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ASYMMETRIC FIRST-PRICE AUCTIONS—A PERTURBATION APPROACH

GADI FIBICH AND ARIEH GAVIOUS

We use perturbation analysis to obtain explicit approximations of the equilibrium bids in asymmetric first-price auctions with n bidders, in which bidders' valuations are independently drawn from different distribution functions. Several applications are presented: explicit approximations of the seller's expected revenue, the maximal bid, the optimal reserve price, inefficiency, and a consequence of stochastic dominance. We also suggest an improved numerical method for calculating the seller's expected revenue.

1. Introduction. The theory of independent private-value auction has mostly dealt with *symmetric auctions*, in which the valuations of all bidders are drawn according to the same distribution function. In this case the mathematical model is given by a single ordinary-differential question, which can be solved explicitly for the equilibrium strategies for all standard private-value auction mechanisms. Perhaps the most important result on (symmetric) auctions is the revenue equivalence theorem, which states that the seller's expected revenue is independent of the auction mechanism (Riley and Samuelson 1981, Myerson 1981). For a comprehensive review of (the mostly symmetric) auction theory, see Klemperer (1999) and Krishna (2002).

In practice, however, it often happens that bidders' valuations are drawn from different distribution functions (i.e., *asymmetric auctions*). In such cases the mathematical model is given by a system of coupled nonlinear ordinary-differential equations that cannot be solved explicitly for the equilibrium strategies, except for very simple models. As a result, analysis of asymmetric auctions is considerably more complex than for symmetric ones, and relatively little is known at present on asymmetric auctions. In situations like this, where it is difficult or even impossible to obtain exact solutions, much insight can be gained by employing *perturbation analysis*, whereby one calculates an *explicit approximation* to the solution. In this paper we adopt this approach and use perturbation analysis to calculate the equilibrium bid strategies in first-price auctions. As we shall see, these explicit approximations are quite insightful, making the sacrifice of "exactness" worthwhile.

The paper is organized as follows: In §2 we formulate the model of asymmetric first-price auctions. In §3 we review the symmetric model and analyze the effect of a symmetric perturbation. In §4 we use perturbation analysis to calculate the equilibrium bids in asymmetric first-price auctions with n bidders. In §5 we illustrate how these explicit expressions can be used to analyze asymmetric first-price auctions: We generalize the result of Lebrun (1998) and Maskin and Riley (2000a) on stochastic dominance to the case of n weakly asymmetric bidders; we derive explicit approximations of the seller's expected revenue, the optimal reserve price, and inefficiency. In §6 we compare the results to those for asymmetric second-price auctions. In §7 we compare our explicit approximations with results of numerical simulations. Although the perturbation analysis is formally valid only for weak asymmetry, this comparison shows that its results are quite accurate even when asymmetry

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is not small. Finally, in §8 we introduce several improvements to the numerical method of Marshall et al. (1994) for calculating the equilibrium bids and the seller's expected revenue in asymmetric first-price auctions.

2. Asymmetric first-price auctions. Throughout this paper we consider n risk-neutral bidders bidding for an indivisible object in a first-price auction. Let us denote by v_i $(i = 1, \ldots, n)$ the valuation of the ith bidder for the object, which is private information to bidder i. We assume that v_i is drawn independently from a twice continuously differentiable distribution function $F_i(v_i)$ whose support is common to all bidders, i.e., $v_i \in [\underline{v}, \bar{v}]$, $F_i(\underline{v}) = 0$, and $F_i(\bar{v}) = 1$. We also denote by $f_i = F_i'$ the corresponding density functions.

Let $b_i = b_i(v_i)$ be the bid function of bidder i in equilibrium. (For existence of an equilibrium, see Maskin and Riley 2000b and Lebrun 1996, 1999.) Because the equilibrium bids are strictly monotonic (Maskin and Riley 2000b), we can define the inverse bid functions $v_i = v_i(b_i)$. The maximization problem for bidder i reads

$$\max_b U_i(b; v_i) = (v_i - b) \prod_{\substack{j=1 \\ j\neq i}}^{n} F_j(v_j(b)), \qquad i = 1, \ldots, n, \tag{1}$$

where v_i is given and fixed. Therefore, the (inverse) bid functions are solutions of

$$\frac{\partial U_i(b; v_i)}{\partial b} = (v_i - b) \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(\prod_{\substack{k=1 \\ k\neq i,j}}^{n} F_k(v_k(b)) \right) f_j(v_j(b)) v_j'(b) - \prod_{\substack{j=1 \\ j\neq i}}^{n} F_j(v_j(b)) = 0,$$

or

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{f_j(v_j(b)) v_j'(b)}{F_j(v_j(b))} = \frac{1}{v_i(b) - b}, \qquad i = 1, \ldots, n. \tag{2}$$

To bring these equations to a more standard form, we first sum (2) over all bidders to get

$$\sum_{j=1}^{n} \frac{f_j(v_j(b)) v_j'(b)}{F_j(v_j(b))} = \frac{1}{n-1} \sum_{j=1}^{n} \frac{1}{(v_j(b) - b)}. \tag{3}$$

Taking the difference of Equations (3) and (2) gives

$$v_i'(b) = \frac{F_i(v_i(b))}{f_i(v_i(b))} \left[\left(\frac{1}{n-1} \sum_{j=1}^{n} \frac{1}{(v_j(b) - b)} \right) - \frac{1}{(v_i(b) - b)} \right], \qquad i = 1, \ldots, n. \tag{4}$$

Because all bidders have the same domain of values $[\underline{v}, \bar{v}]$, it follows that $b_i(\underline{v}) = \underline{v}$ for all i (see Maskin and Riley 2000a). Hence, the initial condition for the system (4) is given by

$$v_i(b = \underline{v}) = \underline{v}, \qquad i = 1, \ldots, n. \tag{5}$$

The equilibrium strategies also satisfy the condition that all bidders with the highest valuation $\bar{v}$ place the same (unknown) maximal bid, denoted by $\bar{b}$ (Maskin and Riley 2000a). Hence,

$$v_i(\bar{b}) = \bar{v}, \qquad i = 1, \ldots, n. \tag{6}$$

Condition (6) is trivially satisfied in symmetric auctions, yet it plays an important role in the asymmetric case (see Appendix A).

3. Symmetric case. In the symmetric case, $F_i = F$ for $i = 1, \dots, n$. In that case the inverse bid functions are identical; i.e., $v_i(b) = v_{\text{sym}}(b)$ for all i. Equations (4)–(6) thus reduced to

$$v'_{\text{sym}}(b) = \frac{[F(v_{\text{sym}}(b))]}{(n-1)f(v_{\text{sym}}(b))} \frac{1}{v_{\text{sym}}(b) - b}, \qquad v_{\text{sym}}(\underline{v}) = \underline{v}. \tag{7}$$

This equation can be easily solved (see, e.g., Vickrey 1961, Riley and Samuelson 1981), yielding

$$b_{\text{sym}}(v) = v - \frac{1}{F^{n-1}(v)} \int_{\underline{v}}^{v} F^{n-1}(s)\, ds. \tag{8}$$

Hence, the maximal bid $\bar{b}_{\text{sym}} = b_{\text{sym}}(\bar{v})$ is given by

$$\bar{b}_{\text{sym}} = \bar{v} - \int_{\underline{v}}^{\bar{v}} F^{n-1}(s)\, ds. \tag{9}$$

We now use perturbation analysis to calculate the leading-order effect of a small identical change in the distribution functions of all bidders on the equilibrium strategies.

LEMMA 1. *Let*

$$F_i(v) = F(v) + \epsilon H(v), \qquad i = 1, \dots, n, \tag{10}$$

where $F(\underline{v}) = 0$, $F(\bar{v}) = 1$, $H(\underline{v}) = H(\bar{v}) = 0$ *and* $|\epsilon| \ll 1$. *Then, the* (*symmetric*) *equilibrium bid function is given by*

$$b(v; \epsilon) = b_{\text{sym}}(v) + \epsilon U(v; H) + O(\epsilon^2),$$

where $b_{\text{sym}}(v)$ *is given by* (8) *and*

$$U(v; H) = (n-1)\left[\frac{H(v)}{F^n(v)} \int_{\underline{v}}^{v} F^{n-1}(s)\, ds - \frac{1}{F^{n-1}(v)} \int_{\underline{v}}^{v} F^{n-2}(s) H(s)\, ds\right]. \tag{11}$$

In particular, the maximal bid $\bar{b}(\epsilon) = b(\bar{v}; \epsilon)$ *satisfies*

$$\bar{b}(\epsilon) = \bar{b}_{\text{sym}} - \epsilon(n-1) \int_{\underline{v}}^{\bar{v}} F^{n-2}(s) H(s)\, ds + O(\epsilon^2), \tag{12}$$

where $\bar{b}_{\text{sym}}$ *is given by* (9).

