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A Communication Multiplexer Problem: Two Alternating Queues with Dependent Randomly-Timed Gated Regime

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Abstract. Two random traffic streams are competing for the service time of a single server (multiplexer). The streams form two queues, primary (queue 1) and secondary (queue 0). The primary queue is served exhaustively, after which the server switches over to queue 0. The duration of time the server resides in the secondary queue is determined by the dynamic evolution in queue 1. If there is an arrival to queue 1 while the server is still working in queue 0, the latter is immediately gated, and the server completes service there only to the gated jobs, upon which it switches back to the primary queue. We formulate this system as a two-queue polling model with a single alternating server and with randomly-timed gated (RTG) service discipline in queue 0, where the timer there depends on the arrival stream to the primary queue. We derive Laplace–Stieltjes transforms and generating functions for various key variables and calculate numerous performance measures such as mean queue sizes at polling instants and at an arbitrary moment, mean busy period duration and mean cycle time length, expected number of messages transmitted during a busy period and mean waiting times. Finally, we present graphs of numerical results comparing the mean waiting times in the two queues as functions of the relative loads, showing the effect of the RTG regime.

Keywords: two queues, alternating service, multiplexer, polling, randomly-timed gated regime, timers, queue length, busy period, waiting times

1. Introduction

Two streams of traffic are competing for the service time of a single-channel (multiplexer) communication system. Each arrival stream consists of a Poisson flow of messages to be transmitted over the common channel (the server). Messages are of random length and therefore require variable transmission durations. The two streams have different priorities with regard to the order of transmission (service). Type-1 messages are of high priority and form a *primary* queue. The other type of messages, denoted type-0, form a *secondary* queue. It is desired that the primary messages be transmitted with small delays. The secondary messages may suffer larger delays. This dictates the following operating scheme: When attended by the server, the primary queue is served *exhaustively*, i.e. the server switches to queue 0 (incurring switch-over time) only when there are no messages left to be transmitted in queue 1.

After switching in, the duration of time the server resides in the lower-priority queue, queue 0, (i.e. the length of a 'busy period' there) is determined by the dynamic evolution in the *primary* queue. Suppose the server has *completed switching into queue* 0 at time τ (there may be arrivals to queue 1 during the switching time). Let time $\tau + T_1$ denote the instant *thereafter* when the *first* arrival to queue 1 occurs. At that time all messages present in queue 0 are marked (i.e. queue 0 is 'gated') and the server continues serving only those marked messages. When finished, it switches back to the primary queue (incurring another switch-over time). Messages arriving to queue 0 *after* it has been gated will be served only during the next visit of the server to this queue.

We formulate this system as a two-queue model with a single alternating server, performing an exhaustive service discipline in queue 1 and a randomly-timed gated (RTG) service discipline ([10], see also description below) in queue 0, where, in contrast to [10], the length of the busy period in the latter queue *depends* on the arrival rate to the primary queue, making the queues coupled.

Two-queue alternating-service systems have been treated by many authors in the literature, under various assumptions on their operating schemes. Avi-Izhak et al. [1] were the first to study such a configuration, assuming exhaustive service discipline in each queue and zero switchover times. They derived the mean queue size and expected waiting time, as well as the first two moments of the busy period, in each queue. Takacs [18] then followed with an extensive analysis, obtaining Laplace–Stieltjes transforms (LST) and probability generating functions (PGF) of key variables. Neuts and Yadin [16] extended the analysis to transient behavior of the system. Eisenberg [8] investigated the same model but with nonzero changeover times, and further studied a two-queue model [9] where both queues are served according to the 1-limited regime. This system was treated also by Boxma and Groenendijk [3] who obtained the stationary distributions of the queue-length at 'polling' instants, the waiting times and the server's cycle time. Ozawa [17] obtained the mean waiting times when one of the queues is served according to the K-limited regime ($K \geqslant 1$).

Katayama and Takahashi [14] analyzed a two-queue-model where one of the queues follows the 1-limited regime while the other follows a Bernoulli schedule according to which the server serves the next available job in the queue with probability p, or it switches to the other queue with the complementary probability. They derived the PGF of the joint queue-length distribution, as well as the LST of the waiting time distributions. The case where both queues are served according to the Bernoulli schedule was investigated by Feng et al. [12] who obtained the PGF of the joint stationary queue-length distribution at service completion times, as well as the LST and means of the waiting times.

Threshold service disciplines, where one queue is served exhaustively while the other is served only until either the work there is completed or the queue size in the other ('primary') queue hits a given threshold, were studied by Lee [15], Boxma et al. [4,5] and Boxma and Down [2]. In [4] the service times are exponentially distributed and services at queue 0 are preemptively interrupted when the threshold at queue 1 is reached, while in [5] the service process at queue 0 is nonpreemtively interrupted when the threshold is reached. [2] extends the analysis in [5] to the case where service times are generally distributed, and treats both cases of zero and nonzero switchover times. Exact expressions for the joint queue-length distributions at customer departure epochs and for the steady-state queue length and sojourn time distributions are derived. Lee [15] deals with a similar model and gives light and heavy traffic analyses.

The literature on ($N \geqslant 2$) queues with alternating service, termed 'polling systems', got a boost with Takagi's book [19], where many models are presented and analyzed. A following survey [20] augmented the material in [19].

