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Gap-soliton bullets in waveguide gratings

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Abstract

We derive a perturbed two-dimensional nonlinear Schrödinger equation which describes the propagation of gap-soliton bullets in nonlinear periodic waveguides at frequencies close to the gap for Bragg reflection. Analysis and simulations of this equation show that the bullets amplitude undergoes stable focusing–defocusing cycles.

Keywords: Gap soliton; Waveguide gratings; Nonlinear Schrödinger equation

1. Introduction

Gap solitons are realizations of a balance between nonlinearity and grating assisted dispersion in optical fibers. At high powers this balance is achieved within a few centimeters of pulse propagation and has the unique feature of having a full range of velocities from zero to the speed of light in the uniform medium, depending on the location of the frequency of the electromagnetic field with respect to the gap created by the grating. For a general review of gap solitons, see [1,2].

Two particular limits that have attracted much attention are when the soliton frequency is (1) inside the band-gap, corresponding to the formation of slow or even stationary gap solitons, and (2) close to and outside the band-gap, in which case the soliton is reminiscent of that of the integrable one-dimensional nonlinear Schrödinger equation (NLSE). In fact, using asymptotic methods one can show that in this regime the governing coupled mode equations are well approximated by the NLSE. There have been experimental demonstrations of gap solitons in both regimes [3,4] with velocities as low as 50% of the speed of light and in gratings of no more than 20 cm of length. To date, however, no experiments have been reported demonstrating zero velocity gap solitons. This is not surprising, given the intrinsic difficulty in trapping light through a region of strong linear back reflection. There have been interesting proposals on how to achieve this, such as the use of defects or chirped gratings [5] or through a Raman downshift mechanism [6]. The potential application to light storage devices makes this effort quite worthwhile.

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In recent times research on gap-soliton phenomena has extended not only to other classical optical media (e.g., quadratic dielectrics [7]) but also to condensed matter waves [8] and quantum systems [9] to name some.[1] What is common to these works is that they are all in one-dimensional geometries (i.e., the distance of propagation (z) plays the role of 'time', whereas physical time plays the role of a spatial variable). Very little is known at present on existence of stable two-dimensional or three-dimensional gap solitons [11–14]. In [11,12] research has focused on two-dimensional periodic structures in the directions transverse to propagation. In contrast, in this study we consider propagation in a planar waveguide Kerr medium with a periodic refractive index profile in the direction of propagation.

Under the envelope approximation, the equations for fully localized optical pulses (optical bullets) propagating in Kerr-type planar waveguides or in bulk media are the two-dimensional and the three-dimensional cubic NLSE, respectively. In both cases the NLSE model predicts at low intensities a dispersion/diffraction dominated dynamics, whereas at high intensities the field amplitude reaches infinite values in finite propagation distances (collapse). In particular, in both cases optical bullets are unstable [15,16].

We recall that the two-dimensional cubic NLSE is the *critical* case for collapse, in the sense that it is the "boundary" between the *subcritical* one-dimensional case where stable soliton dynamics occurs and the *supercritical* three-dimensional case for which collapse is typically not arrested by small perturbations. In contrast, it has been shown that collapse in the two-dimensional critical NLS can be arrested by various mechanisms (e.g., nonlinear saturation, vectorial effect and nonparaxiality) even when they are small [17]. Therefore, it is more reasonable to try to realize stable bullet propagation (i.e., no collapse) in the planar waveguide two-dimensional case rather than in the bulk media three-dimensional case.

In this paper we present the first asymptotic study of localized pulse dynamics in a grating waveguide. We use a careful multiple-scales analysis to derive a perturbed two-dimensional NLS for the amplitude of the gap bullets. We then use asymptotic analysis and simulations to show that solutions of this perturbed NLS do not collapse. Thus, a unique feature of this model is that collapse arrest is solely due to the specific dispersion relation associated with the grating. While this study is just a first step in the overall study of gap-soliton dynamics, we believe that it indicates the potential for realizing gap solitons in planar waveguides.

2. Slowly-varying envelope approximation

We consider electromagnetic waves propagating in a Kerr medium with a planar waveguide geometry, i.e., the field is confined in one (y) transverse direction and diffracts in the other (x) transverse direction. We assume the usual envelope approximation, which in this case consists of two components (E_+, E_-) each describing the envelope of plane waves propagating in the $\pm z$-directions, respectively.