PROOF. Substituting (10) in (8) and collecting terms gives (11). Substituting $v = \bar{v}$ gives (12). □

The result of Lemma 1 is used in the calculation of the equilibrium bids in asymmetric first-price auctions (see Proposition 2).

4. Calculation of equilibrium bids. In contrast to the case of symmetric auctions, there are no explicit expressions for the equilibrium strategies in asymmetric first-price auctions (i.e., solutions of Equations (4)–(6)), such as (8). One can, however, obtain approximate expressions in the case of a weak asymmetry by using perturbation analysis. To do that, let us first consider the case where

$$F_i(v) = F(v) + \epsilon H_i(v) \qquad i = 1, \dots, n, \tag{13}$$

such that

$$(14)\quad F(\underline{v})=0,\quad F(\bar{v})=1,\quad H_i(\underline{v})=H_i(\bar{v})=0,\quad \max_{\underline{v}\le v\le \bar{v}}|H_i(v)|\le 1 \qquad i=1,\dots,n.$$

(The assumption that the distribution functions are of the form (13) is not restrictive. Indeed, we can bring any family of distribution functions $\{F_i\}_{i=1}^n$ to this form by defining $F=(1/n)\sum_{i=1}^n F_i$, $\epsilon=\max_i \max_v |F_i-F|$ and $H_i=(F_i-F)/\epsilon$.) Because ϵ measures the asymmetry level, *weak asymmetry* corresponds to $|\epsilon|\ll 1$. In that case we can use perturbation analysis to obtain explicit approximations of the equilibrium bids. We begin with the special case in which the arithmetic mean of the distribution functions is not affected by asymmetry, i.e., when $(1/n)\sum_{i=1}^n F_i\equiv F$ (or equivalently, $\sum_{i=1}^n H_i(v)\equiv 0$).

PROPOSITION 1. *Let* (13) *and* (14) *hold and assume that, in addition, for all* v

$$(15)\qquad \sum_{i=1}^{n} H_i(v)\equiv 0.$$

Then, the equilibrium bid functions in first-price auction are given by

$$(16)\qquad b_i(v)=b_{\mathrm{sym}}(v)+\epsilon B_i(v;H_i)+O(\epsilon^2),\qquad i=1,\dots,n,$$

where $b_{\mathrm{sym}}(v)$ *is given by* (8), *and*

$$(17)\qquad B_i(v;H_i)=\frac{-(n-1)}{F^{n-1}(v)}\left[\int_{\underline{v}}^{v}F^{n-1}(s)\,ds\right]^n\int_v^{\bar{v}}\frac{1}{\left[\int_{\underline{v}}^{x}F(s)^{n-1}\,ds\right]^{n-1}}\frac{d}{dx}\left(\frac{H_i(x)}{F(x)}\right)dx.$$

The maximal bid $\bar{b}(\epsilon)$ *is given by* $\bar{b}=\bar{b}_{\mathrm{sym}}+O(\epsilon^2)$, *where* $\bar{b}_{\mathrm{sym}}$ *is given by* (9).

PROOF. See Appendix A. □

When Assumption (15) of Proposition 1 does not hold, we can rewrite (13) as

$$F_i=F+\epsilon(H+\widetilde{H}_i),$$

where

$$(18)\qquad H=\frac{1}{n}\sum_{i=1}^{n}H_i,\qquad \widetilde{H}_i=H_i-H.$$

Because by construction $\sum_{i=1}^n \widetilde{H}_i(v)\equiv 0$, we have expressed the asymmetric perturbations $\{\epsilon H_i\}_{i=1}^n$ as a sum of:

(1) a symmetric shift by ϵH, and

(2) asymmetric shifts by $\{\epsilon\widetilde{H}_i\}_{i=1}^n$ that satisfy (15).

The leading-order effects of these two shifts are given by Lemma 1 and Proposition 1, respectively. Hence, we immediately have the following result.

PROPOSITION 2. *Let* (13) *and* (14) *hold. Then, the equilibrium bid functions in first-price auction are given by*

$$b_i(v)=b_{\mathrm{sym}}(v)+\epsilon[U(v;H)+B_i(v;\widetilde{H}_i)]+O(\epsilon^2),\qquad i=1,\dots,n,$$

where $b_{\mathrm{sym}}(v)$, U, *and* B_i *are given by* (8), (11), *and* (17), *respectively*; *and* H *and* $\widetilde{H}_i$ *are given by* (18). *In addition, the maximal bid* $\bar{b}(\epsilon)$ *is given by* (12).

PROOF. See Appendix B. □

Let us now consider the general case with n distribution functions $\{F_i\}_{i=1}^n$, i.e., when they are not given in the form (13). We can define the *average distribution*

$$F_{\text{avg}}(v) = \frac{1}{n}\sum_{i=1}^{n} F_i(v), \tag{19}$$

the *asymmetry parameter*

$$\varepsilon = \max_{1\le i\le n} \max_{\underline{v}\le v\le \bar{v}} |F_i(v) - F_{\text{avg}}(v)|, \tag{20}$$

and the auxiliary functions

$$H_i(v) = \frac{F_i(v) - F_{\text{avg}}(v)}{\epsilon}. \tag{21}$$

Note that by construction

$$F_i(v) = F_{\text{avg}}(v) + \epsilon H_i(v), \qquad H_i(\underline{v}) = H_i(\bar{v}) = 0 \qquad i = 1, \ldots, n, \tag{22}$$

$\sum_{i=1}^{n} H_i(v) \equiv 0$, and $|H_i| \le 1$ in $[\underline{v}, \bar{v}]$ for all i. Therefore, from Proposition 1 we immediately have the following result.

PROPOSITION 3. *Consider an asymmetric first-price auction with n bidders with distribution functions $\{F_i\}_{i=1}^n$ satisfying the conditions outlined at the beginning of §2. Let F_{avg}, ϵ, and H_i be given by* (19)–(21). *Assume that $|\epsilon| \ll 1$. Then, the equilibrium bid functions and the maximal bid in first-price asymmetric auctions are as in Proposition* 1 *with $F = F_{\text{avg}}$.*

We have thus achieved the goal of obtaining explicit approximations of the equilibrium strategies in asymmetric first-price auctions.

5. Applications. The explicit approximations of the equilibrium bids that we obtained using perturbation analysis can lead to a considerable simplification of the analysis of asymmetric first-price auctions. Below we give several such examples.

5.1. Stochastic dominance. We recall that a distribution F_1 is said to be *conditionally stochastic dominant* over F_2 (denoted by $F_1 \succ F_2$) if $d(F_1(v)/F_2(v))/dv > 0$ for $\underline{v} < v < \bar{v}$. Because $F_1(\bar{v}) = F_2(\bar{v}) = 1$, it follows that $F_1 < F_2$ for all $\underline{v} < v < \bar{v}$, i.e., Bidder 1 is stronger than Bidder 2. Lebrun (1998) and Maskin and Riley (2000a) showed that in equilibrium the stronger bidder bids less aggressively than the weaker one.

LEMMA 2 (LEBRUN 1998, MASKIN AND RILEY 2000a). *Consider an asymmetric first-price auction with two bidders. If $F_1 \succ F_2$ then $b_1(v) < b_2(v)$ for all $\underline{v} < v < \bar{v}$.*

An immediate application of the explicit expressions of the equilibrium bids obtained in §4 is the following generalization of Lemma 2 to the case of n bidders with weak asymmetry.