Dynamical optimal control of polling systems is a difficult problem with only partial solutions available. A two queue setup is studied by Hofri and Ross [13]. Browne and Yechiali [6] studied the general model with $N \geqslant 2$ queues and with general service and switchover times. For both the gated and exhaustive regimes they formulated the dynamic control problem as a semi-Markov decision process and derived Bellman's optimal equations for the N-dimensional system. However, the solution of this set of equations seems to be untractable. Nevertheless, a novel approach leads to a semi-dynamic optimal control procedure which is easy to implement. Optimal dynamic and static control policies for various polling systems are presented in [21].

Randomly-timed gated (RTG) regimes were introduced by Eliazar and Yechiali [10,11] to deal with various telecommunication systems were the server's residence times in the various queues are determined by random 'Timers.' In [10] a single server $M/G/1$ queue with server's general Intermission Intervals (INT) is studied. When the server re-enters the system after an INT, a random exponential Timer is activated. If the server empties the queue before the Timer's expiration, it immediately leaves for another INT. Otherwise (if the Timer expires while there is still work in the queue), the server obeys one of the following rules, each leading to a different model. (1) The server completes *all* the work accumulated up to time T ('bank' model), and leaves. (2) The server completes only the service of the job *currently* being served (nonpreemtive discipline, as in [5]), and leaves. (3) The server leaves immediately (preemptive discipline). The analysis is achieved through a general solution of an infinite set of linear equations where the unknowns are the state-dependent joint transforms of the length of a busy period starting with r jobs ($r = 0, 1, 2, \ldots$), and the number of jobs left behind at the end of such a busy period. Performance measures are derived for all three models.

The RTG model was then applied in [11] to analyze a general N-queue polling system where each queue is controlled by an *independent* Timer.

Returning to our original problem, one cannot treat the secondary queue in isolation as if its stochastic evolution can be described by the above $M/G/1$ queue with RTG regime and server's intermissions. In our case, the queues are coupled, as the Timer's duration is determined by the arrival process to the primary queue and thus makes the queues dependent on one another.

Nevertheless, we'll exploit ideas of the RTG model when studying the behavior of the secondary queue. We therefore present in section 2 an analysis, different from the one given in [10], of the RTG model. In section 3 we then present the full analysis of the two-queue system, using results from the single-queue RTG system. We derive various performance measures such as mean queue sizes at polling instants and at an arbitrary moment, mean busy period and cycle time, expected number of messages transmitted during a busy period and mean waiting times. It turns out that some of the results depend on the unknown PGF of the number of messages at polling instant of the secondary queue, evaluated at a certain point which is the value of the LST of the busy period of a $M/G/1$ queue for the secondary queue, itself evaluated at a point which is the parameter of the inter-arrival time to the primary queue. We give an approximation to this function which enables us to obtain explicit (yet approximated) values for all performance measure that depend on the above PGF.

Formal description of the model

Before starting with the analysis of the RTG regime, we give a formal description of our two-queue communication system.

There are two queues (channels), labeled $i = 0, 1$, and a single server that alternates its visits among the channels. Switchover times from queue 0 to queue 1 or backwards are independent random variables denoted by $D^{(0)}$ and $D^{(1)}$, respectively, with corresponding LSTs $\widetilde{D}^{(0)}(\cdot)$ and $\widetilde{D}^{(1)}(\cdot)$.

Arrivals to channel i are according to an independent Poisson process with rate λ_i. Each message (job) in channel i demands an independent service time V_i with LST $\widetilde{V}_i(\cdot)$. The traffic flow rate into queue i is $\rho_i = \lambda_i E[V_i]$ and we assume henceforth that $\rho_0 + \rho_1 < 1$.

The service discipline in channel 1 is exhaustive, i.e. the server leaves channel 1 (and switches to channel 0, incurring switch-over time $D^{(1)}$) only when the former becomes empty. However, the server's sojourn time in channel 0 is a modified RTG regime in which the Timer *depends* on the arrival process to channel 1. Specifically, *after* the server polls channel 0 and starts serving messages there (if none, it switches back to channel 1), it waits for the instant of next type-1 arrival to queue 1. At that instant queue 0 is 'gated'. That is, all jobs present there are marked and the server resides in queue 0 only until all marked jobs are served. At that moment the server switches back to channel 1, incurring a switch-over time $D^{(0)}$.

We call this operating-scheme a 'bank-type' procedure: like in a bank's branch, when doors are closed at the end of the day, only customers still present will be served.

2. Analysis of the RTG regime

2.1. Definitions

Let θ_r denote the length of a busy period starting with r awaiting jobs in a *regular* $M/G/1$ queue with arrival rate λ, service times distributed as V and $\rho \triangleq \lambda E[V] < 1$. For the RTG model, define by B_r the length of a busy period initiated by r awaiting jobs, by N_r the number of jobs served during that busy period, and set $\Delta_r \triangleq (B_r - T)^+$. $\Delta_r = 0$ if the Timer expires after the busy period B_r. Otherwise, Δ_r is the remaining time within the busy period beyond the Timer's expiration. Note that jobs arriving during Δ_r will *not* be served during B_r. Let the Timer be a random variable T, exponentially distributed with parameter μ.