Nonlinear wave phenomena results from the balance between dispersion and nonlinearity and in many instances the description is given by the NLSE. In this section we show this is the case for gap bullets, at least within a certain range of frequencies.

Our starting point are the nonlinear coupled mode equations in a waveguide configuration [13]:

$$\mathrm{i}(\partial_T + c_\mathrm{g}\partial_z)E_+(T,x,z) + \kappa E_- + \partial^2_{x^2}E_+ + \Gamma(|E_+|^2 + 2|E_-|^2)E_+ = 0,$$

$$\mathrm{i}(\partial_T - c_\mathrm{g}\partial_z)E_-(T,x,z) + \kappa E_+ + \partial^2_{x^2}E_- + \Gamma(|E_-|^2 + 2|E_+|^2)E_- = 0.$$

[1] Ref. [10] highlights several of these applications.

In the linear case (i.e., $\Gamma = 0$) the solution of these equations is given by

$$\begin{pmatrix} E_+ \\ E_- \end{pmatrix} = \mathbf{U}\, \mathrm{e}^{\mathrm{i}(k_z z + k_x x - \Omega T)} + \text{c.c.}, \quad \mathbf{U} = \begin{pmatrix} U_+ \\ U_- \end{pmatrix},$$

where Ω, k_z, k_x satisfy the dispersion relation $(\Omega - k_x^2)^2 = \kappa^2 + c_{\mathrm{g}} k_z^2$. In particular, when $k_x = k_z = 0$ then $\Omega = \pm\kappa$, and the linear problem has the solution:

$$\begin{pmatrix} E_+ \\ E_- \end{pmatrix} = c \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mathrm{e}^{-\mathrm{i}\kappa T} + \text{c.c.}$$

We note that this solution satisfies

$$L \begin{pmatrix} E_+ \\ E_- \end{pmatrix} = 0,$$

where L is the operator

$$L = \begin{bmatrix} \mathrm{i}\partial_T & \kappa \\ \kappa & \mathrm{i}\partial_T \end{bmatrix}.$$

2.1. Weakly nonlinear theory

Our goal is to derive the envelope equation for frequencies in the vicinity of $\Omega = \kappa, k_x = k_z = 0$, when dispersion and nonlinearity are of the same order. Let ϵ be the distance between Ω and κ, i.e., $\Omega = \kappa + \mathrm{O}(\epsilon)$. From the dispersion relation it follows that $k_x, k_z = \mathrm{O}(\epsilon^{1/2})$. Therefore, using the method of multiple scales (see, e.g., [18]) we look for solutions of the form

$$\begin{pmatrix} E_+ \\ E_- \end{pmatrix} = \epsilon^{1/2} A(\tau_1, \tau_2, X, Z) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mathrm{e}^{-\mathrm{i}\kappa T} + \epsilon \mathbf{U}_1 + \epsilon^{3/2} \mathbf{U}_2 + \epsilon^2 \mathbf{U}_3 + \cdots,$$

where $\tau_1 = \epsilon T$, $\tau_2 = \epsilon^2 T$, $X = \epsilon^{1/2} x$ and $Z = \epsilon^{1/2} z$.

We now proceed to solve for (E_+, E_-) for successive orders in ϵ. Balancing the $\mathrm{O}(\epsilon)$ terms gives

$$L\mathbf{U_1} = -\mathrm{i} c_{\mathrm{g}} \partial_Z A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{-\mathrm{i}\kappa T}.$$

This linear problem has the solution

$$\mathbf{U_1} = -\mathrm{i}\frac{c_{\mathrm{g}}}{2\kappa} \partial_Z A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{-\mathrm{i}\kappa T}.$$

In order to go to higher orders we first need a careful computation of the nonlinear terms. We have that

$$(|E_+|^2 + 2|E_-|^2)E_+ = \left(\left|\epsilon^{1/2} A - \mathrm{i}\epsilon \frac{c_{\mathrm{g}}}{2\kappa} \partial_Z A\right|^2 + 2\left|-\epsilon^{1/2} A - \mathrm{i}\epsilon \frac{c_{\mathrm{g}}}{2\kappa} \partial_Z A\right|^2 \right) \left(\epsilon^{1/2} A - \mathrm{i}\epsilon \frac{c_{\mathrm{g}}}{2\kappa} \partial_Z A|^2\right) \mathrm{e}^{-\mathrm{i}\kappa T}.$$

