 shows that $p_{\mathrm{cr}}(m)$ is well approximated by $4\sqrt{3}m$, and that the approximation improves as m increases.

We now consider the critical power of various vortex input profiles, and ask under what condition the critical power is close to the lower bound $p_{\mathrm{cr}}(m)$. As in the vortex-free case, the only input profiles that can collapse with input power exactly equal to $p_{\mathrm{cr}}(m)$ are $\psi_0=\lambda R_m(\lambda r)e^{im\theta}$. We first calculate the critical power of the Laguerre-Gaussians profiles

TABLE I. The values of $p_{\mathrm{cr}}(m)$ for $m=1,\ldots,6$.

m	1	2	3	4	5	6
$p_{\mathrm{cr}}(m)/p_{\mathrm{cr}}$	4.12	7.65	11.3	15.0	18.7	22.4

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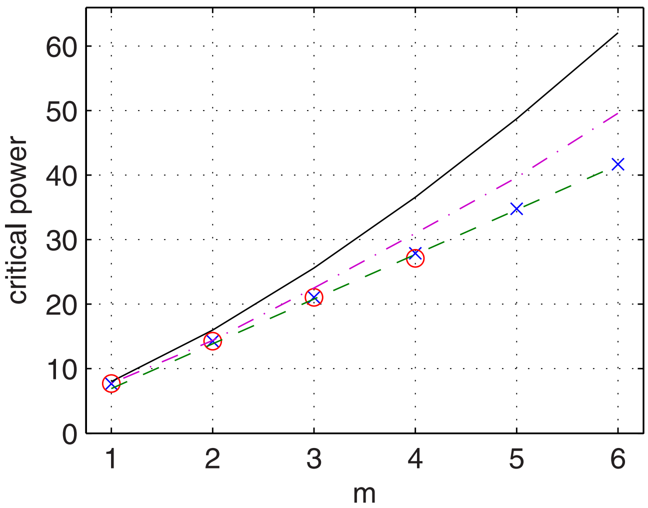

FIG. 1. (Color online) Critical power $p_{\rm cr}(m)=\int_0^\infty R_m^2 rdr$ ($\times$), the approximation $4\sqrt{3}m$ (dashed line), numerical estimate of the critical power as a function of m ($\bigcirc$, data taken from [6]), the critical power for collapse of Laguerre-Gaussians (dash-dotted line), and the analytic estimate $I_{\rm cr}^{(m)}$ ([6], solid line).

$$\psi_0^{LG} = {\rm c}r^m e^{-r^2} e^{im\theta},$$

which are the vortex modes of the linear Schrödinger equation. To do that, we solve the NLS with the initial condition ψ_0^{LG} and gradually increase c until, at c_{th}, the solution collapses. In this case, the critical power is close to $p_{\rm cr}(m)$ for $m=1$ but as m increases, the excess power above $p_{\rm cr}(m)$ needed for collapse increases (see Table II). Similarly, for the sech input profile

$$\psi_0^{\rm sech} = {\rm c}r^2\ {\rm sech}(r-5)e^{im\theta},$$

the critical power is close to $p_{\rm cr}(m)$ only for $m=2,3,4$ (see Table II).

To better understand these results, let us consider the vortex profile of the form $\psi_0 = cf(r)e^{im\theta}$, where

$$f(r) = Q(\rho), \quad \rho = \frac{r - r_{\rm max}}{L},$$

and $Q(\rho)$ attains its maximum at $\rho=0$. This ring profile is characterized by the ring width L and radius $r_{\rm max}$. As in the vortex-free case, the closer f is to a member of the one-parameter family $\lambda R_m(\lambda r)$, the smaller the excess power above $p_{\rm cr}(m)$ needed for collapse. By Eq. (2), the family $\lambda R_m(\lambda r)$ is characterized by

$$\text{radius/width} = \sqrt{3}m. \tag{3}$$

Therefore, $f(r)$ has to satisfy Eq. (3) to "leading order" to be close to $\lambda R_m(\lambda r)$.

