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An elementary proof of the common maximal bid in asymmetric first-price and all-pay auctions

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HIGHLIGHTS

- We generalize the conditions under which the maximal bids in asymmetric auctions are the same.
- We show that the bidders do not have to be risk neutral.
- We show that the distributions of their valuations do not have to be independent.

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ABSTRACT

We prove that the maximal bid in asymmetric first-price and all-pay auctions is the same for all bidders. Our proof is elementary, and does not require that bidders are risk neutral, or that the distribution functions of their valuations are independent or smooth.

1. Introduction

Consider n bidders that compete for a single item. Each bidder's valuation for the item is private information to that bidder and is drawn according to a distribution function that is common knowledge. The bidders are asymmetric, i.e., their valuations are drawn according to different distribution functions.

In the case of asymmetric first-price auctions, Maskin and Riley (2000) and Lebrun (1999) proved the existence and uniqueness of an equilibrium when bidders are risk neutral and their distribution functions are independent and have the same compact support. In particular, they showed that the maximal bids of all bidders are the same. In the case of asymmetric all-pay auctions with risk-averse bidders whose distribution functions are independent and have the same compact support, Parreiras and Rubinchik (2010) proved that the maximal bids of all bidders in equilibrium are the same.

In the case of two asymmetric bidders, a simple argument shows that the maximal bid of both bidders in equilibrium has to be identical. Indeed, if the maximal bid of bidder 1 is larger than that of bidder 2, she can lower her maximal bid slightly and still win with probability 1 but pay less, which contradicts the definition of an equilibrium. In this study we provide an elementary proof for $n \geq 3$, which is essentially a revealed-preference argument from the perspective of the highest type of two bidders whose maximum bids differ. This argument enables us to prove this result for the following general settings.

1. The auction mechanism can be a first-price or an all-pay one.
2. The distribution functions of bidders' valuations can be dependent.
3. The bidders do not have to be risk neutral, and they can have a nonlinear utility function $U(x)$.
4. The distribution functions of bidders' valuations can be nonsmooth.

The main novelty of this study is that it removes the assumption that the distribution functions are independent. Thus, the proof

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of Parreiras and Rubinchik (2010) can be extended to first-price auctions and to nonsmooth distributions, but it requires the independence assumption.[1] We also note that one aspect in which this study is less general than that of Lebrun (1999) is that it only considers pure strategy equilibria, while Lebrun (1999) allows for mixed strategies.

2. The model

Consider n players that bid for an indivisible object, in which the highest bidder wins the object and pays her bid, and all other bidders pay s times their bids. Thus, $s = 0$ is the first-price auction, and $s = 1$ is the all-pay auction. The value of bidder i, denoted by v_i, is private information to herself and it is drawn from the interval $[0, 1]$ according to a distribution function F_i which is common knowledge ($i = 1, \dots, n$). All bidders have the same utility function $U(x)$, where $U(0) = 0$ and $U' > 0$, so the utility of bidder i when her value is v_i and she submits a bid of b_i is $U(v_i - b_i)$ if she wins the object, $U(0) = 0$ if she does not win and pays nothing, and $U(-b_i)$ if she does not win and pays her bid. Denote by $\{b_i(v_i)\}_{i=1}^n$ the equilibrium bidding strategies. If bidder i has value v_i and she submits a bid of b_i, her expected utility (assuming that all other bidders follow their equilibrium strategies) is

$$P\left(\max_{m\neq i} b_m(v_m) < b_i\right) U(v_i - b_i) + s\left(1 - P\left(\max_{m\neq i} b_m(v_m) < b_i\right)\right) U(-b_i).$$

Theorem 1. *Assume that the following hold.*

1. *The joint density of the values is strictly positive, i.e.,*
$$f(v_1, \dots, v_n) > 0, \quad (v_1, \dots, v_n) \in (0, 1)^n. \tag{1}$$
2. *The utility function satisfies $U(0) = 0$ and $U' > 0$.*
3. *The equilibrium strategies $\{b_i(v_i)\}_{i=1}^n$ exist and are strictly monotonically increasing.*

Then the maximal bids of all bidders are the same, i.e., there exists $\bar{b}$ such that $b_i(1) = \bar{b}$ for $i = 1, \dots, n$.