Expanding the square modulus terms gives

$$\begin{aligned} \left|\epsilon^{1/2} A - \mathrm{i}\epsilon \frac{c_{\mathrm{g}}}{2\kappa} \partial_Z A\right|^2 &= \left(\epsilon^{1/2} A - \mathrm{i}\epsilon \frac{c_{\mathrm{g}}}{2\kappa} \partial_Z A\right) \left(\epsilon^{1/2} A^* + \mathrm{i}\epsilon \frac{c_{\mathrm{g}}}{2\kappa} \partial_Z A^*\right) \\ &= \epsilon |A|^2 + \mathrm{i}\epsilon^{3/2} \frac{c_{\mathrm{g}}}{2\kappa} (A \partial_Z A^* - A^* \partial_Z A) + \epsilon^2 \frac{c_{\mathrm{g}}^2}{4\kappa^2} \partial_Z A \partial_Z A^*. \end{aligned}$$

Similarly,

$$2\left|\epsilon^{1/2}A+\mathrm{i}\epsilon\frac{c_{\mathrm{g}}}{2\kappa}\partial_Z A\right|^2=2\left(\epsilon^{1/2}A+\mathrm{i}\epsilon\frac{c_{\mathrm{g}}}{2\kappa}\partial_Z A\right)\left(\epsilon^{1/2}A^*-\mathrm{i}\epsilon\frac{c_{\mathrm{g}}}{2\kappa}\partial_Z A^*\right)$$
$$=2\left(\epsilon|A|^2+\mathrm{i}\epsilon^{3/2}\frac{c_{\mathrm{g}}}{2\kappa}(A^*\partial_Z A-A\partial_Z A^*)+\epsilon^2\frac{c_{\mathrm{g}}^2}{4\kappa^2}\partial_Z A\partial_Z A^*\right).$$

Thus the nonlinear terms read

$$(|E_+|^2+2|E_-|^2)E_+=\left(3\epsilon|A|^2+\mathrm{i}\epsilon^{3/2}\frac{c_{\mathrm{g}}}{2\kappa}(A^*\partial_Z A-A\partial_Z A^*)+3\epsilon^2\frac{c_{\mathrm{g}}^2}{4\kappa^2}\partial_Z A\partial_Z A^*\right)$$
$$\times\left(\epsilon^{1/2}A-\mathrm{i}\epsilon\frac{c_{\mathrm{g}}}{2\kappa}\partial_Z A\right)\mathrm{e}^{-\mathrm{i}\kappa T}+\mathrm{c.c.}$$
$$=\left(3\epsilon^{3/2}|A|^2A-\mathrm{i}\epsilon^2\frac{c_{\mathrm{g}}}{2\kappa}(2|A|^2\partial_Z A+A^2\partial_Z A^*)\right.$$
$$\left.+\epsilon^{5/2}\left(\frac{c_{\mathrm{g}}^2}{2\kappa^2}A\partial_Z A\partial_Z A^*+\frac{c_{\mathrm{g}}^2}{4\kappa^2}A^*(\partial_Z A)^2\right)-\mathrm{i}\epsilon^3\frac{3c_{\mathrm{g}}^3}{8\kappa^3}(\partial_Z A)^2(\partial_Z A^*)\right)\mathrm{e}^{-\mathrm{i}\kappa T}+\mathrm{c.c.}$$

Similarly,

$$(|E_-|^2+2|E_+|^2)E_-=-\left(3\epsilon^{3/2}|A|^2A+\mathrm{i}\epsilon^2\frac{c_{\mathrm{g}}}{2\kappa}(2|A|^2\partial_Z A+A^2\partial_Z A^*)\right.$$
$$\left.+\epsilon^{5/2}\left(\frac{c_{\mathrm{g}}^2}{2\kappa^2 A}\partial_Z A\partial_Z A^*+\frac{c_{\mathrm{g}}^2}{4\kappa^2}A^*(\partial_Z A)^2\right)+\mathrm{i}\epsilon^3\frac{3c_{\mathrm{g}}^3}{8\kappa^3}(\partial_Z A)^2(\partial_Z A^*)\right)\mathrm{e}^{-\mathrm{i}\kappa T}+\mathrm{c.c.}$$

We now continue computing higher order corrections to (E_+,E_-). Balancing the $\mathrm{O}(\epsilon^{3/2})$ terms gives

$$L\mathbf{U_2}=\left(-\mathrm{i}\partial_{\tau_1}A-\partial_{X^2}^2A-\frac{c_{\mathrm{g}}^2}{2\kappa}\partial_{Z^2}^2A-3\Gamma|A|^2A\right)\begin{pmatrix}1\\-1\end{pmatrix}\mathrm{e}^{-\mathrm{i}\kappa T}+\mathrm{c.c.}$$

