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A modulation method for self-focusing in the perturbed critical nonlinear Schrödinger equation

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Abstract

In this Letter we introduce a systematic perturbation method for analyzing the effect of small perturbations on critical self-focusing by reducing the perturbed critical nonlinear Schrödinger equation (PNLS) to a simpler system of modulation equations that do not depend on the transverse variables. The modulation equations can be further simplified depending on whether PNLS is power conserving or not. An important and somewhat surprising result is that various small defocusing perturbations lead to a canonical form for the modulation equations, whose solutions have slowly decaying focusing–defocusing oscillations. © 1998 Elsevier Science B.V.

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1. Introduction

The perturbed critical nonlinear Schrödinger equation (PNLS)

$$i\psi_z + \Delta_\perp\psi + |\psi|^2\psi + \epsilon F(\psi, \psi_z, \nabla_\perp\psi, \dots) = 0,$$
$$\psi(0,x,y) = \psi_0(x,y) \qquad (1)$$
$$\Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad 0 < \epsilon \ll 1,$$

arises in various physical models in nonlinear optics [3], plasma physics and fluid dynamics (see, e.g., Table 1). When $\epsilon = 0$, Eq. (1) reduces to the critical nonlinear Schrödinger equation (CNLS)

$$i\psi_z + \Delta_\perp\psi + |\psi|^2\psi = 0 \ . \qquad (2)$$

We recall that for the nonlinear Schrödinger equation with a general nonlinearity σ and transverse dimension D,

$$i\psi_z + \left(\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_D^2}\right)\psi + |\psi|^{2\sigma}\psi = 0;$$

we distinguish between three different cases.

(i) When $\sigma D < 2$, the subcritical case, diffraction always dominates and focusing singularities do not form.

(ii) In the supercritical case, $\sigma D > 2$, there is a large class of smooth initial amplitudes for which a fo-

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[3] The amplitude ψ may depend on additional variables, such as t in the case of time dispersion.

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cusing singularity forms with finite distance z. Since in supercritical self-focusing the nonlinearity dominates over diffraction, addition of small perturbations to the equation has a small effect.

(iii) In the critical case, $\sigma D = 2$ (as in the case of Eq. (2)), solutions can also become singular with a finite z. However, in this borderline case between subcritical and supercritical self-focusing, singularity formation is characterized by a near-balance between the focusing nonlinearity and diffraction. As a result, critical self-focusing is extremely sensitive to small perturbations, which can have a large effect and can even lead to the arrest of collapse.

Self-focusing is a genuinely nonlinear phenomenon and standard linearization methods cannot be used to analyze singularity formation in Eqs. (1) and (2). In addition, methods such as the inverse scattering transform (IST), which is so successful in the 1D cubic subcritical case, cannot be applied to Eq. (2), because Eq. (2) is not integrable. Self-focusing in Eq. (1) or (2) is, moreover, a local phenomenon which cannot be accurately captured by global estimates. For these reasons, despite considerable progress the present theory of critical self-focusing in the presence of small perturbations is still far from complete.

In this Letter we present a general method for analyzing the effect of a any deterministic or random perturbation on critical self-focusing. In this method PNLS (1) is reduced to a simpler system of modulation equations which do not depend on the transverse variables. The reduced system is much easier to analyze and to simulate, and it provides insights that are hard to get directly from PNLS.

2. Review of critical self-focusing

CNLS (2) has two important conserved quantities: the power

$$N := \frac{1}{2\pi}\int |\psi|^2 \,\mathrm{d}x\,\mathrm{d}y,$$

and the Hamiltonian

$$H(\psi) := \frac{1}{2\pi}\left(\int |\nabla_\perp \psi|^2 \,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}\int |\psi|^4 \,\mathrm{d}x\,\mathrm{d}y\right),$$

$$\nabla_\perp = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right).$$

A sufficient condition for singularity formation in Eq. (2) is

$$H(\psi_0) < 0,$$

while a necessary condition is

$$N(\psi_0) \geqslant N_c \simeq 1.86.$$

CNLS has waveguide solutions of the form

$$\psi = \exp(\mathrm{i}t) R(r), \quad r = (x^2 + y^2)^{1/2},$$

where $R(r)$ satisfies

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) R - R + R^3 = 0, \qquad R'(0) = 0, \quad \lim_{r\to\infty} R(r) = 0. \tag{3}$$