We have

$$\begin{aligned}
&\text{(i)} \quad B_r = \begin{cases} \theta_r \mathbb{1}_{[T>\theta_r]} + \left(\displaystyle\sum_{i=1}^{r+A(T)} V_i \right) \mathbb{1}_{[T \leqslant \theta_r]}, & r \geqslant 1, \\ 0, & r = 0, \end{cases} \\
&\text{(ii)} \quad \Delta_r = \begin{cases} (B_r - T)^+, & r \geqslant 1, \\ 0, & r = 0, \end{cases} \\
&\text{(iii)} \quad N_0 \equiv 0 \quad \text{and} \quad B_r = \sum_{i=1}^{N_r} V_i \qquad \forall r \geqslant 1,
\end{aligned} \tag{2.1}$$

where $A(t)$ denotes the number of Poisson arrivals during a time interval of length t.

Then, a basic observation that will play a key role in the sequel is the following:

Let $r \geqslant 1$. At time V_1, given $A(V_1)$, if $T > V_1$ then the busy period *re-generates* (at time V_1) with $r + A(V_1) - 1$ awaiting jobs.

2.2. Joint LST of (B_r, Δ_r)

Let $\varphi_r(z, w) \triangleq E[\mathrm{e}^{-zB_r}\mathrm{e}^{-w\Delta_r}]$ be the joint Laplace–Stieltjes Transform (LST) of the r.v.'s (B_r, Δ_r). Since $B_0 = \Delta_0 \equiv 0$, $\varphi_0(z, w) \equiv 1$.

Step 1. Let $r \geqslant 1$. Using the re-generation property we deduce that

$$(B_r, \Delta_r) = \begin{cases} \left(\displaystyle\sum_{i=1}^{r+A(T)} V_i, \sum_{i=1}^{r+A(T)} V_i - T \right), & T \leqslant V_1, \\ (V_1 + B_{r-1+A(V_1)}, \Delta_{r-1+A(V_1)}), & T > V_1, \end{cases}$$

where $(B_{r-1+A(V_1)}, \Delta_{r-1+A(V_1)})\big|_{A(V_1)}$ are the B and Δ that correspond to a *new* busy period beginning at time V_1 with $r - 1 + A(V_1)$ awaiting jobs having service times $\{V_2, V_3, \ldots, V_{r+A(V_1)}\}$.

Now,

$$\varphi_r(z, w) = E\left[\mathrm{e}^{-zB_r - w\Delta_r} \mathbb{1}_{[T \leqslant V_1]}\right] + E\left[\mathrm{e}^{-zB_r - w\Delta_r} \mathbb{1}_{[T > V_1]}\right]. \tag{2.2}$$

Step 2. We write

$$z\sum_{i=1}^{r+A(T)} V_i + w\left(\sum_{i=1}^{r+A(T)} V_i - T\right) = (z+w)\sum_{i=2}^{r+A(T)} V_i + (z+w)V_1 - wT$$

and denote $\sigma = \sum_{i=2}^{r+A(T)} V_i$.

Therefore,

$$\begin{aligned} E\big[\mathrm{e}^{zB_r - w\Delta_r}\mathbb{1}_{[T\le V_1]}\big] &= E\big[\mathrm{e}^{-(z+w)\sigma}\,\mathrm{e}^{-(z+w)V_1}\mathrm{e}^{wT}\mathbb{1}_{[T\leqslant V_1]}\big] \\ &= E\big[\mathrm{e}^{-(z+w)V_1}\,\mathrm{e}^{wT}\mathbb{1}_{[T\leqslant V_1]}E\big[\mathrm{e}^{-(z+w)\sigma}\mid T, V_1\big]\big] \\ &= E\big[\mathrm{e}^{-(z+w)V_1}\,\mathrm{e}^{wT}\mathbb{1}_{[T\leqslant V_1]}E\big[\mathrm{e}^{-(z+w)\sigma}\mid T\big]\big]. \end{aligned}$$

Now,

$$\begin{aligned} E\big[\mathrm{e}^{-(z+w)\sigma}\mid T\big] &= E\big[E\big[\mathrm{e}^{-(z+w)\sigma}\mid A(T)\big]\mid T\big] \\ &= E\left[\prod_{i=2}^{r+A(T)} E\big[\mathrm{e}^{-(z+w)V_i}\mid A(T)\big]\mid T\right] = E\big[\widetilde{V}(z+w)^{r-1+A(T)}\mid T\big] \\ &= \widetilde{V}(z+w)^{r-1}E\big[\widetilde{V}(z+w)^{A(T)}\mid T\big] = \widetilde{V}(z+w)^{r-1}\mathrm{e}^{-\lambda(1-\widetilde{V}(z+w))T}. \end{aligned}$$

Hence,

$$E\big[\mathrm{e}^{-zB_r - w\Delta_r}\mathbb{1}[T\leqslant V_1]\big] = \widetilde{V}(z+w)^{r-1}E\big[\mathrm{e}^{-(z+w)V_1}\mathrm{e}^{-(\lambda-w-\lambda\widetilde{V}(z+w))T}\mathbb{1}_{[T\leqslant V_1]}\big]. \tag{2.3}$$