PROPOSITION 4. *Under the conditions of Proposition* 3, *for any two bidders $i \ne j$, if $F_i(v) \succ F_j(v)$, then $b_i(v) < b_j(v)$ for all $\underline{v} < v < \bar{v}$.*

PROOF. Because $F_i(v) \succ F_j(v)$ implies that $(F_i/F_j)' > 0$, we have,

$$\begin{aligned}\left(\frac{F_i}{F_j}\right)' &= \left(\frac{F_{\text{avg}} + \varepsilon H_i}{F_{\text{avg}} + \varepsilon H_j}\right)' = \left[\left(1 + \varepsilon\frac{H_i}{F_{\text{avg}}}\right)\left(1 - \varepsilon\frac{H_j}{F_{\text{avg}}}\right)\right]' + O(\varepsilon^2)\\ &= \varepsilon\left[\left(\frac{H_i}{F_{\text{avg}}}\right)' - \left(\frac{H_j}{F_{\text{avg}}}\right)'\right] + O(\varepsilon^2),\end{aligned}$$

and for ϵ sufficiently small $(H_i/F_{\text{avg}})' > (H_j/F_{\text{avg}})'$. Therefore, from Proposition 3 and (17) we have that $B_i(v; H_i) < B_j(v; H_i)$, hence that $b_i(v) < b_j(v)$. $\square$

When $n = 2$, Proposition 4 reduces to Lemma 2. Note, however, that Proposition 4 follows almost trivially from the explicit expressions which were calculated using the perturbation analysis, whereas the proof of Lemma 2 in Lebrun (1998) and in Maskin and Riley (2000a) is considerably more complex. Unlike Proposition 4, however, Lemma 2 is valid even when the asymmetries are not small.

5.2. Seller's expected revenue. The seller's expected revenue in first-price auctions with n bidders is given by

$$R^{1\text{st}} = E[\max\{b_1(v_1), \dots, b_n(v_n)\}]. \tag{23}$$

Let us first derive a simpler expression for $R^{1\text{st}}$ that will be used in the subsequent analysis (Proposition 5) and also in the simulations (see §8).

LEMMA 3. *The seller's expected revenue in a first-price auction is equal to*

$$R^{1\text{st}} = \bar{b} - \int_{\underline{v}}^{\bar{b}} \prod_{k=1}^{n} F_k(v_k(b))\, db, \tag{24}$$

where $v_k(b)$ are the inverse equilibrium bid functions and $\bar{b}$ is the maximal bid.

PROOF. See Appendix C. $\square$

We recall that the seller's expected revenue in the symmetric first-price auction is given by (see, e.g., Riley and Samuelson 1981)

$$R_{\text{sym}}[F] = \bar{v} + (n-1)\int_{\underline{v}}^{\bar{v}} F^n(v)\, dv - n\int_{\underline{v}}^{\bar{v}} F^{n-1}(v)\, dv. \tag{25}$$

Indeed, this result follows from (9) and (24). There is no such explicit expression in the asymmetric case. We can, however, use Proposition 2 and Lemma 3 to calculate the following approximation to the seller's expected revenue in first-price auction.

PROPOSITION 5. *Let* (13,14) *hold and let $R^{1\text{st}}$ be the seller's expected revenue in asymmetric first-price auction in equilibrium. Then, $R^{1\text{st}} = R_{\text{sym}}[F] + \epsilon \Delta R + O(\epsilon^2)$, where $R_{\text{sym}}[F]$ is given by* (25) *and*

$$\Delta R = -(n-1)\int_{\underline{v}}^{\bar{v}} (1 - F(v))F^{n-2}(v) \sum_{i=1}^{n} H_i(v)\, dv. \tag{26}$$

PROOF. See Appendix D. $\square$

The following is an immediate consequence of Proposition 5.

COROLLARY 1. *In first-price auction, if $\sum_{i=1}^{n} H_i(v) < 0$ (>0) for all $\underline{v} \le v \le \bar{v}$, then the seller's expected revenue increases* (*decreases*) *with ϵ.*

Proposition 5 shows that the effect of asymmetry on the seller's expected revenue is $O(\epsilon)$. In other words, if we approximate the seller's expected revenue with $R_{\text{sym}}[F]$, the approximation error would be first order in ϵ. The following proposition shows that if we approximate the seller's expected revenue with $R_{\text{sym}}[F_{\text{avg}}]$, the approximation error is only $O(\epsilon^2)$.

PROPOSITION 6. *Under the conditions of Proposition* 3, *the seller's expected revenue in a first-price auction is given by*

$$R^{1\text{st}} = R_{\text{sym}}[F_{\text{avg}}] + O(\epsilon^2),$$

where $R_{\text{sym}}[\cdot]$, F_{avg}, and ϵ are given by (25), (19), *and* (20), *respectively.*

PROOF. If $F=(1/n)\sum_{i=1}^n F_i$, then $\sum_{i=1}^n H_i(v)\equiv 0$, and thus $\Delta R=0$. From Proposition 5 we have the result. $\square$

We thus see that the seller's expected revenue in the asymmetric case can be approximated, with $O(\epsilon^2)$ accuracy, with the revenue in the symmetric case in which the distribution function of all bidders is given by the arithmetic mean of the asymmetric distribution functions.

5.3. Reserve prices. Our results can be easily generalized to the case in which the seller sets a reserve price r ($r>\underline{v}$). Recall that in symmetric auctions with reserve prices the seller's expected revenue is given by

$$R_{\text{sym}}[F,r]=-rF^n(r)+\bar{v}+(n-1)\int_r^{\bar{v}}F^n(v)\,dv-n\int_r^{\bar{v}}F^{n-1}(v)\,dv. \tag{27}$$

From the condition $(d/dr)R_{\text{sym}}[F,r]=0$, it follows that the optimal reserve price in a symmetric auction $r_{\text{sym}}^{\text{opt}}$ is the solution of

$$r_{\text{sym}}^{\text{opt}}=\frac{1-F(r_{\text{sym}}^{\text{opt}})}{f(r_{\text{sym}}^{\text{opt}})}. \tag{28}$$

In the case of asymmetric first-price auctions, if bidder i has value $v_i<r$, he will not make a positive profit, and so $b_i=0$. If, however, $v_i\ge r$, bidder i has a positive probability for winning and making a profit, hence $r\le b_i\le v_i$. Taking the limit as v_i approaches r from above shows that

$$v_i(b=r)=r,\qquad i=1,\dots,n. \tag{29}$$

Hence, for $v_i\ge r$, the inverse equilibrium strategies are the solutions of (4), subject to the initial conditions (29) and the boundary condition (6). Because the only change in the mathematical model due to the introduction of a reserve price is that the initial condition (29) is given at $b=r$ rather than at $b=\underline{v}$, the equilibrium bids can be approximated as in Proposition 2.

PROPOSITION 7. *Consider an asymmetric first-price auction in which the seller sets a reserve price r. Then, the equilibrium bid functions are zero for $\underline{v}\le v_i<r$. For $r\le v_i\le\bar{v}$ the equilibrium bid functions are as in Proposition* 2, *the only difference being that in all the expressions where the lower limit of integral is given by $\underline{v}$, it should be replaced with r.*

PROOF. Same as in the case with no reserve prices. $\square$

Thus, for example, the maximal bid can be approximated with

$$\bar{b}=\bar{v}-\int_r^{\bar{v}}F_{\text{avg}}^{n-1}(s)\,ds+O(\epsilon^2). \tag{30}$$

As in Lemma 3, the seller's expected revenue can be written as a one-dimensional integral.

LEMMA 4. *The seller's expected revenue in an asymmetric first-price auction with reserve price r is given by*

$$R^{1\text{st}}[F_1,\dots,F_n,r]=\bar{b}-\int_r^{\bar{b}}\prod_{k=1}^n F_k(v_k(b))\,db+(v_{\text{seller}}-r)\prod_{i=1}^n F_i(r), \tag{31}$$

where $v_k(b)$ are the inverse equilibrium bid functions and $\bar{b}$ is the maximal bid.

PROOF. Similar to the proof of Lemma 3 in Appendix C. $\square$

Applying the result of Proposition 6 and using the fact that $\prod_{i=1}^{n} F_i(r) = F_{\rm avg}^n(r) + O(\epsilon^2)$ gives that $R^{\rm 1st}[F_1, \ldots, F_n, r] = R_{\rm sym}[F_{\rm avg}, r] + O(\epsilon^2)$, where $R_{\rm sym}[F_{\rm avg}, r]$ is defined in (27). Therefore, in the asymmetric case the optimal reserve price satisfies

$$r_{\rm as}^{\rm opt} = \frac{1 - F_{\rm avg}(r_{\rm as}^{\rm opt})}{f_{\rm avg}(r_{\rm as}^{\rm opt})} + O(\epsilon^2).$$

We have thus proved the following result.