The solution of Eq. (3) with the lowest power ("ground state"), sometimes called the Townes soliton, has an important role in self-focusing theory. This positive, monotonically decreasing solution has exactly the critical power for self-focusing

$$\int_0^\infty R^2 r \,\mathrm{d}r = N_c, \tag{4}$$

and its Hamiltonian is equal to zero

$$H(R) = 0. \tag{5}$$

The analysis of blow-up in CNLS is based on the assumption (which is supported by numerical and analytical evidence) that near the singularity the solution is roughly a modulated Townes soliton,

$$\psi \sim \psi_R, \tag{6}$$

where

$$\psi_R := \frac{1}{L} R(\rho) \exp(\mathrm{i}S),$$

$$\rho = \frac{r}{L}, \quad S = \zeta + \frac{L_z}{L}\frac{r^2}{4}, \tag{7}$$

and

$$\frac{\mathrm{d}\zeta}{\mathrm{d}z} = \frac{1}{L^2}. \tag{8}$$

Table 1
Perturbations of critical NLS and their corresponding modulation equations. Here $I_6 = \int_0^\infty R^6 r\,dr$. A † means that f_1 is given by Eq. (16). The last column indicates where the derivation can be found

Perturbed CNLS	Application	Reduced equation	Conservative	Ref.
$i\psi_z + \Delta_\perp\psi + \|\psi\|^2\psi + \epsilon\psi_{xxxx} + \frac{2}{5}\epsilon^2\psi_{xxxxxx} = 0$	fiber arrays	$(y_z)^2 = -\frac{4H_0}{My^2}(y_M - y)(y - y_m)(y - y_3)$	yes	[6]
$i\psi_z + \Delta_\perp\psi + \|\psi\|^2\psi - \epsilon\|\psi\|^4\psi = 0$	quintic nonlinearity	(17) with $C_1 = \frac{4}{3}I_6$	yes†	[4]
$i\psi_z + \Delta_\perp\psi + \frac{1}{2\epsilon}[1 - \exp(-2\epsilon\|\psi\|^2)]\psi = 0$	saturating nonlinearity	(17) with $C_1 = \frac{4}{3}I_6$	yes†	[6]
$i\psi_z + \Delta_\perp\psi + \frac{\|\psi\|^2}{1+\epsilon\|\psi\|^2}\psi = 0$	Saturating nonlinearity	(17) with $C_1 = \frac{4}{3}I_6$	yes†	[6]
$i\psi_z + \Delta_\perp\psi + \|\psi\|^2\psi - \epsilon\phi_x\psi = 0$, $\alpha\phi_{xx} + \phi_{yy} = -(\|\psi\|^2)_x$	Davey-Stewartson	(9), (10)	yes	[6]
$i\psi_z + \Delta_\perp\psi + \|\psi\|^2\psi + \epsilon\psi_{zz} = 0$	nonparaxiality	(17) with $C_1 = 4N_c$	no	[7]
$i\psi_z + \Delta_\perp\psi + \|\psi\|^2\psi + \epsilon(x^2+y^2)h(z)\psi = 0$	h random	$-L^3L_{zz} = \beta_0 + 4\epsilon L^4 h(z)$	yes	[6]
$i\psi_z + \Delta_\perp\psi\|\psi\|^2\psi + \epsilon_1(x^2+y^2) \times h(z)\psi - \epsilon_2\|\psi\|^4\psi = 0$	Quintic nonlinearity + h random	$-L^3L_{zz} = \beta_0 + 4\epsilon_1 L^4 h(z) - \frac{2\epsilon I_6}{3M}\frac{1}{L^2}$	yes	[6]
$i\psi_z + \Delta_\perp\psi + \|\psi\|^2\psi - \epsilon\psi_{tt} = 0$	time dispersion	$\beta_z = \frac{2\epsilon N_c}{M}\zeta_{tt}$, (8), (10)	no	[8]
$i\psi_z + \Delta_\perp\psi + N\psi = 0$, $\epsilon N_t + N = \|\psi\|^2$	Debye relaxation	$-L^3L_{zz} = \beta_0 + \frac{\epsilon C_D}{2M}\frac{L_t}{L}$, $C_D = \int(\nabla_\perp R^2)^2 r^3\,dr \simeq 6.43$	yes	[6]
$i\psi_z + \Delta_\perp\psi + \|\psi\|^2\psi + \epsilon_1\psi_{zz} + \epsilon_2\left(2i\frac{n_0 c_g}{c}(\|\psi\|^2\psi)_t - \psi_{zt}\right) - \epsilon_3\psi_{tt} = 0$	time dispersion + nonparaxiality	$\beta_z = -\frac{2\epsilon_1 N_c}{M}\left(\frac{1}{L^2}\right)_z - \left(6c_g\frac{n_0}{c} - 2\right)\frac{\epsilon_2 N_c}{M}\left(\frac{1}{L^2}\right)_t + \frac{2\epsilon_3 N_c}{M}\zeta_{tt}$, (8), (10)	no	[9]