Step 3. For $p\geqslant 0$, q real, and T distributed exponentially with mean $1/\mu$,

$$\begin{aligned} E\big[\mathrm{e}^{-pV_1}\mathrm{e}^{-qT}\mathbb{1}[T\leqslant V_1]\big] &= E\big[\mathrm{e}^{-pV_1}E\big[\mathrm{e}^{-qT}\mathbb{1}_{[T\leqslant V_1]}\mid V_1\big]\big] \\ &= E\left[\mathrm{e}^{-pV_1}\int_0^{V_1}\mathrm{e}^{-qt}\,\mathrm{d}P[T\leqslant t]\right] \\ &= E\left[\mathrm{e}^{-pV_1}\int_0^{V_1}\mathrm{e}^{-qt}\mu\mathrm{e}^{-\mu t}\,\mathrm{d}t\right] \\ \text{if } \mu+q\neq 0 \quad &= E\left[\mathrm{e}^{-pV_1}\frac{\mu}{\mu+q}\int_0^{V_1}(\mu+q)\mathrm{e}^{-(\mu+q)t}\,\mathrm{d}t\right] \\ &= \frac{\mu}{\mu+q}\big\{\widetilde{V}(p) - \widetilde{V}(\mu+p+q)\big\} \\ \text{if } \mu+q = 0 \quad &= E\left[\mathrm{e}^{-pV_1}\mu\int_0^{V_1}\mathrm{d}t\right] = \mu E\big[V_1\mathrm{e}^{-pV_1}\big] \\ &= -\mu\frac{\mathrm{d}}{\mathrm{d}p}E\big[\mathrm{e}^{-pV_1}\big] = -\mu\widetilde{V}'(p). \end{aligned}$$

Take $p = z+w,\ q = \lambda - w - \lambda\widetilde{V}(z+w)$. Then,

$$-(\mu+q) = -\mu - \lambda + w + \lambda\widetilde{V}(z+w) = (z+w) - \big[\mu + z + \lambda\big(1-\widetilde{V}(z+w)\big)\big].$$

Using step 2, we obtain

$$
\begin{aligned}
&E\big[\mathrm{e}^{-zB_r - w\Delta_r}\mathbb{1}_{[T\leqslant V_1]}\big]\\
&\quad=\begin{cases}
-\mu\dfrac{\widetilde{V}(z+w)-\widetilde{V}[\mu+z+\lambda(1-\widetilde{V}(z+w))]}{(z+w)-[\mu+z+\lambda(1-\widetilde{V}(z+w))]}\widetilde{V}(z+w)^{r-1},\\
\quad \mu+\lambda\neq w+\lambda\widetilde{V}(z+w),\\
-\mu\widetilde{V}'(z+w)\cdot\widetilde{V}(z+w)^{r-1}, \quad \mu+\lambda=w+\lambda\widetilde{V}(z+w).
\end{cases}
\end{aligned}
\tag{2.4}
$$

Step 4.

$$
\begin{aligned}
E\big[\mathrm{e}^{-zB_r-w\Delta_r}\mathbb{1}_{[T>V_1]}\big] &= E\Bigg[\sum_{j=0}^{\infty}\mathrm{e}^{-zB_r-w\Delta_r}\mathbb{1}_{[T>V_1,A(V_1)=j]}\Bigg]\\
&=\sum_{j=0}^{\infty}E\big[\mathrm{e}^{-zB_r-w\Delta_r}\mathbb{1}_{[T>V_1,A(V_1)=j]}\big]
\end{aligned}
$$

and

$$
\begin{aligned}
&E\big[\mathrm{e}^{-zB_r-w\Delta_r}\mathbb{1}_{[T>V_1,A(V_1)=j]}\big]\\
&\quad= E\big[\mathrm{e}^{-zV_1}\mathrm{e}^{-zB_{r-1+j}-w\Delta_{r-1+j}}\mathbb{1}_{[T>V_1,A(V_1)=j]}\big]\\
&\quad= E\big[\mathrm{e}^{-zB_{r-1+j}-w\Delta_{r-1+j}}\big]E\big[\mathrm{e}^{-zV_1}\mathbb{1}_{[T>V_1,A(V_1)=j]}\big]\\
&\quad= \varphi_{r-1+j}(z,w)E\big[\mathrm{e}^{-zV_1}E\big[\mathbb{1}_{[T>V_1,A(V_1)=j]}\mid V_1\big]\big]\\
&\quad= \varphi_{r-1+j}(z,w)E\big[\mathrm{e}^{-zV_1}P\big[T>V_1,A(V_1)=j\mid V_1\big]\big]\\
&\quad= \varphi_{r-1+j}(z,w)E\big[\mathrm{e}^{-zV_1}P\big[T>V_1\mid V_1\big]P\big[A(V_1)=j\mid V_1\big]\big]\\
&\quad= \varphi_{r-1+j}(z,w)E\bigg[\mathrm{e}^{-zV_1}\cdot\mathrm{e}^{-\mu V_1}\cdot\frac{(\lambda V_1)^j}{j!}\mathrm{e}^{-\lambda V_1}\bigg]\\
&\quad= \varphi_{r-1+j}(z,w)E\bigg[\frac{(\lambda V_1)^j}{j!}\mathrm{e}^{-(\mu+\lambda+z)V_1}\bigg].
\end{aligned}
\tag{2.5}
$$