Note that the right-hand side has slowly-varying terms arising from $\mathbf{U_1}$. Since

$$\begin{pmatrix}1\\-1\end{pmatrix}\mathrm{e}^{-\mathrm{i}\kappa T}$$

is in the null space of L, the physical requirement that

$$\mathrm{i}\partial_{\tau_1}A+\partial_{X^2}^2A+\frac{c_{\mathrm{g}}^2}{2\kappa}\partial_{Z^2}^2+3\Gamma|A|^2A=0, \tag{1}$$

accounts for the removal of secular terms, or solvability condition. Once this condition is imposed, one finds $\mathbf{U_2}=\mathbf{0}$.

The derivation of the two-dimensional NLSE for the bullets amplitude is not surprising, given that the one-dimensional NLSE has been previously derived for fiber grating with no diffraction due to the transverse waveguide structure at frequencies close to but outside the gap. We recall that for the focusing critical NLSE

$$\mathrm{i}\psi_t(t,x,y)+\psi_{xx}+\psi_{yy}+|\psi|^2\psi=0,\qquad \psi(0,x,y)=\psi_0(x,y),$$

collapse can occur when the input power $\int |\psi_0|^2 \,\mathrm{d}x\,\mathrm{d}y$ is above the critical power N_c [19], whereas no collapse occurs for the mixed-signs NLSE

$$\mathrm{i}\psi_t(t,x,y) + \psi_{xx} - \psi_{yy} + |\psi|^2\psi = 0,$$

which is similar to the hydrodynamics problem. Therefore, Eq. (1) has blowup solutions for pulses whose center frequency is at $\Omega \sim \kappa$, but not when $\Omega = -\kappa$, in which case the NLSE is given by

$$\mathrm{i}\partial_{\tau_1} A + \partial^2_{X^2} A - \frac{c_g^2}{2\kappa}\partial^2_{Z^2} + 3\Gamma|A|^2 A = 0.$$

Returning to the case of more interest ($\Omega \sim \kappa$), the possibility of collapse indicates a breakdown of the asymptotic expansion. From physical considerations, however, we do not expect the bullet to collapse. Indeed, since pulse compression leads to spectral broadening, at some point the down-frequency side of the spectral pulse will "see" the edge of the gap in the dispersion relation, thus preventing further broadening and most likely arresting the collapse. Of interest is then to see if this could be modeled by considering higher order corrections in the asymptotic expansion. To do so we introduce a slower time variable $\tau_2 = \epsilon^2 T$. Continuing the expansion we have to $\mathrm{O}(\epsilon^2)$,

$$L\mathbf{U_3} = \left(\mathrm{i}\frac{\Gamma c_g}{2\kappa}(2|A|^2\partial_Z A + A^2\partial_Z A^*) - \mathrm{i}\frac{c_g}{2\kappa}\partial_Z(-\mathrm{i}\partial_{\tau_1} - \partial^2_{X^2})A\right)\begin{pmatrix}1\\1\end{pmatrix}\mathrm{e}^{-\mathrm{i}\kappa T} + \mathrm{c.c.}$$

Using (1) we have that

$$\partial_Z(-\mathrm{i}\partial_{\tau_1} - \partial^2_{X^2})A = \frac{c_g^2}{2\kappa}\partial^3_{Z^3}A + \partial_Z(3\Gamma|A|^2 A).$$

Therefore, the equation for $\mathbf{U_3}$ can be rewritten as

$$L\mathbf{U_3} = -\left[\mathrm{i}\frac{\Gamma c_g}{2\kappa}(4|A|^2\partial_Z A + 2A^2\partial_Z A^*) + \frac{c_g^3}{4\kappa^2}\partial^3_{Z^3}A\right]\begin{pmatrix}1\\1\end{pmatrix}\mathrm{e}^{-\mathrm{i}\kappa T} + \mathrm{c.c.},$$

whose solution is

$$\mathbf{U_3} = -\mathrm{i}\frac{c_g}{4\kappa^2}\left[\Gamma(4|A|^2\partial_Z A + 2A^2\partial_Z A^*) + \frac{c_g^2}{2\kappa}\partial^3_{Z^3}A\right]\begin{pmatrix}1\\1\end{pmatrix}\mathrm{e}^{-\mathrm{i}\kappa T} + \mathrm{c.c.}$$