More precisely, near the singularity, ψ_s, the inner part of the solution[4], whose power is slightly above critical, collapses towards the singularity in a quasi-self-similar fashion,

$$\psi_s \sim \frac{1}{L}V(\zeta,\rho)\exp(iS),$$

[4] A possible definition is $\psi_s = \psi$ for $0 \leqslant \rho \leqslant \rho_c$, with $1 \ll \rho_c$ constant.

where

$$V \to R \quad \text{as } z \to Z_c,$$

and Z_c is the location of the singularity. Based on this modulation ansatz, it was shown that near the singularity self-focusing can be described by the reduced system [1–3]

$$\beta_z = -\frac{\nu(\beta)}{L^2}, \tag{9}$$

$$L_{zz} = -\frac{\beta}{L^3}, \tag{10}$$

where

$$\nu(\beta) \sim c\exp\left(\frac{\pi}{\beta^{1/2}}\right), \quad c \simeq 45.1.$$

In order to motivate the system (9), (10), we note that the modulation variable L is the radial width as well as 1/amplitude of the focusing part ψ_s, and that β is proportional to the excess power above critical of ψ_s [4]. Therefore, at the point of blow-up $L(Z_c) = \beta(Z_c) = 0$. The $\nu(\beta)$ term arises from radiation effects (power losses of ψ_s) during self-focusing. Since near the singularity

$$0 \leqslant \beta(z) \ll 1,$$

$\nu(\beta)$ is exponentially small and self-focusing is essentially adiabatic.

2.1. Adiabatic approach

Originally, the reduced system (9), (10) was analyzed by solving Eq. (9) to leading order near Z_c and then using Eq. (10). This leads to the *log–log law* for the rate of critical blow-up [1–3]. However, it turns out that the log–log law does not become valid even after amplification of the peak amplitude by a factor of a billion or more, which is long after the nonlinear Schrödinger equation ceases to be physically relevant. Fortunately, this can be "fixed" by solving Eqs. (9) and (10) using an adiabatic approach. Since changes in β (i.e. the power of ψ_s) are exponentially small compared with the focusing rate, we first solve Eq. (10) with β constant, and only then add the nonadiabatic effects (9) as the next-order correction. Application of this approach leads to an *adiabatic law* for critical self-focusing, which is valid almost from the onset of self-focusing [5].