COROLLARY 2. *The optimal reserve price and the corresponding seller's expected revenue in an asymmetric first-price auction can be approximated, with $O(\epsilon^2)$ accuracy, with the optimal reserve price and the corresponding seller's expected revenue in a symmetric first-price auction with $F = F_{\rm avg}$.*

5.4. Inefficiency. In addition to the expected revenue, another important criterion for comparing auction mechanisms is their efficiency. In the symmetric case, first-price auctions are efficient; i.e., in equilibrium, the bidder with the highest valuation wins with probability one. This is not the case, however, in asymmetric first-price auctions where an $O(\epsilon)$ asymmetry results in an $O(\epsilon)$ inefficiency.

PROPOSITION 8. *Consider an asymmetric first-price auction with $n = 2$ bidders. Then*

$$\Pr(\textit{inefficiency}) = 2\varepsilon \int_{\underline{v}}^{\bar{v}} \left| V^1(b_{\rm sym}(v)) \right| f_{\rm avg}^2(v)\, dv + O(\varepsilon^2),$$

where

$$V^1(b) = \frac{F_{\rm avg}(v_{\rm sym}(b))}{f_{\rm avg}(v_{\rm sym}(b))} \int_{\underline{v}}^{v_{\rm sym}(b)} F_{\rm avg}(s)\, ds \int_{v_{\rm sym}(b)}^{\bar{v}} \frac{1}{\int_{\underline{v}}^{x} F_{\rm avg}(s)\, ds} \frac{d}{dx}\left(\frac{H_1(x)}{F_{\rm avg}(x)} \right) dx.$$

PROOF. See Appendix E. □

6. Comparison with second-price auctions. It is instructive to compare our results for asymmetric first-price auctions with the equivalent ones for second-price auctions. We first recall that asymmetry does not affect the equilibrium bids in second-price auctions, which remain to bid the true type, i.e., $b_i(v_i) = v_i$. Therefore, unlike first-price auctions (see §5.4), second-price auctions remain efficient under asymmetry.

The seller's expected revenue in second-price auctions is given by

$$R^{\rm 2nd} = \bar{v} - \int_{\underline{v}}^{\bar{v}} \prod_{i=1}^{n} F_i(v)\, dv - \sum_{i=1}^{n} \int_{\underline{v}}^{\bar{v}} (1 - F_i(v)) \prod_{\substack{j=1 \\ j \neq i}}^{n} F_j(v)\, dv. \tag{32}$$

Substituting $F_i = F + \epsilon H_i$ and expanding in ϵ gives that

$$R^{\rm 2nd}(\epsilon) = R_{\rm sym}[F] - \epsilon(n-1) \int_{\underline{v}}^{\bar{v}} (1 - F(v)) F^{n-2}(v) \sum_{i=1}^{n} H_i(v)\, dv + O(\epsilon^2).$$

We thus see that

$$R^{\rm 2nd} - R^{\rm 1st} = O(\epsilon^2).$$

The surprising fact that an $O(\epsilon)$ asymmetry leads to $O(\epsilon)$ changes in $R^{\rm 1st}$ and $R^{\rm 2nd}$, yet results in only an $O(\epsilon^2)$ difference between $R^{\rm 2nd}$ and $R^{\rm 1st}$, is a special case of the *revenue equivalence theorem for asymmetric auctions* (Fibich et al. 2003), which says that all asymmetric auctions are revenue equivalent to $O(\epsilon^2)$.

A question that has been open for many years is whether $R^{2\text{nd}}$ is larger or smaller than $R^{1\text{st}}$ under asymmetry (see, e.g., Marshall et al. 1994, Maskin and Riley 2000a, Cantillon 2002). The above analysis addresses this question indirectly by showing that the revenue difference between the two is quite small, as it is only $O(\epsilon^2)$. Therefore, to determine which one is larger using perturbation analysis, one would have to carry out the expansions to $O(\epsilon^2)$.

7. Simulations. In this section we compare the results of the perturbation analysis with numerical simulations. The numerical method used in these simulations is discussed in §8.

7.1. Bids. The explicit expressions obtained in §4 are derived under the assumption of weak asymmetry. A natural question is, thus, how small ϵ should be in order for these approximations to be valid. To see that, let us consider, for example, a first-price auction with two bidders whose valuations are distributed according to

$$F_1(v) = v - \epsilon v(1-v), \qquad F_2(v) = v + \epsilon v(1-v), \qquad \text{where} \quad v \in [0,\ 1].$$

To apply Proposition 3, we first note that in this case $F_{\text{avg}} = v$. Therefore, from (8) the equilibrium bid in the corresponding symmetric case is $b_{\text{sym}} = v/2$. Substitution in (17) yields

$$B_1(v) = -B_2(v) = -\frac{1}{2}(v^2 - v^3).$$

Therefore, by (16) the equilibrium bid functions are given by

$$b_1(v) \approx \frac{v}{2} - \frac{1}{2}\epsilon(v^2 - v^3), \qquad b_2(v) \approx \frac{v}{2} + \frac{1}{2}\epsilon(v^2 - v^3). \tag{33}$$

In Figure 1 we compare the approximations (33) with the exact equilibrium bid, which we calculated numerically. When $\epsilon = 0.1$ and $\epsilon = 0.25$, the approximate and exact solutions are indistinguishable. Even for $\epsilon = 0.5$, which is outside the formal domain of validity of the perturbation analysis, the agreement is quite remarkable.

7.2. Seller's expected revenue. In Table 1 we compare the expected revenue in asymmetric first-price auctions, which we calculated numerically (see §8), with its $O(\epsilon^2)$ analytic approximation $R_{\text{sym}}[F_{\text{avg}}]$, obtained in Proposition 6. We consider various numbers of bidders and various distribution functions. Although the asymmetry is not small, the differences between the exact revenue and its approximation are less than 1%. We thus see again that the results of the perturbation analysis remain accurate even when the asymmetry is not really weak.

7.3. Maximal bid $\bar{b}$. In general, there is no explicit expression for the maximal bid $\bar{b}$ in first-price asymmetric auctions, such as Expression (9). In Proposition 3 we saw that $\bar{b}$ is given by $\bar{b} = \bar{b}_{\text{sym}} + O(\epsilon^2)$, where $\bar{b}_{\text{sym}}$ is the maximal bid in the symmetric auction with $F = F_{\text{avg}}$. Although not exact, this approximation provides an explicit approximation for the maximal bid in asymmetric first-price auctions.

Our approximate solutions are formally derived for the case of a weak asymmetry. As is often the case with perturbation methods, however, the results remain valid outside their formal domain of validity. Indeed, from Table 2 we can see that $\bar{b}_{\text{sym}}$ is an excellent approximation (less than 1% error) for the exact value of $\bar{b}$, which we calculated numerically, even when asymmetry is not weak.

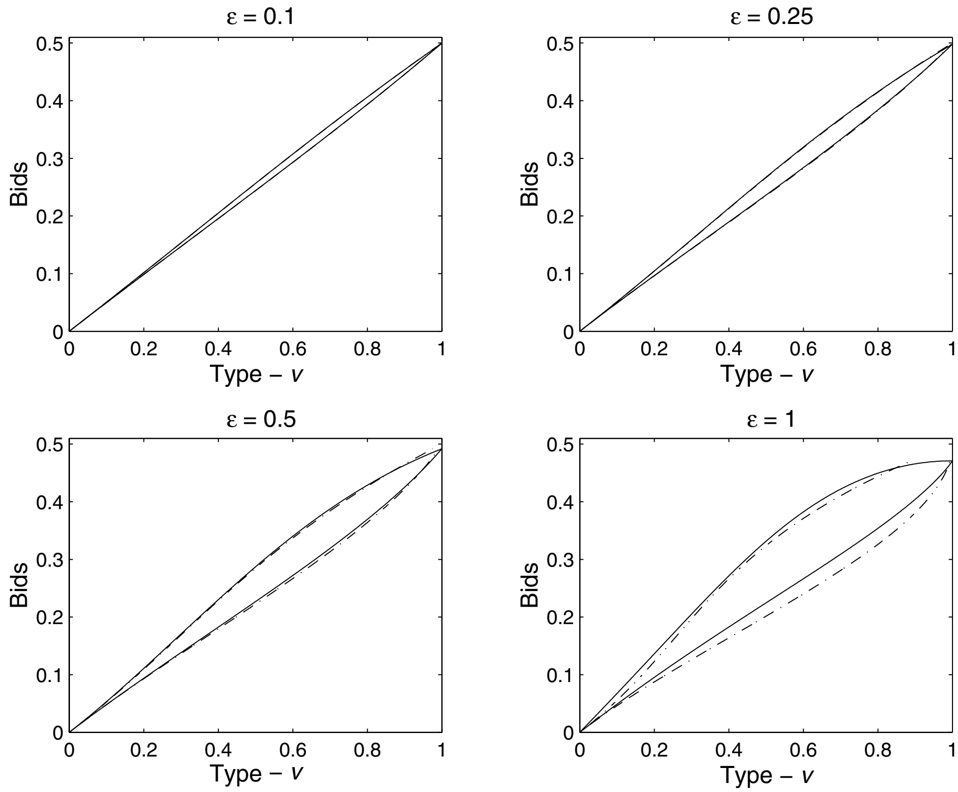

FIGURE 1. Exact (solid) and approximate (dash-dot) equilibrium strategies in first-price auctions.