Step 5. From steps 1, 3 and 4 we have

$$
\begin{cases}
\varphi_r(z,w)=\displaystyle\sum_{j=0}^{\infty}a_j(z)\varphi_{r-1+j}(z,w)+b_r(z,w), \quad r\geqslant 1,\\
\varphi_0(z,w)\equiv 1,
\end{cases}
\tag{2.6}
$$

where

$$
\begin{aligned}
a_j(z) &\stackrel{\Delta}{=} E\bigg[\frac{(\lambda V)^j}{j!}\mathrm{e}^{-(\lambda+\mu+z)V}\bigg], \qquad j\geqslant 0,\\
b_0(z,w) &\stackrel{\Delta}{=} 1,\\
b_r(z,w) &\stackrel{\Delta}{=} -\beta(z,w)\cdot\widetilde{V}(z+w)^{r-1}, \quad r\geqslant 1,
\end{aligned}
$$

$$\beta(z,w) \triangleq \begin{cases} \mu \dfrac{\widetilde{V}(z+w) - \widetilde{V}[\mu + z + \lambda(1-\widetilde{V}(z+w))]}{(z+w) - [\mu + z + \lambda(1-\widetilde{V}(z+w))]}, \\ \quad w \neq \mu + \lambda\big(1-\widetilde{V}(z+w)\big), \\ \mu\widetilde{V}'(z+w), \quad w = \mu + \lambda\big(1-\widetilde{V}(z+w)\big). \end{cases}$$

Theorem 2.1. Let θ be the length of a busy period in a regular $M/G/1$ queue with arrival rate λ and service times distributed as V. Then, $\forall z,\ w \geqslant 0$ and $\forall r = 0, 1, 2, \ldots$ and with $\widetilde{\theta}(s) = \widetilde{V}(s + \lambda(1 - \widetilde{\theta}(s)))$,

$$\varphi_r(z,w) = \begin{cases} \widetilde{\theta}(\mu+z)^r - \mu\widetilde{\theta}'(\mu+z)\cdot r\widetilde{\theta}(\mu+z)^{r-1}, \\ \quad w = \mu + \lambda\big(1-\widetilde{\theta}(\mu+z)\big), \\ \big(1-\alpha(z,w)\big)\widetilde{\theta}(\mu+z)^r + \alpha(z,w)\widetilde{V}(z+w)^r, \\ \quad w \neq \mu + \lambda\big(1-\widetilde{\theta}(\mu+z)\big), \end{cases}$$

where $\alpha(z,w) = \mu/(\mu - w + \lambda(1-\widetilde{V}(z+w)))$, $w \neq \mu + \lambda(1-\widetilde{\theta}(\mu+z))$.

Proof. In [10, theorem 1] it is shown that a system of equations having the form (2.6) admits a unique solution. Applying that result to our specific problem yields theorem 2.1 above. □

Since $\widetilde{B}_r(z) = \varphi_r(z,0)$ and $\widetilde{\Delta}_r(w) = \varphi_r(0,w)$ we also obtain:

Corollary 2.2. $\forall z \geqslant 0,\ \forall r = 0, 1, 2, \ldots$

$$\widetilde{B}_r(z) = \big(1-\beta(z)\big)\widetilde{\theta}(\mu+z)^r + \beta(z)\widetilde{V}(z)^r$$

where $\beta(z) = \mu/(\mu + \lambda(1-\widetilde{V}(z)))\ \forall z \geqslant 0$.

Corollary 2.3. $\forall w \geqslant 0,\ \forall r = 0, 1, 2, \ldots$

$$\widetilde{\Delta}_r(w) = \begin{cases} \widetilde{\theta}(\mu)^r - \mu\widetilde{\theta}'(\mu)\cdot r\widetilde{\theta}(\mu)^{r-1}, & w = \mu + \lambda\big(1-\widetilde{\theta}'(\mu)\big), \\ \big(1-\delta(w)\big)\widetilde{\theta}(\mu)^r + \delta(w)\widetilde{V}(w)^r, & w \neq \mu + \lambda\big(1-\widetilde{\theta}(\mu)\big), \end{cases}$$

where $\delta(w) = \mu/(\mu - w + \lambda(1-\widetilde{V}(w)))$, $w \neq \mu + \lambda(1-\widetilde{\theta}(\mu))$.

2.3. *State-dependent performance measures*

Since

$$E[B_r] = -\frac{\mathrm{d}}{\mathrm{d}z}\widetilde{B}_r(z)\bigg|_{z=0} \quad \text{and} \quad E[\Delta_r] = -\frac{\mathrm{d}}{\mathrm{d}w}\widetilde{\Delta}_r(w)\bigg|_{w=0}$$

we can use corollaries 2.2, 2.3 to compute $E[B_r]$ and $E[\Delta_r]$.