3. Modulation theory for self-focusing in the perturbed CNLS

The adiabatic law, which provides an accurate description of critical self-focusing in the domain of physical interest, is obtained in two stages: (i) derivation of the reduced modulation equations (9) and (10), which do not depend on the transverse variables and (ii) solving these equations with the radiation term $\nu(\beta)$ neglected to leading order (adiabatic approach). In this section we extend this approach to self-focusing in PNLS: (i) The modulation ansatz (6) is used in Proposition 3.1 to reduce Eq. (1) to the system (11), and (ii) the reduced system is analyzed with the adiabatic approach (Propositions 3.2 and 3.3). More details, as well as the proof of the results are published elsewhere [6].

For modulation theory to be valid, the following *three conditions* must hold.

(i) The focusing part of the solution is close to the asymptotic profile (6).

(ii) The power of the focusing part is close to critical,

$$\left|\frac{1}{2\pi}\int |\psi_s(z,x,y)|^2\, dx\, dy - N_c\right| \ll 1,$$

or equivalently,

$$|\beta(z)| \ll 1.$$

(iii) The perturbation is small:

$$|\epsilon F| \ll |\triangle_\perp \psi|, |\epsilon F| \ll |\psi|^3.$$

In general, at the onset of self-focusing only condition (iii) holds. As the solution approaches Z_c (the blow-up point in the absence of the perturbation), conditions (i) and (ii) are also satisfied and modulation theory becomes valid.

The main result of modulation theory is the following.

Proposition 3.1. If conditions (i)–(iii) hold and if F is an even function in x and y, self-focusing in PNLS (1) is given to leading order by the reduced system

$$\beta_z(z) + \frac{\nu(\beta)}{L^2} = \frac{\epsilon}{2M}(f_1)_z - \frac{2\epsilon}{M} f_2,$$

$$L_{zz}(z) = -\frac{\beta}{L^3}. \tag{11}$$

The auxiliary functions f_1 and f_2 are given by

$$f_1(z) = 2L(z)\,\mathrm{Re}\left[\frac{1}{2\pi}\int F(\psi_R)\exp(-\mathrm{i}S) \times [R(\rho) + \rho\nabla_\perp R(\rho)]\,\mathrm{d}x\,\mathrm{d}y\right], \qquad (12)$$

$$f_2(z) = \mathrm{Im}\left[\frac{1}{2\pi}\int \psi_R^* F(\psi_R)\,\mathrm{d}x\,\mathrm{d}y\right], \qquad (13)$$

where

$$M = \tfrac{1}{4}\int_0^\infty r^3 R(r)\,\mathrm{d}r \simeq 0.55.$$

We note that assuming that we can carry out the transverse integration, f_1 and f_2 are known functions of the modulation variables L, β, ζ and their derivatives.

3.1. Conservative and nonconservative perturbations

A considerable simplification is achieved if we distinguish between conservative perturbations, i.e. those for which the power remains conserved,

$$\frac{\mathrm{d}}{\mathrm{d}z}\int |\psi(z,x,y,\cdot)|^2\,\mathrm{d}x\,\mathrm{d}y \equiv 0$$

in Eq. (1), and nonconservative perturbations.

Proposition 3.2. Let conditions (i)–(iii) hold.
(i) If F is a *conservative perturbation*, i.e.

$$\mathrm{Im}\left[\int \psi^* F(\psi)\,\mathrm{d}x\,\mathrm{d}y\right] \equiv 0,$$

then $f_2 \equiv 0$, and to leading order Eq. (11) reduces to

$$-L^3 L_{zz} = \beta_0 + \frac{\epsilon}{2M} f_1,$$
$$\beta_0 = \beta(0) - \frac{\epsilon}{2M} f_1(0), \qquad (14)$$

where β_0 is independent of z.
(ii) If F is a *nonconservative perturbation*, i.e.

$$\mathrm{Im}\left[\int \psi^* F(\psi)\,\mathrm{d}x\,\mathrm{d}y\right] \not\equiv 0,$$