7.4. Reserve prices. In a recent study, Marshall and Schulenberg (2002) calculated numerically the optimal reserve price and the corresponding seller's expected revenue in asymmetric first-price and second-price auctions when $F_1(v) = v^\alpha$ and $F_2(v) = v^\beta$. In Table 3 we compare their numerical values with our $O(\epsilon^2)$ explicit approximations $R^{1\text{st}}[F_1, F_2, r] \approx R_{\text{sym}}[F_{\text{avg}}, r]$ and $r_{\text{as}}^{\text{opt}} \approx r_{\text{sym}}^{\text{opt}}[F_{\text{avg}}]$, where $R_{\text{sym}}[F_{\text{avg}}, r]$ is given by (27), $r_{\text{sym}}^{\text{opt}}[F_{\text{avg}}]$ is the solution of (28) with $F = F_{\text{avg}}$, and $F_{\text{avg}} = (v^\alpha + v^\beta)/2$. The agreement between the two is excellent for a weak asymmetry and good for strong asymmetry.

We note that our finding—that the difference between the seller's expected revenue in asymmetric first-price and second-price auctions is only $O(\epsilon^2)$—remains true when reserve prices are introduced. Indeed, inspection of the values calculated by Marshall and Schulenberg (2002) shows that the difference between the two is typically in the third digit. A key finding of Marshall and Schulenberg (2002) is that once optimal reserve prices are allowed, first-price auctions no longer dominate second-price auctions in terms of the seller's expected revenue. Because the $O(\epsilon^2)$ difference between the two is below the $O(\epsilon)$

TABLE 1. Expected revenue $R^{1\text{st}}[F_1, \ldots, F_n]$ and its explicit approximation $R_{\text{sym}}[F_{\text{avg}}]$.

n	Distributions	$R^{1\text{st}}$	$R_{\text{sym}}[F_{\text{avg}}]$	$(R_{\text{sym}}[F_{\text{avg}}] - R^{1\text{st}})/R^{1\text{st}}$
2	$F_{1,2} = v \pm 0.4v^2(1-v^2)$	0.3325	0.3333	0.25%
2	$F_1 = v, F_2 = v^2$	0.426	0.425	−0.3%
3	$F_1 = v^2, F_2 = v^4, F_3 = v^5$	0.7996	0.7936	−0.76%
4	$F_i = v^i,\ i = 1, \ldots, 4$	0.7944	0.7865	−1%

TABLE 2. Maximal bid $\bar{b}$ and its explicit approximation $\bar{b}_{\text{sym}}[F_{\text{avg}}]$.

n	Distributions	$\bar{b}$	$\bar{b}_{\text{sym}}[F_{\text{avg}}]$	$(\bar{b}_{\text{sym}}[F_{\text{avg}}]-\bar{b})/\bar{b}$
2	$F_{1,2}=v\pm 0.4v^2(1-v^2)$	0.494	0.500	1%
2	$F_1=v, F_2=v^2$	0.578	0.583	0.9%
3	$F_1=v^2, F_2=v^4, F_3=v^5$	0.8740	0.8736	−0.04%
4	$F_i=v^i,\ i=1,\dots,4$	0.8753	0.8752	−0.008%

resolution of our expansions, explaining this finding using perturbation analysis would require carrying out the expansion to higher orders.

8. Numerical method. The (inverse) equilibrium strategies in asymmetric first-price auctions can be calculated by solving the differential equations (4) with the boundary conditions (5)–(6) numerically. Following Marshall et al. (1994), we apply a shooting method in which one "guesses" the value of $\bar{b}$ and then solves Equations (2) backwardly to $\underline{v}$. The value of $\bar{b}$ is modified until the numerical solution satisfies the initial condition (5) with a prescribed tolerance (e.g., 10^{-5} in our simulations).

We have added the following improvements to the algorithm of Marshall et al. (1994):

1. As the initial guess, we use the approximation (9) with $F=F_{\text{avg}}$, i.e.,

$$\bar{b}\approx\bar{v}-\int_{\underline{v}}^{\bar{v}}F_{\text{avg}}^{n-1}(v)\,dv.$$

Because this approximation is already close to $\bar{b}$, it can reduce the number of iterations needed for convergence.

2. In Marshall et al. (1994) the expected revenue is calculated from its definition (23), which is a multidimensional integral whose integrand is given in terms of the equilibrium strategies. Because, however, one solves numerically for the *inverse* equilibrium strategies, the calculation of (23) was done using Monte Carlo methods. In the case of a large number of bidders, Monte Carlo methods require an enormous amount of sampling to achieve high accuracy. In such cases, the new expression for the expected revenue (24) that we derive allows for a considerable reduction of the computational costs. Indeed, let us define the auxiliary function $A(b)$ to be the solution of the differential equation

$$\frac{dA(b)}{db}=-\prod_{k=1}^{n}F_k(v_k(b)),\qquad A(\bar{b})=\bar{b}.$$

From (24) it follows that $A(\underline{v})=R^{1\text{st}}$. Therefore, we solve the inverse bid equations (2) and the equation for A simultaneously. The additional costs of solving for A are minimal,

TABLE 3. Exact (numerical) and approximate (analytical) values of the optimal reserve price and the corresponding seller's expected revenue in asymmetric first-price auctions with $F_1(v)=v^\alpha$ and $F_2(v)=v^\beta$.

α	β	$r_{\text{as}}^{\text{opt}}$	$r_{\text{sym}}^{\text{opt}}[F_{\text{avg}}]$	$\left(r_{\text{as}}^{\text{opt}}-r_{\text{sym}}^{\text{opt}}[F_{\text{avg}}]\right)/r_{\text{as}}^{\text{opt}}$	$R^{1\text{st}}$	$R_{\text{sym}}[F_{\text{avg}}]$	$\left(R^{1\text{st}}-R_{\text{sym}}[F_{\text{avg}}]\right)/R^{1\text{st}}$
2	3	0.609	0.607	0.3%	0.633	0.632	0.2%
1	4	0.659	0.623	5.5%	0.608	0.589	3.1%
3	8	0.742	0.709	4.5%	0.774	0.764	1.3%
4	7	0.723	0.710	1.8%	0.785	0.782	0.4%
5	6	0.716	0.711	0.7%	0.791	0.790	0.1%

Exact values are taken from Marshall and Schulenberg (2002).

the sampling rate is dictated only by the ODE solver being used, and the error control is much more reliable than in Monte Carlo methods. Indeed, using this approach allowed us to calculate the seller's expected revenue with 4 asymmetric bidders (last line of Table 1) at almost the same computational costs as for two bidders.

Appendix A: Proof of Proposition 1. We can expand the inverse equilibrium strategy of bidder k as

$$v_k(b) = v_{\mathrm{sym}}(b) + \epsilon V^k(b) + O(\epsilon^2), \qquad k = 1, \dots, n, \tag{34}$$

where $v_k(\underline{v}) = \underline{v}$, $v_k(\bar{b}) = \bar{v}$ and v_{sym} is the inverse function of (8). We now prove two auxiliary lemmas.