The $\mathrm{O}(\epsilon^{5/2})$ terms give

$$L\mathbf{U_4} = -\left(\mathrm{i}\partial_{\tau_2}A + \frac{c_g^2}{4\kappa^2}(\Gamma\partial_Z 4|A|^2\partial_Z A + 2A^2\partial_Z A^*) + \frac{c_g^4}{8\kappa^3}\partial^4_{Z^4}A + \frac{\Gamma c_g^2}{4\kappa^2}2A\partial_Z A\partial_Z A^* + A^*(\partial_Z A)^2\right) \times \begin{pmatrix}1\\-1\end{pmatrix}\mathrm{e}^{-\mathrm{i}\kappa T} + \mathrm{c.c.}$$

As in the $\mathrm{O}(\epsilon^{3/2})$ case, in order to prevent secular terms we impose the condition

$$\mathrm{i}\partial_{\tau_2}A + \frac{\Gamma c_g^2}{4\kappa^2}(5A^*(\partial_Z A)^2 + 10A\partial_Z A\partial_Z A^* + 4|A|^2\partial^2_{Z^2}A + 2A^2\partial^2_{Z^2}A^*) + \frac{c_g^4}{8\kappa^3}\partial^4_{Z^4}A = 0. \tag{2}$$

We finish by adding Eqs. (1) and (2) and defining the slow time $\tau = \tau_1 + \epsilon\tau_2$, leading to the perturbed two-dimensional NLSE

$$\mathrm{i}\partial_\tau A + \partial^2_{X^2}A + \frac{c_g^2}{2\kappa}\partial^2_{Z^2}A + 3\Gamma|A|^2 A = -\epsilon F(A), \tag{3}$$

$$F(A) = \frac{\Gamma c_{\rm g}^2}{4\kappa^2}(5A^*(\partial_Z A)^2 + 10A\partial_Z A\partial_Z A^* + 4|A|^2\partial_{Z^2}^2 A + 2A^2\partial_{Z^2}^2 A^*) + \frac{c_{\rm g}^4}{8\kappa^3}\partial_{Z^4}^4 A.$$

3. Analysis of the 2D perturbed NLSE

In the previous section we derived the perturbed NLSE (3) as the slowly-varying envelope approximation of the coupled mode equations. While we do not expect Eq. (3) to remain valid over very long distances, a natural question is whether solutions of (3) can collapse. We now show that collapse is arrested in (3), resulting instead in stable focusing–defocusing oscillations.

In order to bring Eq. (3) to the standard form we make the change of variables

$$y = \frac{\sqrt{2\kappa}}{c_{\rm g}}z, \qquad \psi = \sqrt{3\Gamma}A.$$

This leads to the nondimensional equation

$$\mathrm{i}\psi_\tau(\tau, x, y) + \Delta\psi + |\psi|^2\psi = -\frac{\epsilon}{12\kappa}\left(10\psi|\psi_y|^2 + 5\psi^*\psi_y^2 + 4|\psi|^2\psi_{yy} + 2\psi^2\psi_{yy}^* + \frac{1}{2}\psi_{yyyy}\right), \tag{4}$$

where $\Delta = \partial_{xx} + \partial_{yy}$. When the input power is close to the critical power we can analyze the effects of the small terms on the right hand-side of (4) on collapse and on the propagation dynamics using *modulation theory* [17]. Modulation theory is based on the observation that a self-focusing pulse rearranges itself as a modulated Townesian, i.e., $|\psi| \sim L^{-1}(z)R(r/L(z))$, where $R(r)$, the so-called *Townes soliton*, is the ground-state positive solution of

$$\Delta R(r) - R + R^3 = 0, \quad R'(0) = 0, \quad \lim_{r\to\infty} R(r) = 0. \tag{5}$$

Application of modulation theory to Eq. (4) leads to the following result.