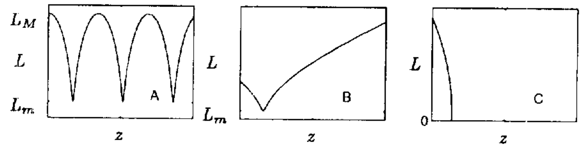

Fig. 1. The leading-order effect of the canonical conservative perturbation (16). (a) Defocusing perturbation and $H_0 < 0$ (Proposition 3.3ia); (b) defocusing perturbation, $H_0 > 0$ and $L_z(0) < 0$ (Proposition 3.3ib1); (c) focusing perturbation and $L_z(0) < 0$ (Proposition 3.3ii). In all cases $\beta_0 > 0$ (i.e. power above critical).

then to leading order Eq. (11) reduces to

$$\beta_z = -\frac{2\epsilon}{M} f_2, \quad L_{zz} = -\frac{\beta}{L^3}. \qquad (15)$$

Note that in both cases, nonadiabatic effects disappear from the leading-order behavior of Eq. (11).

3.2. Canonical effect of conservative perturbations

It has been observed that various seemingly different small perturbations have the same effect: arrest of collapse, followed by focusing–defocusing cycles (see Fig. 1a). In the next Proposition we use modulation theory to explain this observation, by showing that all conservative perturbations for which f_1 is of the form [5]

$$f_1 \sim -\frac{C_1}{L^2}, \quad C_1 = \text{const}, \qquad (16)$$

have the same qualitative effect on self-focusing.

Proposition 3.3. When self-focusing is given by Eq. (14) and f_1 is given by Eq. (16), then $y := L^2$ satisfies the canonical oscillator equation

$$(y_z)^2 = -\frac{4H_0}{M}\frac{1}{y}(y_M - y)(y - y_m), \qquad (17)$$

where

$$y_M = -\frac{\sqrt{\beta_0^2 + \epsilon C_1 H_0/M^2} + \beta_0}{2H_0/M}$$
$$= -\frac{M\beta_0}{H_0}\left[1 + O\left(\frac{\epsilon H_0}{\beta_0^2}\right)\right] \qquad (18)$$

[5] For example, those marked in Table 1 with a †.

$$y_{\mathrm{m}} = \frac{\epsilon C_1}{2M}\frac{1}{\sqrt{\beta_0^2+\epsilon C_1 H_0/M^2}+\beta_0} = \frac{\epsilon C_1}{4M\beta_0}\left[1+\mathrm{O}\left(\frac{\epsilon H_0}{\beta_0^2}\right)\right], \tag{19}$$

$$\beta_0 = \beta(0) + \frac{\epsilon C_1}{2ML^2(0)},$$

$$H_0 \sim H(0) + \frac{\epsilon C_1}{4}\frac{1}{L^4(0)}.$$

Let us define $L_{\mathrm{m}} := y_{\mathrm{m}}^{1/2}$, $L_{\mathrm{M}} := y_{\mathrm{M}}^{1/2}$.

(i) If the perturbation is *defocusing*, i.e.

$$\epsilon C_1 > 0, \tag{20}$$

then it will arrest blow-up in Eq. (14), i.e. L (and y) will remain positive for all z.

(ia) If, in addition to Eq. (20), $\beta_0 > 0$ and $H_0 < 0$, then $0 < L_{\mathrm{m}} < L_{\mathrm{M}}$, and L will go through periodic oscillations between L_{m} and L_{M} (Fig. 1a). The period of the oscillations is

$$\Delta Z = 2\sqrt{-\frac{My_{\mathrm{M}}}{H_0}}E\left(1-\frac{y_{\mathrm{m}}}{y_{\mathrm{M}}}\right), \tag{21}$$

where $E(m) = \int_0^{\pi/2}(1-m\sin^2\theta)^{1/2}\,\mathrm{d}\theta$ is the complete elliptic integral of the second kind.

(ib) If in addition to Eq. (20), $\beta_0 > 0$ and $H_0 > 0$, then

(ib1) If $L_z(0) < 0$, self-focusing is arrested when $L = L_{\mathrm{m}} > 0$, after which L will increase monotonically to infinity (Fig. 1b).