Using $\beta(0) = 1$, and

$$\frac{\mathrm{d}}{\mathrm{d}z}\beta(z)\bigg|_{z=0} = -\mu\big(\mu + \lambda - \lambda\widetilde{V}(z)\big)^{-2}\big(-\lambda\widetilde{V}'(z)\big)\big|_{z=0} = -\frac{\lambda E[V]}{\mu},$$

we get

$$E[B_r] = E[V]\left\{r + \frac{\lambda}{\mu}\big(1 - \widetilde{\theta}(\mu)^r\big)\right\}. \tag{2.7}$$

Since $\delta(0) = 1$, and

$$\frac{\mathrm{d}}{\mathrm{d}w}\delta(w) = -\mu\big(\mu - w + \lambda\big(1 - \widetilde{V}(w)\big)\big)^{-2}\big(-1 - \lambda\widetilde{V}'(w)\big),$$
$$\frac{\mathrm{d}}{\mathrm{d}w}\delta(w)\bigg|_{w=0} = \frac{1 - \lambda E[V]}{\mu},$$

we get

$$E[\Delta_r] = E[V]r - \frac{1 - \lambda E[V]}{\mu}\big(1 - \widetilde{\theta}(\mu)^r\big). \tag{2.8}$$

We can state:

Corollary 2.4.

(i) $E[B_r] = E[V]r + \rho(1 - \widetilde{\theta}(\mu)^r)/\mu$,

(ii) $E[\Delta_r] = E[V]r - (1 - \rho)(1 - \widetilde{\theta}(\mu)^r)/\mu$,

$\forall r = 0, 1, 2, \ldots.$

Note that corollary 2.4 implies that $E[B_r - \Delta_r] = (1 - \widetilde{\theta}(\mu)^r)/\mu$. This can also be obtained directly as follows. By the definition of B_r and Δ_r we have

$$B_r - \Delta_r = \mathrm{Min}(T, \theta_r).$$

Hence,

$$\begin{aligned}
E[B_r - \Delta_r] &= E\big[\mathrm{Min}(T, \theta_r)\big] = E[T\mathbb{1}_{[T\leqslant\theta_r]} + \theta_r\mathbb{1}_{[T>\theta_r]}]\\
&= E\big[E[T\mathbb{1}_{[T\leqslant\theta_r]} \mid \theta_r]\big] + E\big[E[\theta_r\mathbb{1}_{[T>\theta_r]} \mid \theta_r]\big]\\
&= E\left[\int_0^{\theta_r} t\mu\mathrm{e}^{-\mu t}\,\mathrm{d}t\right] + E\big[\theta_r P[T > \theta_r \mid \theta_r]\big]\\
&= E\left[-\theta_r\mathrm{e}^{-\mu\theta_r} + \frac{1}{\mu}\big(1 - \mathrm{e}^{-\mu\theta_r}\big)\right] + E\big[\theta_r\mathrm{e}^{-\mu\theta_r}\big]\\
&= \frac{1}{\mu}\big(1 - E\big[\mathrm{e}^{-\mu\theta_r}\big]\big) = \frac{1 - \widetilde{\theta}_r(\mu)}{\mu} = \frac{1 - \widetilde{\theta}(\mu)^r}{\mu}.
\end{aligned}$$

Therefore, we have

$$E\big[\mathrm{Min}(T,\theta_r)\big] = E[B_r - \Delta_r] = \frac{1-(\widetilde{\theta}(\mu))^r}{\mu} \quad \forall r = 0,1,2,\ldots. \tag{2.9}$$

We conclude this section by computing the expected value of N_r. Recall from (2.1) that $N_0 \equiv 0$ and $B_r = \sum_{i=1}^{N_r} V_i \ \forall r \geqslant 1$.

By Wald's lemma [7], $E[B_r] = E[N_r]\cdot E[V]$, $\forall r \geqslant 1$. For $r=0$, $E[B_0] = 0 = 0\cdot E[V] = E[N_0]\cdot E[V]$. Thus,

$$E[N_r] = \frac{E[B_r]}{E[V]} = r + \lambda\frac{1-\theta(\mu)^r}{\mu}, \quad \forall r = 0,1,2,\ldots. \tag{2.10}$$

2.4. *The RTG queue in steady state*

The analysis so far dealt with a state-dependent single busy period generated by a *fixed* number of awaiting jobs. We now extend the analysis to a busy period generated by a *random* number of awaiting jobs and obtain results (needed for the study of the communication two-queue system) when the RTG queue is in steady state.

We begin with some definitions and observations.

Polling instant: A moment where the server enters the system, following an intermission interval.

$C =$ cycle length. The time interval between two consecutive polling instants.

$B =$ length of the busy period during a cycle.

$I =$ length of the idle period (i.e. intermission interval) during a cycle.

$N =$ number of jobs served during a cycle.

$\Delta =$ length of the time interval within a cycle during which the service of newcoming jobs is being deferred to the next cycle.

$X =$ queue size at polling instants.

Observe that

(i) $C = B + I$,

(ii) $A(\Delta + I) = X$,

(iii) $(B,\Delta,N) = \sum_{r=0}^{\infty}(B_r,\Delta_r,N_r)\mathbb{1}_{[X=r]}$.