LEMMA 5. *Under the conditions of Proposition* 1,

$$\sum_{k=1}^{n} V^k(b) \equiv 0 \qquad \textit{for all } b. \tag{35}$$

PROOF. Using (34) we have that

$$\begin{aligned} F_k(v_k(b)) &= F(v_k(b)) + \epsilon H_k(v_k(b)) = F(v_{\mathrm{sym}} + \epsilon V^k) + \epsilon H_k(v_{\mathrm{sym}} + \epsilon V^k) + O(\epsilon^2) \\ &= F(v_{\mathrm{sym}}) + \epsilon V^k f(v_{\mathrm{sym}}) + \epsilon H_k(v_{\mathrm{sym}}) + O(\epsilon^2). \end{aligned} \tag{36}$$

Similarly,

$$f_k(v_k(b)) = f(v_{\mathrm{sym}}) + \epsilon V^k f'(v_{\mathrm{sym}}) + \epsilon H_k'(v_{\mathrm{sym}}) + O(\epsilon^2). \tag{37}$$

Substituting (34), (36) and (37) in (4) gives

$$\begin{aligned} (v_{\mathrm{sym}})' + \epsilon(V^k)' = {} & \frac{F(v_{\mathrm{sym}}) + \epsilon V^k f(v_{\mathrm{sym}}) + \epsilon H_k(v_{\mathrm{sym}})}{f(v_{\mathrm{sym}}) + \epsilon V^k f'(v_{\mathrm{sym}}) + \epsilon H_k'(v_{\mathrm{sym}})} \\ & \times \left[\left(\frac{1}{n-1} \sum_{j=1}^{n} \frac{1}{v_{\mathrm{sym}} + \epsilon V^j - b} \right) - \frac{1}{v_{\mathrm{sym}} + \epsilon V^k - b} \right] + O(\epsilon^2). \end{aligned}$$

Expanding the fractions in Taylor series in ϵ yields

$$\begin{aligned} (v_{\mathrm{sym}})' + \epsilon(V^k)' = {} & \frac{F(v_{\mathrm{sym}}) + \epsilon V^k f(v_{\mathrm{sym}}) + \epsilon H_k(v_{\mathrm{sym}})}{f(v_{\mathrm{sym}})} \left(1 - \epsilon \frac{V^k f'(v_{\mathrm{sym}}) + H_k'(v_{\mathrm{sym}})}{f(v_{\mathrm{sym}})} \right) \\ & \times \left[\left(\frac{1}{n-1} \sum_{j=1}^{n} \frac{1}{v_{\mathrm{sym}} - b} \left(1 - \epsilon \frac{V^j}{v_{\mathrm{sym}} - b} \right) \right) - \frac{1}{v_{\mathrm{sym}} - b} \left(1 - \epsilon \frac{V^k}{v_{\mathrm{sym}} - b} \right) \right] \\ & + O(\epsilon^2). \end{aligned}$$

The equation for the $O(1)$ terms is automatically satisfied. Collecting the $O(\epsilon)$ terms gives

$$\begin{aligned} (V^k)'(b) = {} & \frac{F(v_{\mathrm{sym}})}{f(v_{\mathrm{sym}})} \left[-\frac{1}{n-1} \sum_{j=1}^{n} \frac{V^j}{(v_{\mathrm{sym}} - b)^2} + \frac{V^k}{(v_{\mathrm{sym}} - b)^2} \right] \\ & - \frac{F(v_{\mathrm{sym}})}{f(v_{\mathrm{sym}})} \frac{V^k f'(v_{\mathrm{sym}}) + H_k'(v_{\mathrm{sym}})}{f(v_{\mathrm{sym}})} \frac{1}{(n-1)(v_{\mathrm{sym}} - b)} \\ & + \frac{V^k f(v_{\mathrm{sym}}) + H_k(v_{\mathrm{sym}})}{f(v_{\mathrm{sym}})} \frac{1}{(n-1)(v_{\mathrm{sym}} - b)}, \end{aligned} \tag{38}$$

subject to $V^k(\underline{v}) = 0$. Summing (38) over $k = 1, \dots, n$ and substituting (15) gives the following equation for $u(b) = \sum_{k=1}^n V^k(b)$:

$$u'(b) = -Au, \qquad u(\underline{v}) = 0,$$

where

$$A = \frac{F(v_{\mathrm{sym}})}{(n-1)f(v_{\mathrm{sym}})(v_{\mathrm{sym}} - b)^2} + \frac{F(v_{\mathrm{sym}})f'(v_{\mathrm{sym}})}{(n-1)f^2(v_{\mathrm{sym}})(v_{\mathrm{sym}} - b)} - \frac{1}{(n-1)(v_{\mathrm{sym}} - b)}.$$

The solution of this linear first-order ODE is $u = C\exp\left(\int_b^{\bar{b}_{\mathrm{sym}}} A\right)$, where C is a constant. From (7) and (8) we have that

$$\begin{aligned}\int_b^{\bar{b}_{\mathrm{sym}}} A(b)\,db &= \int_b^{\bar{b}_{\mathrm{sym}}} \left[\frac{F^{n-1}(v_{\mathrm{sym}})}{\int_{\underline{v}}^{v_{\mathrm{sym}}} F^{n-1}(s)\,ds} + \frac{f'(v_{\mathrm{sym}})}{f(v_{\mathrm{sym}})} - \frac{f(v_{\mathrm{sym}})}{F(v_{\mathrm{sym}})}\right] v'_{\mathrm{sym}}(b)\,db \\ &= \ln\left[\frac{f(v)}{F(v)}\int_{\underline{v}}^{v_{\mathrm{sym}}} F^{n-1}(s)\,ds\right]_{v_{\mathrm{sym}}(b)}^{\bar{v}}.\end{aligned}$$

Hence,

$$\exp\left(\int_b^{\bar{b}_{\mathrm{sym}}} A\right) = \frac{f(\bar{v})F(v_{\mathrm{sym}}(b))\int_{\underline{v}}^{\bar{v}} F^{n-1}(s)\,ds}{f(v_{\mathrm{sym}}(b))\int_{\underline{v}}^{v_{\mathrm{sym}}(b)} F^{n-1}(s)\,ds}. \tag{39}$$

Because $\lim_{b\to\underline{v}}\exp\left(\int_b^{\bar{b}_{\mathrm{sym}}} A\right) = \infty$, we conclude that $C = 0$, and thus that $u(b) \equiv 0$. $\square$

LEMMA 6. *Let*

$$\bar{b}(\epsilon) = \bar{b}_{\mathrm{sym}} + \epsilon b^1 + O(\epsilon^2). \tag{40}$$

Then, under the conditions of Proposition 1, $b^1 = 0$. *In addition,*

$$V^k(\bar{b}_{\mathrm{sym}}) = 0, \qquad k = 1, \dots, n. \tag{41}$$

PROOF. From (6), (34), and (40) we have that for any k

$$\bar{v} = v_k(\bar{b}) = v_k(\bar{b}_{\mathrm{sym}} + \epsilon b^1 + O(\epsilon^2)) = v_{\mathrm{sym}}(\bar{b}_{\mathrm{sym}}) + \epsilon[b^1 v'_{\mathrm{sym}}(\bar{b}_{\mathrm{sym}}) + V^k(\bar{b}_{\mathrm{sym}})] + O(\epsilon^2).$$

If we balance the $O(\epsilon)$ terms, we get that

$$b^1 v'_{\mathrm{sym}}(\bar{b}_{\mathrm{sym}}) + V^k(\bar{b}_{\mathrm{sym}}) = 0. \tag{42}$$

Summing (42) over $k = 1, \dots, n$ and using (35) gives that $b^1 = 0$. Using this in (42) proves (41). $\square$

We now turn to the proof of Proposition 1. Substituting (35) in (38) gives

$$(V^k)'(b) + B(b)V^k = D_k(b), \qquad V^k(\underline{v}) = 0,$$

where

$$\begin{aligned}B(b) &= -\frac{F(v_{\mathrm{sym}})}{f(v_{\mathrm{sym}})(v_{\mathrm{sym}} - b)^2} + \frac{F(v_{\mathrm{sym}})f'(v_{\mathrm{sym}})}{f^2(v_{\mathrm{sym}})}\frac{1}{(n-1)(v_{\mathrm{sym}} - b)} \\ &\quad - \frac{1}{(n-1)(v_{\mathrm{sym}} - b)}, \\ D_k(b) &= -\frac{1}{(n-1)(v_{\mathrm{sym}} - b)}\frac{F^2(v_{\mathrm{sym}})}{f^2(v_{\mathrm{sym}})}\left(\frac{H_k(v_{\mathrm{sym}})}{F(v_{\mathrm{sym}})}\right)'.\end{aligned} \tag{43}$$