(ib2) If $L_z(0) > 0$, L will increase monotonically to infinity.

(ii) If the perturbation is *focusing*, i.e. $\epsilon C_1 < 0$, and if in addition $\beta_0 > 0$ and one of the following two conditions holds, (1) $H_0 > 0$ and $L_z(0) < 0$ or (2) $H_0 < 0$, then the solution of Eq. (14) will blow up at a finite distance (Fig. 1c), i.e.

$$\exists Z_* \quad \text{such that } 0 < Z_* < \infty \text{ and } L(Z_*) = 0.$$

3.3. Non-adiabatic effects

The results of the previous section show that the exponentially small radiation term $\nu(\beta)$ disappears from the leading-order behavior of perturbed CNLS. In the nonconservative case the effect of $\nu(\beta)$ is even smaller than the $(f_1)_z$ term, which is also ignored. However, in the conservative case when $\beta_0 > 0$ and $H_0 < 0$, a defocusing perturbation can lead to periodic oscillations (as in Proposition 3.3ia). In this case, the nonadiabatic radiation effect $\nu(\beta)$ provides the only mechanism for decay of the oscillations. It can be shown that if ϵ is moderately small, the total power loss during one oscillation is small and the focusing–defocusing oscillations are slowly decreasing, but that for sufficiently small ϵ the quasi-periodic picture breaks down and focusing is completely arrested after a few oscillations [7]. Further analysis of nonadiabatic effects in Eq. (17) can be found in Ref. [4].

3.4. Modulation theory for multiple perturbations

In some cases, one is interested in the combined effect of several small perturbations, e.g. randomness and quintic nonlinearity or time dispersion and nonparaxiality (see Table 1). Modulation theory can easily handle these cases, since the modulation equations are linear in F. Therefore, one simply adds the contribution of each perturbation to the modulation equations.

4. Applications

The modulation approach was used by Malkin to study the effect of a small defocusing fifth power nonlinearity [4]. In Ref. [8], Fibich et al. analyzed the effect of small normal time dispersion, using for the first time a systematic approach that is generalized in this Letter. A similar approach was also used by Fibich to analyze the effect of beam nonparaxiality [7] and the unperturbed CNLS [5] and by Fibich and Papanicolaou to analyze the combined effect of time dispersion and nonparaxiality [9]. Additional applications, listed in Table 1, are derived in Ref. [6].

Direct numerical confirmation of the validity of the modulation equations and the adiabatic approach was carried out in the case of the unperturbed NLS [5] and in the case of small normal time dispersion [8]. In many other cases, there is qualitative agreement between the predictions of the modulation equations and the results of numerical simulations of the corresponding PNLS. For example, the behavior of decaying focusing–defocusing oscillations was observed numerically for fiber arrays [10], saturating nonlinearities [11,12] and nonparaxiality [13].

5. Conclusion

In this Letter we have introduced a modulation theory for analyzing the leading-order effects of small perturbations on critical self-focusing. This theory is able to capture the delicate balance between nonlinearity and diffraction in critical self-focusing, because it is based on perturbations of the solution around modulated Townes solitons (ψ_R). We note that the validity of other studies of PNLS in which the derivation of reduced equations is based on modulated Gaussians is questionable, because modulated Gaussians cannot capture the delicate balance in critical self-focusing (e.g. Gaussians cannot simultaneously satisfy Eqs. (4) and (5)). Moreover, the derivation of reduced equations with our ψ_R-based modulation theory is just as easy as with modulated Gaussians. In fact, all that is needed is to carry out the transverse integration in evaluating f_1 and f_2.

We have already remarked that modulation theory becomes valid near the blow-up point Z_c. For some perturbations (e.g. nonparaxiality, saturating nonlinearities) one can show that the modulation equations remain valid for all z [6,7]. However, in other cases (e.g. small normal time dispersion) it is not clear for how long modulation theory remains valid, and further analysis may be needed for the advanced stages of self-focusing.

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