Theorem 2.5. Let $\varphi(z,w) = E[\varphi_X(z,w)]$. Then, for all $z, w \geqslant 0$,

$$\varphi(z,w) = \begin{cases} \widehat{X}\big(\widetilde{\theta}(\mu+z)\big) - \mu\widetilde{\theta}'(\mu+z)\widehat{X}'\big(\widetilde{\theta}(\mu+z)\big), \\ \quad w = \mu + \lambda\big(1-\widetilde{\theta}(\mu+z)\big), \\ \big(1-\alpha(z,w)\big)\widehat{X}\big(\widetilde{\theta}(\mu+z)\big) + \alpha(z,w)\widehat{X}\big(\widetilde{V}(z+w)\big), \\ \quad w \neq \mu + \lambda\big(1-\widetilde{\theta}(\mu+z)\big), \end{cases}$$

where

$$\alpha(z,w)=\frac{\mu}{\mu-w+\lambda(1-\widetilde{V}(z+w))},\qquad w\neq\mu+\lambda\bigl(1-\widetilde{\theta}(\mu+z)\bigr).$$

Proof. If $w=\mu+\lambda(1-\widetilde{\theta}(\mu+z))$ we get, by theorem 2.1,

$$\begin{aligned}\varphi(z,w)&=E\bigl[\bigl(\widetilde{\theta}(\mu+z)\bigr)^X\bigr]-E\bigl[\mu\widetilde{\theta}'(\mu+z)\cdot X\widetilde{\theta}(\mu+z)^{X-1}\bigr]\\&=\widehat{X}\bigl(\widetilde{\theta}(\mu+z)\bigr)-\mu\widetilde{\theta}'(\mu+z)E\bigl[Xt^{X-1}\bigr]\big|_{t=\widetilde{\theta}(\mu+z)}\\&=\widehat{X}\bigl(\widetilde{\theta}(\mu+z)\bigr)-\mu\widetilde{\theta}'(\mu+z)\widehat{X}'\bigl(\widetilde{\theta}(\mu+z)\bigr).\end{aligned}$$

If $w\neq\mu+\lambda(1-\widetilde{\theta}(\mu+z))$ we get,

$$\begin{aligned}\varphi(z,w)&=\bigl(1-\alpha(z,w)\bigr)E\bigl[\bigl(\widetilde{\theta}(\mu+z)\bigr)^X\bigr]+\alpha(z,w)E\bigl[\bigl(\widetilde{V}(z+w)\bigr)^X\bigr]\\&=\bigl(1-\alpha(z,w)\bigr)\widehat{X}\bigl(\widetilde{\theta}(\mu+z)\bigr)+\alpha(z,w)\widehat{X}\bigl(\widetilde{V}(z+w)\bigr).\end{aligned}$$ □

Since $\widetilde{B}(z)=\varphi(z,0)$, $\widetilde{\Delta}(w)=\varphi(0,w)$, we obtain:

Corollary 2.6. $\forall z\geqslant 0$

$$\widetilde{B}(z)=\bigl(1-\beta(z)\bigr)\widehat{X}\bigl(\widetilde{\theta}(\mu+z)\bigr)+\beta(z)\widehat{X}\bigl(\widetilde{V}(z)\bigr)$$

where $\beta(z)=\mu/(\mu+\lambda(1-\widetilde{V}(z)))$.

Corollary 2.7. $\forall w\geqslant 0$

$$\widetilde{\Delta}(w)=\begin{cases}\widehat{X}\bigl(\widetilde{\theta}(\mu)\bigr)-\mu\widetilde{\theta}'(\mu)\widehat{X}'\bigl(\widetilde{\theta}'(\mu)\bigr), & w=\mu+\lambda\bigl(1-\widetilde{\theta}(\mu)\bigr),\\ \bigl(1-\delta(w)\bigr)\widehat{X}\bigl(\widetilde{\theta}(\mu)\bigr)+\delta(w)\widehat{X}\bigl(\widetilde{V}(w)\bigr), & w\neq\mu+\lambda\bigl(1-\widetilde{\theta}(\mu)\bigr),\end{cases}$$

where $\delta(w)=\mu/(\mu-w+\lambda(1-\widetilde{V}(w)))$.

We denote $\gamma\triangleq(1-\widehat{X}(\widetilde{\theta}(\mu)))/\mu$ and get:

Theorem 2.8.

(i) $E[B]=E[V]E[X]+\rho\gamma$,

(ii) $E[\Delta]=E[V]E[X]-(1-\rho)\gamma$,

(iii) $E[N]=E[B]/E[V]=E[X]+\lambda\gamma$,

(iv) $E[\mathrm{Min}(T,\theta_X)]=E[B-\Delta]\triangleq\gamma$.

Proof. By corollary 2.4

$$\text{(i)}\quad E[B]=E\bigl[E[B|X]\bigr]=E\biggl[E[V]X+\rho\frac{1-(\widetilde{\theta}(\mu))^X}{\mu}\biggr]=E[V]E[X]+\rho\gamma.$$

$$\text{(ii)} \quad E[\Delta] = E\big[E[\Delta \mid X]\big] = E\left[E[V]X - (1-\rho)\frac{1-(\widetilde{\theta}(\mu))^X}{\mu}\right]$$
$$= E[V]E[X] - (1-\rho)\gamma.$$

From (2.10) and (2.9)

$$\text{(iii)} \quad E[N] = E\left[X + \lambda\frac{1-(\widetilde{\theta}(\mu))^X}{\mu}\right] = E[X] + \lambda\gamma = \frac{E[B]}{E[V]}.$$
$$\text{(iv)} \quad E\big[\mathrm{Min}(T, \theta_X)\big] = E\left[\frac{1-(\widetilde{\theta}(\mu))^X}{\mu}\right] = \gamma = E[B] - E[\Delta]. \qquad \square$$

Since the system is in steady state, we can introduce two additional performance measures:

$P_{\text{busy}} \triangleq$ the probability that the system is busy at an arbitrary moment of time.