Proposition 1. *When $\epsilon/\kappa \ll 1$, self-focusing dynamics in Eq. (4) is given, to leading order, by the reduced system of ODEs*

$$L_{\tau\tau}(\tau) = -\frac{\beta}{L^3}, \qquad \beta_\tau(\tau) = -\frac{\epsilon}{\kappa}C_{\rm GS}\frac{N_{\rm c}}{2M}\left(\frac{1}{L^2}\right)_\tau, \tag{6}$$

where

$$N_{\rm c} = \int_0^\infty R^2 r\,\mathrm{d}r \approx 1.86, \qquad M = \frac{1}{4}\int_0^\infty r^2R^2r\,\mathrm{d}r \approx 0.55 \quad \textit{and} \quad C_{\rm GS} \approx 55/48.$$

To prove Proposition 1, we first rewrite (4) as the perturbed NLS

$$\mathrm{i}\psi_\tau + \Delta_\perp\psi + |\psi|^2\psi + \frac{\epsilon}{12\kappa}\sum_{i=1}^{5}F^i[\psi] = 0, \tag{7}$$

where

$$F^1[\psi] = 10|\psi_y|^2\psi, \qquad F^2[\psi] = 5(\psi_y)^2\psi^*, \qquad F^3[\psi] = 4|\psi|^2\psi_{yy},$$
$$F^4[\psi] = 2\psi^2\psi_{yy}^*, \qquad F^5[\psi] = \tfrac{1}{2}\psi_{yyyy}. \tag{8}$$

It is known that in critical collapse the focusing core approaches the asymptotic profile $\psi_R(r,\tau)$ which is given by

$$\psi_R(r,\tau) := \frac{1}{L(\tau)} R(\rho)\, \mathrm{e}^{\mathrm{i}S}, \tag{9}$$

where $R(\rho)$ is defined in Eq. (5),

$$\rho(r,\tau) := \frac{r}{L(\tau)}, \qquad S(r,\tau) := \zeta(\tau) + \frac{r^2 L_\tau(\tau)}{4L(\tau)}, \qquad \zeta_\tau(\tau) := \frac{1}{L^2(\tau)}. \tag{10}$$

Under these assumptions, self-focusing dynamics of the perturbed NLS equation (7) is described, to leading order, by Fibich and Papanicolaou [17, Proposition 4.3]

$$L_{\tau\tau}(\tau) = -\frac{\beta}{L^3}, \qquad \beta_\tau(\tau) = \frac{\epsilon}{2M}\left(\sum_{i=1}^{5} f_{1,\tau} - 4\sum_{i=1}^{5} f_2\right), \tag{11}$$

where the auxiliary functions for $F[\psi]$ are given by

$$f_1^i(\tau) = \frac{L}{\pi}\,\mathrm{Re}\int F^i[\psi_R][R(\rho)+\rho R'(\rho)]\,\mathrm{e}^{-\mathrm{i}S}\,\mathrm{d}x\,\mathrm{d}y, \qquad f_2^i(\tau) = \frac{1}{2\pi}\,\mathrm{Im}\int F^i[\psi_R]\psi_R^*\,\mathrm{d}x\,\mathrm{d}y. \tag{12}$$

The leading-order behavior of the auxiliary functions is given in the following lemma.

Lemma 1. *The auxiliary functions corresponding to* Eq. (8) *satisfy*:

$$f_{1,\tau}^1 \sim \frac{10N_c}{3}\left(\frac{1}{L^2}\right)_\tau, \qquad f_2^1 = 0,$$

$$f_{1,\tau}^2 \sim \frac{5N_c}{3}\left(\frac{1}{L^2}\right)_\tau, \qquad f_2^2 = \frac{5N_c}{4}\left(\frac{1}{L^2}\right)_\tau,$$

$$f_{1,\tau}^3 \sim \left(4N_c - \frac{8}{3}I_6\right)\left(\frac{1}{L^2}\right)_\tau, \qquad f_2^3 = -N_c\left(\frac{1}{L^2}\right)_\tau,$$

$$f_{1,\tau}^4 \sim \left(2N_c - \frac{4}{3}I_6\right)\left(\frac{1}{L^2}\right)_\tau, \qquad f_2^4 = \frac{N_c}{2}\left(\frac{1}{L^2}\right)_\tau,$$

$$f_{1,\tau}^5 \sim \frac{1}{4}(3I_6 - 9N_c)\left(\frac{1}{L^2}\right)_\tau, \qquad f_2^5 = 0,$$

where $I_6 \approx 6N_c$.