The solution of this equation is

$$V^k = \exp\left(\int_b^{\bar{b}_{\text{sym}}} B\right)\left[C_k - \int_b^{\bar{b}_{\text{sym}}} D_k(x)\exp\left(-\int_x^{\bar{b}_{\text{sym}}} B\right)dx\right], \tag{44}$$

where C_k is a constant. Similarly to (39), we have that

$$\exp\left(\int_b^{\bar{b}_{\text{sym}}} B\right) = \frac{f(\bar{v})}{\left[\int_{\underline{v}}^{\bar{v}} F^{n-1}(s)ds\right]^{n-1}}\frac{F(v_{\text{sym}}(b))\left[\int_{\underline{v}}^{v_{\text{sym}}(b)} F^{n-1}(s)\,ds\right]^{n-1}}{f(v_{\text{sym}}(b))}. \tag{45}$$

Hence, for any C_k the initial condition $V^k(\underline{v}) = 0$ is automatically satisfied. The value of C_k is determined from the boundary condition (6). Because this condition reduces to (41), we conclude that $C_k = 0$. Therefore, by (44) and (45),

$$\begin{aligned} V^k(b) = &-\frac{F(v_{\text{sym}}(b))}{f(v_{\text{sym}}(b))}\left[\int_{\underline{v}}^{v_{\text{sym}}(b)} F^{n-1}(s)\,ds\right]^{n-1} \\ &\times \int_b^{\bar{b}_{\text{sym}}} D_k\left[\frac{f(v_{\text{sym}}(x))}{F(v_{\text{sym}}(x))\left[\int_{\underline{v}}^{v_{\text{sym}}(x)} F^{n-1}(s)\,ds\right]^{n-1}}\right]dx. \end{aligned}$$

Substituting (43) and using (7) gives

$$\begin{aligned} V^k(b; H_i) = &\frac{F(v_{\text{sym}}(b))}{f(v_{\text{sym}}(b))}\left[\int_{\underline{v}}^{v_{\text{sym}}(b)} F^{n-1}(s)\,ds\right]^{n-1} \\ &\times \int_{v_{\text{sym}}(b)}^{\bar{v}} \frac{1}{\left[\int_{\underline{v}}^{x} F(s)^{n-1}\,ds\right]^{n-1}}\frac{d}{dx}\left(\frac{H_k(x)}{F(x)}\right)dx, \qquad i = 1, \ldots, n. \end{aligned} \tag{46}$$

Differentiating the identity $v = b_k^{-1}(b_k(v;\epsilon);\epsilon)$ with respect to ϵ and substituting $\epsilon = 0$ gives

$$0 = \left.\frac{\partial b_k^{-1}}{\partial \epsilon}\right|_{\epsilon=0} + (b_k^{-1})'\left.\frac{\partial b_k}{\partial \epsilon}\right|_{\epsilon=0}.$$

Observe that $\partial b_k^{-1}/\partial\epsilon|_{\epsilon=0} = V^k(b_{\text{sym}}(v); H_i)$, $\partial b_k/\partial\epsilon|_{\epsilon=0} = B^k(v; H_i)$, and $(b_k^{-1})'|_{\epsilon=0} = v'_{\text{sym}}(b(v)) = 1/(b'_{\text{sym}}(v))$. Therefore,

$$B^k(v; H_i) = -b'_{\text{sym}}(v)V^k(b_{\text{sym}}(v); H_i). \tag{47}$$

Substituting $b'_{\text{sym}}(v)$ yields the results.

Appendix B: Proof of Proposition 2. Let $\widetilde{F} = F + \epsilon H$. Then, by Lemma 1,

$$b_i(v; \widetilde{F}) = b_{\text{sym}}(v) + \epsilon U(b; H) + O(\epsilon^2).$$

Let $F_i = \widetilde{F} + \epsilon\widetilde{H}_i$. Then, by Proposition 1,

$$b_i(v; F_1, \ldots, F_n) = b_i(v; \widetilde{F}) + \epsilon B_i(v; \widetilde{H}_i) + O(\epsilon^2).$$

The result follows from the last two equations.

Appendix C: Proof of Lemma 3. Define $\tilde{b}(v_1, \ldots, v_n) = \max[b_1(v_1), \ldots, b_n(v_n)]$. The distribution of $\tilde{b}$ is given by

$$F^{1\mathrm{st}}(\tilde{b}) = \Pr\big(b_1(v_1) \le \tilde{b}, \ldots, b_n(v_n) \le \tilde{b}\big) = \prod_{j=1}^{n} F_j\big(v_j(\tilde{b})\big).$$

Therefore,

$$R^{1\mathrm{st}} = \int_{\underline{v}}^{\bar{b}} \tilde{b} \frac{dF^{1\mathrm{st}}(\tilde{b})}{d\tilde{b}}\, d\tilde{b} = \tilde{b} F^{1\mathrm{st}}(\tilde{b})\Big|_{\underline{v}}^{\bar{b}} - \int_{\underline{v}}^{\bar{b}} F^{1\mathrm{st}}(\tilde{b})\, d\tilde{b} = \bar{b} - \int_{\underline{v}}^{\bar{b}} \prod_{j=1}^{n} F_j(v_j(\tilde{b}))\, d\tilde{b}.$$

Appendix D: Proof of Proposition 5. Let us first rewrite the result of Proposition 2 in terms of the *inverse* equilibrium bid functions; i.e.,

$$v_k(b) = v_{\mathrm{sym}}(b) + \epsilon[U(b; H) + V^k(b; \widetilde{H}_i)] + O(\epsilon^2), \qquad i = 1, \ldots, n,$$

where $v_{\mathrm{sym}}(b)$ is the inverse function of (8),

$$U(b; H) = \frac{1}{f(v_{\mathrm{sym}}(b))} \frac{1}{v_{\mathrm{sym}}(b) - b} \left[\frac{1}{F^{n-2}(v_{\mathrm{sym}}(b))} \int_{\underline{v}}^{v_{\mathrm{sym}}(b)} F^{n-2}(s) H(s)\, ds - \frac{H(v_{\mathrm{sym}}(b))}{F^{n-1}(v_{\mathrm{sym}}(b))} \int_{\underline{v}}^{v_{\mathrm{sym}}(b)} F^{n-1}(s)\, ds \right], \tag{48}$$

V^k is given by (46), and H and $\widetilde{H}_i$ are given by (18). Therefore, we can expand $F_k(v_k(b))$ as

$$F_k(v_k(b)) = F(v_{\mathrm{sym}}(b)) + \epsilon\big[V^k(s; \widetilde{H}_k) + U(b)\big] f(v_{\mathrm{sym}}(b)) + \epsilon H_k(v_{\mathrm{sym}}(b)) + O(\epsilon^2), \tag{49}$$

where the relations between $\widetilde{H}_k$, H_k, and H are given by (18). Substituting (49) and (40) in (24) gives

$$\begin{aligned} R^{1\mathrm{st}} &= \bar{b} - \int_{\underline{v}}^{\bar{b}} \prod_{k=1}^{n} F_k(v_k(b))\, db \\ &= \bar{b}_{\mathrm{sym}} + \epsilon b^1 - \int_{\underline{v}}^{\bar{b}_{\mathrm{sym}} + \epsilon b^1} \prod_{k=1}^{n} \big[F(v_{\mathrm{sym}}(b)) + \epsilon[V^k(s; \widetilde{H}_k) + U(b)] f(v_{\mathrm{sym}}(b)) \\ &\qquad + \epsilon H_k(v_{\mathrm{sym}}(b))\big]\, db + O(\epsilon^2) \\ &= \bar{b}_{\mathrm{sym}} + \epsilon b^1 - \int_{\underline{v}}^{\bar{b}_{\mathrm{sym}} + \epsilon b^1} \big[F(v_{\mathrm{sym}}(b))\big]^n\, db - \epsilon \int_{\underline{v}}^{\bar{b}_{\mathrm{sym}}} \big[F(v_{\mathrm{sym}}(b))\big]^{n-1} \\ &\quad \times \sum_{k=1}^{n} \big[\big[V^k(s; \widetilde{H}_k) + U(b)\big] f(v_{\mathrm{sym}}(b)) + H_k(v_{\mathrm{sym}}(b))\big]\, db + O(\epsilon^2). \end{aligned}$$