The Laguerre-Gaussian modes ψ_0^{LG} are characterized by radius/width$=\sqrt{m/2}$. This ratio is close to Eq. (3) only for $m\approx 1$, explaining why the critical power of Laguerre-Gaussian modes is close to $p_{\rm cr}(m)$ only for $m=1$. Similarly, the sech profile $\psi_0^{\rm sech}$ is characterized by radius/width$=5$. Since the radius/width of $\lambda R_m(\lambda r)$ is equal to $\sqrt{3}m$, this ratio is close to 5 for $m=\frac{5}{\sqrt{3}}\approx 2.88$ [see Eq. (3)]. This explains why the threshold power of the sech profile $\psi_0^{\rm sech}$ is closest to $p_{\rm cr}(m)$ for $m=3$. As a final confirmation of this observation, we "fix" the sech profile $\psi_0^{\rm sech}$ so that "it behaves like a $\lambda R_m(\lambda r)$ profile," i.e., that it satisfies Eq. (3) to leading order, as follows:

$$\psi_0^{m-{\rm sech}} = \sqrt{2}\left(\frac{r}{\sqrt{3}m}\right)^2 {\rm sech}(r-\sqrt{3}m)e^{im\theta}. \tag{4}$$

Indeed, the threshold power of the "modified" sech profile (4) is less than 1% above the critical power for $m=1,\ldots,6$ (see Table II).

In [6], Kruglov *et al.* estimated the critical power for vortex collapse to be equal to

$$I_c^{(m)} = \frac{2^{2m+1} m!\,(m+1)!}{(2m)!}. \tag{5}$$

In [6], they also estimated the critical power numerically for $m=1$, 2, 3, and 4. These numerical results agree with our analytic calculation of $p_{\rm cr}(m)$, but not with their own estimate $I_c^{(m)}$ (see Fig. 1). To understand why this is the case, we note that the derivation of $I_c^{(m)}$ was based on the assumption that the collapsing vortex has a self-similar Laguerre-Gaussian profile. As noted before, the Laguerre-Gaussian modes are not a good approximation of the one-parameter family $\lambda R_m(\lambda r)$, and as m increases this approximation becomes less and less accurate. In addition, the assumption that the solution undergoes an aberrationless (adiabatic) self-similar collapse is known to lead to overestimates of the critical power [5]. Indeed, even for Laguerre-Gaussian input beams, the critical power is closer to $p_{\rm cr}(m)$ than to $I_{cm}(m)$ (see Fig. 1).

Most studies on optical vortices considered stationary vortices. Recently, there has been a growing interest in the dynamics of collapsing vortices. Berge *et al.* showed that for vortices with input power $P\approx I_c^{(m)}$, symmetry breaking noise causes the vortex ring to break into $2m+1$ filaments [9]. Vuong *et al.* generalized this result for vortices with power larger above $I_c^{(m)}$ [10]. We now show that these azimuthal instabilities can occur even for vortices with dimensionless power less than $I_c^{(m)}$ and even less than the lower bound $p_{\rm cr}(m)$. To do that, we solve the NLS with the slightly elliptic Laguerre-Gaussian input profile

TABLE II. Excess power above $p_{\rm cr}(m)$ needed for collapse.

m	Input beams		
	ψ_0^{LG}	$\psi_0^{\rm sech}$	$\psi_0^{m-{\rm sech}}$
1	0.65%	20%	0.13%
2	0.80%	4.5%	0.91%
3	7%	1.9%	0.71%
4	11%	1.9%	0.32%
5	14%	9%	0.17%
6	19%	14%	0.34%

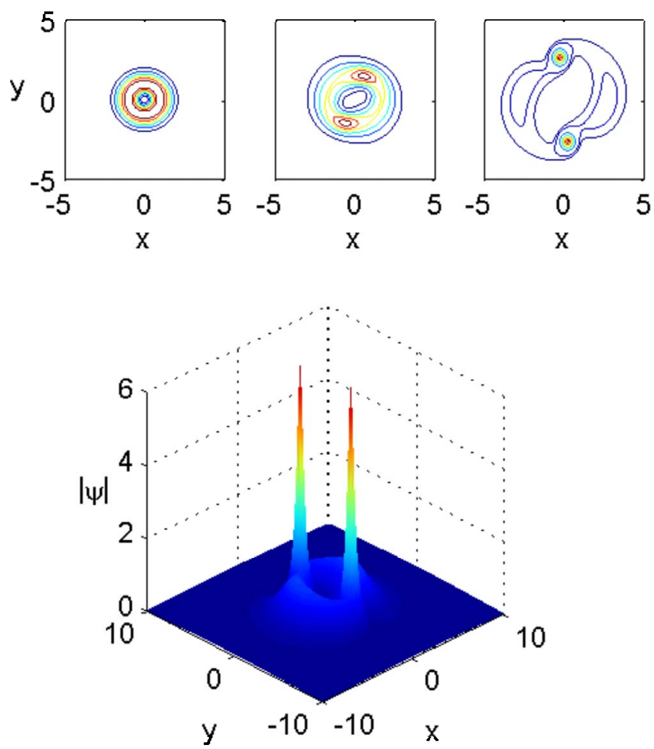

FIG. 2. (Color online) Solution of the NLS with input beam (6). Top: Levels set at $z=0$, 0.5, and $z=1$ (from left to right). Bottom: Surface plot at $z=1$.