The auxiliary functions of F^1 and F^2 are calculated in [20] and the auxiliary functions of F^5 are calculated in [17]. To calculate the other auxiliary functions we note that (see [20, Appendices])

$$S_{yy} = \frac{L_\tau}{2L}, \qquad R_y = \frac{R'\cos\theta}{L}, \qquad R_{yy} = \frac{1}{L^2}\left(R''\cos^2\theta + \frac{1}{\rho}R'\sin^2\theta\right),$$

$$\psi_{R,yy} = L^{-1}\,\mathrm{e}^{\mathrm{i}S}[(R_{yy} - RS_y^2) + \mathrm{i}(2R_yS_y + RS_{yy})],$$

$$\int R^4\rho\,\mathrm{d}\rho = 2N_c,$$

$$I_1 := 3\int R^2R'^2\rho\,\mathrm{d}\rho, \qquad I_3 := 3\int \rho R^2R'R''\rho\,\mathrm{d}\rho, \qquad I_4 := 3\int R^3R''\rho\,\mathrm{d}\rho, \qquad I_6 := \int R^6\rho\,\mathrm{d}\rho,$$

$$I_1 = I_6 - 2N_c, \qquad I_3 = -N_c, \qquad I_4 = 6N_c - 3I_6 + \tfrac{3}{4}R^4(0).$$

Therefore, for $f_1^3[\psi]$ we have that

$$\begin{aligned} f_1^3 &:= \frac{L}{\pi}\,\mathrm{Re}\int F^3[\psi_R](R+\rho R')\,\mathrm{e}^{-\mathrm{i}S}\,\mathrm{d}x\,\mathrm{d}y = \frac{4L}{\pi}\mathrm{Re}\int |\psi_R|^2\psi_{R,yy}(R+\rho R')\,\mathrm{e}^{-\mathrm{i}S}\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{4}{\pi L^2}\int R^2(R_{yy}-RS_y^2)(R+\rho R')\,\mathrm{d}x\,\mathrm{d}y \approx \frac{4}{\pi L^2}\int R^2 R_{yy}(R+\rho R')\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{4}{\pi L^2}\int R^2\left(R''\cos^2\theta + \frac{1}{\rho}R'\sin^2\theta\right)(R+\rho R')\rho\,\mathrm{d}\rho\,\mathrm{d}\theta = \frac{4}{L^2}\int R^2\left(R''+\frac{1}{\rho}R'\right)(R+\rho R')\rho\,\mathrm{d}\rho \\ &= \frac{4}{L^2}\int\left(R^3R''+\rho R^2R'R''+R^2R'^2+\frac{1}{\rho}R^3R'\right)\rho\,\mathrm{d}\rho \\ &= \frac{4}{3}\left[I_4+I_3+I_1-\frac{3}{4}R^4(0)\right]\frac{1}{L^2} = \left(4N_c-\frac{8}{3}I_6\right)\frac{1}{L^2}. \end{aligned}$$

Similarly, for $f_1^4[\psi]$ we get that

$$\begin{aligned} f_1^4 &:= \frac{L}{\pi}\,\mathrm{Re}\int F^4[\psi_R](R+\rho R')\,\mathrm{e}^{-\mathrm{i}S}\,\mathrm{d}x\,\mathrm{d}y = \frac{2L}{\pi}\,\mathrm{Re}\int \psi_R^2\psi_{R,yy}^*(R+\rho R')\,\mathrm{e}^{-\mathrm{i}S}\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{2}{\pi L^2}\int R^2(R_{yy}-RS_y^2)(R+\rho R')\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}f_1^3 \sim \left(2N_c-\frac{4}{3}I_6\right)\frac{1}{L^2}. \end{aligned}$$

For $f_2^3[\psi]$ we have that

$$\begin{aligned} f_2^3 &:= \frac{2}{\pi}\mathrm{Im}\int |\psi_R|^2\psi_{R,yy}\psi_R^*\,\mathrm{d}x\,\mathrm{d}y = \frac{2}{\pi L^4}\int R^3(2R_yS_y+RS_{yy})\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{2}{\pi L^4}\int\left[\frac{1}{2}(R^4)_yS_y+R^4S_{yy}\right]\mathrm{d}x\,\mathrm{d}y = \frac{1}{\pi L^4}\int R^4S_{yy}\,\mathrm{d}x\,\mathrm{d}y = \frac{L_\tau}{L^3}\int R^4\rho\,\mathrm{d}\rho = -N_c\left(\frac{1}{L^2}\right)_\tau. \end{aligned}$$

Similarly, for $f_2^4[\psi]$ we get that

$$f_2^4 := \frac{1}{\pi}\mathrm{Im}\int \psi_R^2\psi_{R,yy}^*\psi_R^*\,\mathrm{d}x\,\mathrm{d}y = -\frac{1}{\pi L^4}\int R^3(2R_yS_y+RS_{yy})\,\mathrm{d}x\,\mathrm{d}y = -\frac{1}{2}f_2^3 = \frac{N_c}{2}\left(\frac{1}{L^2}\right)_\tau.$$