Proof. Let k denote the number of bidders that attain the maximal bid in equilibrium, and denote these bidders by $i_1, i_2, \dots, i_k$. Thus

$$\bar{b}_{i_1} = \cdots = \bar{b}_{i_k} > \max_{i\neq i_1,\dots,i_k} \bar{b}_i, \qquad \bar{b}_i := b_i(1) = \max_{0\le v_i\le 1} b_i(v_i).$$

- If $k = n$, we are done.
- Assume by negation that $k = 1$, i.e., that the maximal bid is attained by the single bidder i_1. If bidder i_1 has value 1 and she submits a bid which is slightly below $\bar{b}_{i_1}$ but still above $\max_{i\neq i_1} \bar{b}_i$, she still wins with probability 1 but pays less, and so her utility increases, which contradicts the definition of an equilibrium.
- Assume by negation that $1 < k < n$. Denote by j a bidder with the second-highest maximal bid. Thus,
$$\bar{b}_{i_1} = \cdots = \bar{b}_{i_k} > \bar{b}_j \ge \max_{i\neq i_1,\dots,i_k,j} \bar{b}_i.$$
Since bidder j is in equilibrium, her expected utility when her value is 1 and she bids $\bar{b}_j$ is higher or equal than her expected utility when her value is 1 and she increases her bid to $\bar{b}_{i_1}$ (and thus wins with probability 1). Therefore,
$$P\left(\max_{i\neq j} b_i < \bar{b}_j\right) U(1 - \bar{b}_j) + s\left(1 - P\left(\max_{i\neq j} b_i < \bar{b}_j\right)\right) U(-\bar{b}_j) \ge U(1 - \bar{b}_{i_1}).$$
Similarly, since bidder i_1 is in equilibrium, her expected utility when her value is 1 and she bids $\bar{b}_{i_1}$ (and thus wins with probability 1) is higher or equal than her expected utility when her value is 1 and she lowers her bid to $\bar{b}_j$, i.e.,
$$U(1 - \bar{b}_{i_1}) \ge P\left(\max_{i\neq i_1} b_i < \bar{b}_j\right) U(1 - \bar{b}_j) + s\left(1 - P\left(\max_{i\neq i_1} b_i < \bar{b}_j\right)\right) U(-\bar{b}_j).$$
Combining the last two inequalities gives
$$P\left(\max_{i\neq j} b_i < \bar{b}_j\right) U(1 - \bar{b}_j) + s\left(1 - P\left(\max_{i\neq j} b_i < \bar{b}_j\right)\right) U(-\bar{b}_j) \ge P\left(\max_{i\neq i_1} b_i < \bar{b}_j\right) U(1 - \bar{b}_j) + s\left(1 - P\left(\max_{i\neq i_1} b_i < \bar{b}_j\right)\right) U(-\bar{b}_j).$$
Since $\bar{b}_j < 1$, $U(1 - \bar{b}_j) > 0$, and so we have that $U(1 - \bar{b}_j) - sU(-\bar{b}_j) > 0$. Therefore, it follows that
$$P\left(\max_{i\neq j} b_i < \bar{b}_j\right) \ge P\left(\max_{i\neq i_1} b_i < \bar{b}_j\right).$$
Since $\bar{b}_j$ is the second-highest maximal bid, it can only be exceeded by bidders $i_1, i_2, \dots, i_k$. Hence, the last inequality can be rewritten as
$$P\left(\max\{b_{i_1}, b_{i_2}, \dots, b_{i_k}\} < \bar{b}_j\right) \ge P\left(\max\{b_{i_2}, \dots, b_{i_k}\} < \bar{b}_j\right).$$
Therefore, it follows that
$$P\left(b_{i_1} > \bar{b}_j, \max\{b_{i_2}, \dots, b_{i_k}\} < \bar{b}_j\right) = 0,$$
which is in contradiction with (1). ⊡

Condition (1) holds, in particular, if $\{F_i\}$ are independent and monotonically increasing. Note that the $\{F_i\}$ do not need to be smooth or even continuous. Furthermore, from the proof it immediately follows that Condition (1) can be replaced with the weaker condition that there exists $0 < \delta \ll 1$ such that

$$f(v_1, \dots, v_n) > 0 \quad \text{if } v_i \in (0, \delta) \cup (1 - \delta, 1) \text{ for } i = 1, \dots, n,$$

i.e., that the joint density is strictly positive for any combination of low and high types.

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[1] This is because in the proof of their Lemma 3 they used the independence assumption to obtain the identity $W_i(b)G_i(b) = W_j(b)G_j(b)$.