$$\psi_0 = \psi_0^{LG}[\sqrt{x^2 + (1.05y)^2}], \tag{6}$$

with $m=2$ and with input power equal to $\frac{3}{4}p_{\rm cr}(m=2)$. Although the power of this vortex beam is below $p_{\rm cr}(m)$, it breaks into two filaments which subsequently undergo collapse (see Fig. 2). This effect of symmetry breaking is very different from the case of peak-type nonvortex solutions, where deviations from radial symmetry increase the critical power for collapse [5]. This is because peak-type solutions collapse with the modulated Townes profile, (i.e., $\psi \sim \frac{1}{L(z)}R(\frac{r}{L(z)})$, where $L \to 0$ at the singularity) which is stable under azimuthal perturbations, as was demonstrated experimentally and numerically in [11], and analytically in [12]. In contrast, vortices collapse with a ring profile, which breaks into a ring of filaments under azimuthal perturbations [10]. Since these filaments do not collapse at the phase singularity point $r=0$, each filament can collapse with the Townes profile, hence with the critical power $p_{\rm cr}=p_{\rm cr}(m=0)<p_{\rm cr}(m)$ [13]. Note, that these filaments continue to rotate around $r=0$, so that total helicity is preserved.

Our results are also relevant for beams which are not linearly polarized. Let $\psi_\pm$ be the amplitudes of the circular components $\hat{e}_\pm = (\hat{x} \pm i\hat{y})/\sqrt{2}$. The equation for each circular component is

$$i\frac{\partial \psi_\pm}{\partial z} + \Delta\psi_\pm + \frac{2}{3}[|\psi_\pm|^2 + 2|\psi_\mp|^2]\psi_\pm = 0.$$

In the case of a pure circular polarization (CP) state ($\psi_- \equiv 0$), this equation reduces to

$$i\frac{\partial \psi_+}{\partial z} + \Delta\psi_+ + \frac{2}{3}|\psi_+|^2\psi_+ = 0. \tag{7}$$

Since the Kerr effect is smaller by a factor 2/3 compared to the NLS (1) for a linear polarization state, the critical power for collapse is larger by a factor of 3/2 [15]. In particular, the lower bound for the critical power of a CP vortex beam $\psi_+ = e^{im\theta}A_0(r)$ is given by

$$p_{\rm cr}^{CP}(m) = \frac{3}{2}p_{\rm cr}(m) \approx 6\sqrt{3}m.$$

Similarly, consider the cases of radial polarization (RP)

$$\psi^{RP} = A(r,t)[e^{i\theta}\hat{e}_- + e^{-i\theta}\hat{e}_+],$$

and azimuthal polarization (AP)

$$\psi^{AP} = iA(r,t)[e^{i\theta}\hat{e}_- - e^{-i\theta}\hat{e}_+].$$

Since $|\psi_+|=|\psi_-|=|A|$, the equation for each component is

$$i\frac{\partial \psi_\pm}{\partial z} + \Delta\psi_\pm + 2|\psi_\pm|^2\psi_\pm = 0.$$

The Kerr effect is larger by a factor of 2, hence the critical power for collapse for each component is smaller by a factor of $\frac{1}{2}$, i.e., $p_{\rm cr}(\psi_+)=p_{\rm cr}(\psi_-)=\frac{1}{2}p_{\rm cr}(m=1)$. In addition, the power of ψ^{AP} and ψ^{RP} is the sum of the power of ψ_+ and of ψ_-. Hence,

$$p_{\rm cr}^{RP} = p_{\rm cr}^{AP} = p_{\rm cr}(\psi_+) + p_{\rm cr}(\psi_-) = p_{\rm cr}(m=1) \approx 4.12p_{\rm cr},$$

in agreement with recent numerical simulations [16].

In summary, we showed that the critical power for collapse of radially symmetric vortex beams is typically a few percent above $P_{\rm cr}(m) = \frac{\lambda^2}{4\pi n_0 n_2}p_{\rm cr}(m)$, where $p_{\rm cr}(m) = \int_0^\infty R_m^2 r dr \approx 4\sqrt{3}m$. Deviations from radial symmetry do not increase the critical power, but rather lead to disintegration into collapsing nonvortex filaments.

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