Because $F(v_{\mathrm{sym}}(\bar{b}_{\mathrm{sym}})) = 1$, we have that

$$\int_{\underline{v}}^{\bar{b}_{\mathrm{sym}} + \epsilon b^1} \big[F(v_{\mathrm{sym}}(b))\big]^n\, db = \int_{\underline{v}}^{\bar{b}_{\mathrm{sym}}} [F(v_{\mathrm{sym}}(b))]^n\, db + \epsilon b^1 + O(\epsilon^2).$$

Using (25) and $\sum_{k=1}^{n} V^k(b; \widetilde{H}_k) = 0$ gives

$$R^{1\mathrm{st}} = R_{\mathrm{sym}}[F] - \epsilon \int_{\underline{v}}^{\bar{b}_{\mathrm{sym}}} \left[n f(v_{\mathrm{sym}}(b)) U(b) + \sum_{k=1}^{n} H_k(v_{\mathrm{sym}}(b)) \right] F^{n-1}(v_{\mathrm{sym}}(b))\, db + O(\epsilon^2).$$

Substituting (48) and $\sum_{k=1}^{n} H_k(v_{\text{sym}}(b)) = nH(v_{\text{sym}}(b))$, we have

$$R^{1\text{st}} = R_{\text{sym}}[F] - \epsilon n \int_{\underline{v}}^{\bar{b}_{\text{sym}}} \left\{ \frac{1}{v_{\text{sym}}(b) - b} \left[\frac{1}{F^{n-2}(v_{\text{sym}}(b))} \int_{\underline{v}}^{v_{\text{sym}}(b)} F^{n-2}(s) H(s)\, ds \right.\right.$$
$$\left.\left. - \frac{H(v_{\text{sym}}(b))}{F^{n-1}(v_{\text{sym}}(b))} \int_{\underline{v}}^{v_{\text{sym}}(b)} F^{n-1}(s)\, ds \right] + H(v_{\text{sym}}(b)) \right\}$$
$$\times F^{n-1}(v_{\text{sym}}(b))\, db + O(\epsilon^2).$$

From (8) we have that $1/(v_{\text{sym}}(b) - b) = F^{n-1}(v_{\text{sym}}(b)) / \int_{\underline{v}}^{\bar{v}_{\text{sym}}} F^{n-1}(s)\, ds$. Therefore,

$$R^{1\text{st}} = R_{\text{sym}}[F] - \epsilon \int_{\underline{v}}^{\bar{b}_{\text{sym}}} \frac{nF^n(v_{\text{sym}}(b))}{\int_{\underline{v}}^{v_{\text{sym}}(b)} F^{n-1}(s)\, ds} \left[\int_{\underline{v}}^{v_{\text{sym}}(b)} F^{n-2}(s) H(s)\, ds \right] db + O(\epsilon^2).$$

From (7)–(8) we have that $v'_{\text{sym}}(b) = F^n(v_{\text{sym}}(b)) / \big((n-1) f(v_{\text{sym}}(b)) \int_{\underline{v}}^{v_{\text{sym}}(b)} F^{n-1}(s)\, ds\big)$. Therefore, we can make a change of variables in the integral, yielding

$$R^{1\text{st}} = R_{\text{sym}}[F] - \epsilon (n-1) \int_{\underline{v}}^{\bar{v}} f(v) \left[\int_{\underline{v}}^{v} F^{n-2}(s) nH(s)\, ds \right] dv + O(\epsilon^2).$$

Integrating by parts and substituting $\sum_{k=1}^{n} H_k(v_{\text{sym}}(b)) = nH(v_{\text{sym}}(b))$ completes the proof. □

Appendix E: Proof of Proposition 8. We first note that

$$\Pr(\text{inefficiency}) = \Pr(b_2(v_2) < b_1(v_1), v_1 < v_2) + \Pr(b_1(v_1) < b_2(v_2), v_2 < v_1).$$

Now,

$$\begin{aligned}
\Pr(b_2(v_2) < b_1(v_1), v_1 < v_2) &= E_{v_2} \Pr(b_2(v_2) < b_1(v_1), v_1 < v_2 | v_2) \\
&= E_{v_2} \Pr(b_1^{-1}(b_2(v_2)) < v_1 < v_2 | v_2) \\
&= E_{v_2} [F_1(v_2) - F_1(b_1^{-1}(b_2(v_2)))]_+ \\
&= \int_{\underline{v}}^{\bar{v}} [F_1(v_2) - F_1(b_1^{-1}(b_2(v_2)))]_+ f_2(v_2)\, dv_2,
\end{aligned}$$

where $[x]_+ = \max\{x, 0\}$. It should be noted that $F_1(v_2) = F_{\text{avg}}(v_2) + \varepsilon H_1(v_2) + O(\varepsilon^2)$ and $F_1(b_1^{-1}(b_2(v_2))) = F_{\text{avg}}(v_1(b_2(v_2))) + \varepsilon H_1(v_1(b_2(v_2))) + O(\varepsilon^2)$. Furthermore, by (34),

$$\begin{aligned}
v_1(b_2(v_2)) &= v_1(b_{\text{sym}}(v_2) + \varepsilon B_2(v_2)) + O(\varepsilon^2) \\
&= v_{\text{sym}}(b_{\text{sym}}(v_2) + \varepsilon B_2(v_2)) + \varepsilon V_1(b_{\text{sym}}(v_2)) + O(\varepsilon^2) \\
&= v_{\text{sym}}(b_{\text{sym}}(v_2)) + \varepsilon B_2(v_2) v'_{\text{sym}}(b_{\text{sym}}(v_2)) + \varepsilon V_1(b_{\text{sym}}(v_2)) + O(\varepsilon^2) \\
&= v_2 + \varepsilon \big(B_2(v_2) v'_{\text{sym}}(b_{\text{sym}}(v_2)) + V_1(b_{\text{sym}}(v_2))\big) + O(\varepsilon^2).
\end{aligned}$$

Thus,

$$\begin{aligned}
F_1(b_1^{-1}(b_2(v_2))) &= F_{\text{avg}}(v_1(b_2(v_2))) + \varepsilon H_1(v_1(b_2(v_2))) + O(\varepsilon^2) \\
&= F_{\text{avg}}(v_2 + \varepsilon B_2(v_2) v'_{\text{sym}}(b_{\text{sym}}(v_2)) + V_1(b_{\text{sym}}(v_2))) + \varepsilon H_1(v_2) + O(\varepsilon^2) \\
&= F_{\text{avg}}(v_2) + \varepsilon \big[B_2(v_2) v'_{\text{sym}}(b_{\text{sym}}(v_2)) + V_1(b_{\text{sym}}(v_2))\big] f_{\text{avg}}(v_2) \\
&\quad + \varepsilon H_1(v_2) + O(\varepsilon^2).
\end{aligned}$$

Combining the above and using (47) gives

$$\begin{aligned}\Pr(b_2(v_2) < b_1(v_1), v_1 < v_2) &= \varepsilon \int_{\underline{v}}^{\bar{v}} \left[-B_2(v_2)v'_{\mathrm{sym}}(b_{\mathrm{sym}}(v_2)) - V_1(b_{\mathrm{sym}}(v_2))\right]_+ \\ &\qquad \times f^2_{\mathrm{avg}}(v_2)\, dv_2 + O(\varepsilon^2) \\ &= 2\varepsilon \int_{\underline{v}}^{\bar{v}} \left[-V_1(b_{\mathrm{sym}}(v_2))\right]_+ f^2_{\mathrm{avg}}(v_2)\, dv_2 + O(\varepsilon^2).\end{aligned}$$

Similarly, $\Pr(b_1(v_1) < b_2(v_2), v_2 < v_1) = 2\varepsilon \int_{\underline{v}}^{\bar{v}} \left[V_1(b_{\mathrm{sym}}(v_2))\right]_+ f^2_{\mathrm{avg}}(v_2)\, dv_2 + O(\varepsilon^2)$. Thus, the result follows. □

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G. Fibich: School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel; e-mail: fibich@math.tau.ac.il

A. Gavious: Faculty of Engineering Sciences, Department of Industrial Engineering and Management, Ben Gurion University, P.O. Box 653, Beer-Sheva 84105, Israel; e-mail: ariehg@bgumail.bgu.ac.il