To obtain Eq. (6) we substitute the auxiliary functions from Lemma 1 in (11). Therefore, to leading order we obtain

$$\begin{aligned} \beta_\tau &= \frac{\epsilon}{12\kappa}\left[\frac{10N_c}{3}+\frac{5N_c}{3}+\left(4N_c-\frac{8}{3}I_6\right)+\left(2N_c-\frac{4}{3}I_6\right)\right. \\ &\qquad\left.+\frac{1}{4}(3I_6-9N_c)-5N_c+4N_c-2N_c\right]\frac{1}{2M}\left(\frac{1}{L^2}\right)_\tau \\ &= \frac{1}{48}(23N_c-13I_6)\frac{\epsilon}{\kappa}\frac{1}{2M}\left(\frac{1}{L^2}\right)_\tau \approx -\frac{55}{48}\frac{\epsilon}{\kappa}\frac{N_c}{2M}\left(\frac{1}{L^2}\right)_\tau, \end{aligned}$$

from which we obtain the second equation in (6) with $C_{GS} \approx 55/48$.

In [17] it was shown that (6) is the generic reduced equation in critical self-focusing. Analysis of this reduced system shows that its solutions do not collapse. Rather, they undergo stable focusing–defocusing oscillations. Indeed, we observe this behavior when we solve Eq. (4) numerically (see Fig. 1).

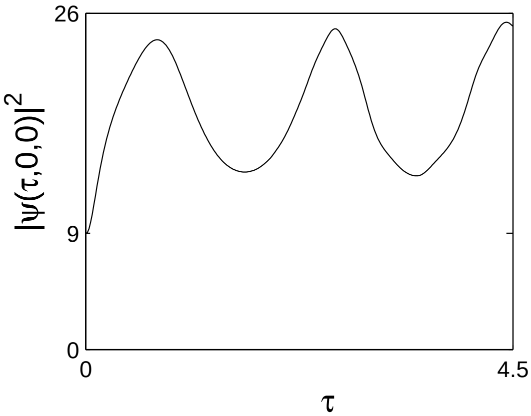

Fig. 1. On-axis intensity of the solution of Eq. (4) with $\epsilon/\kappa = 0.075$. The initial condition is $\psi(\tau = 0, x, y) = 2\sqrt{N(0)}\,e^{-(x^2+y^2)}$, and $N(0) = 1.2N_c$.

4. Numerical methods

Because the small perturbation terms in Eq. (4) depend only on the y derivatives, this equation is anisotropic in the (x, y) plane. Hence, even though we use a radially-symmetric Gaussian for the initial conditions, the solution does not remain cylindrically-symmetric during propagation. We therefore solve Eq. (4) on a rectangular Cartesian domain, using a finite-difference scheme with fourth-order accuracy in space. Time-stepping is achieved using a fourth-order Runge–Kutta algorithm. We impose Dirichlet boundary conditions at the outer boundaries. Since Dirichlet conditions are reflective, special care is taken to assure that reflections from the numerical boundaries have no effect on the solution. This is especially important when solving Eq. (4), because the radiation can propagate quickly in the y-direction. We therefore take sufficiently large numerical boundaries in the y-direction and verify the validity of our results by using a larger domain.

5. Conclusions

We have used a multiple-scales analysis to derive the extended NLSE approximation for optical bullet dynamics in a waveguide with a grating in the direction of propagation. We then showed that in this extended NLSE, collapse is arrested. An intuitive explanation for this is as follows. When the field frequency lies outside of the gap in the dispersion relation, the dynamics is initially well approximated by the two-dimensional NLSE and for sufficiently high input power a collapse-type dynamics occurs (see Fig. 1). Since critical collapse occurs in a radially symmetric way [21,22], the resulting spectral broadening occurs in all wavenumber directions. Our analysis thus shows that the fact that broadening of the z-component of the wavenumber is prevented by the presence of the band-gap is enough to prevent the collapse,[2] leading instead to nontrivial dynamics which is yet to be fully understood. This suggests, nevertheless, that the periodic waveguide structure is a suitable candidate for first observing gap-soliton bullets in short propagation, if only as a metastable object. Better understanding of the overall dynamics, however, will require a direct analysis of the coupled mode equations.

[2] It remains to see whether such a small effect would be able to prevent the collapse in bulk media, where collapse is supercritical.

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