P_{busy} is the so-called "busy fraction", and is given by $E[B]/E[C] = E[B]/(E[B]+E[I])$.

$P_{\text{lucky}} \triangleq$ the probability that a newly arriving job finds the system busy *and* its service is *not* postponed to the next cycle.

P_{lucky} is given by $E[B-\Delta]/E[C] = (E[B]-E[\Delta])/(E[B]+E[I])$.

We claim:

Theorem 2.9.

(i) $E[C] = E[N]/\lambda = E[X]/\lambda + \gamma$,

(ii) $P_{\text{busy}} = \rho$,

(iii) $P_{\text{lucky}} = \gamma/E[C] = \lambda\gamma/(E[X]+\lambda\gamma)$.

Proof. Since $A(\Delta + I) = X$, by applying theorem 2.8,

$$\lambda E[C] - \lambda\gamma = \lambda\big(E[B]+E[I]\big) - \lambda\big(E[B]-E[\Delta]\big) = \lambda\big(E[\Delta]+E[I]\big)$$
$$= \lambda E[\Delta + I] = E\big[A(\Delta+I)\big] = E[X].$$

Thus,

$$\text{(i)} \quad \lambda E[C] = E[X] + \lambda\gamma = E[N].$$
$$\text{(ii)} \quad \frac{E[B]}{E[C]} = \frac{\lambda E[B]}{\lambda E[C]} = \frac{\lambda E[V]E[N]}{E[N]} = \rho.$$
$$\text{(iii)} \quad \frac{\gamma}{E[C]} = \frac{E[B-\Delta]}{E[C]} = \frac{\lambda E[B-\Delta]}{\lambda E[C]} = \frac{\lambda\gamma}{E[X]+\lambda\gamma}.$$

Finally, the mean queue size $E[L]$ and the mean waiting time $E[W]$ for the RTG regime are given in theorem 2.10 below. $\square$

Theorem 2.10.

$$\text{(i)}\quad E[L] = \lambda E[V] - \frac{\lambda}{\mu}\big(1 - \lambda E[V]\big) + \frac{(1/2)(1+\lambda E[V])E[X(X-1)] + (\lambda/\mu)E[X]}{E[N]},$$

$$\text{(ii)}\quad E[W] = \frac{E[L]}{\lambda}.$$

Proof. The expression for $E[L]$ has been derived by Eliazar and Yechiali [10, equation (4.52)]. The expression for $E[W]$ is Little's law. $\square$

Remark. As was indicated in [10], the exhaustive regime with server's intermission intervals is a limiting case of the general RTG regime. It is obtained by letting the Timer's duration approach infinity, i.e. by letting $\mu \to 0$. Similarly, the results for the Gated $M/G/1$ queue with server's Intermission Intervals can be derived from the RTG model by letting $\mu \to \infty$.

3. The two-queue system

3.1. Notation

We use the following notation:

- i-cycle: the time interval between two consecutive polling instants to channel i ($i = 0, 1$).
- $C^{(i)}$: length of an i-cycle.
- $B^{(i)}$: length of a busy period in channel i. That is, the time interval, during an i-cycle, in which the server is busy serving jobs in channel i.
- $N^{(i)}$: number of type-i jobs served during an i-cycle.
- $\Delta^{(0)} = (B^{(0)} - T_1)^+$, where $T_1 \sim \mathrm{Exp}(\lambda_1)$ is the inter-arrival time to queue 1. $\Delta^{(0)}$ is the time interval in which new arrivals to queue 0 are accumulated, only to be served during the *next* cycle.
- $\theta^{(i)}$: length of a busy period in a regular $M/G/1$ queue with arrival rate λ_i and general service times V_i. It is well known that the LST of $\theta^{(i)}$ is given by
$$\widetilde{\theta}^{(i)}(s) = \widetilde{V}_i\big(s + \lambda_i\big(1 - \widetilde{\theta}^{(i)}(s)\big)\big).$$
- $\mathbf{X} = (X_0, X_1)$: system state at polling instant of channel 0.
- $\mathbf{Y} = (Y_0, Y_1)$: system state at polling instant of channel 1.
- $\theta^{(1)}_{Y_1} \triangleq \sum_{k=1}^{Y_1} \theta^{(1)}_k$, where $\theta^{(1)}_k \sim \theta^{(1)}$ and are i.i.d.
- $\theta^{(1)}_{Y_1}$ is the length of a busy period in channel 1 when starting with Y_1 jobs. Clearly, the LST of $\theta^{(1)}_{Y_1}$ is given by $\widetilde{\theta}^{(1)}_{Y_1}(\omega) = [\widetilde{\theta}^{(1)}(\omega)]^{Y_1}$.

3.2. Law of motion

Proposition 3.1. The system's law of motion is given by

$$\begin{pmatrix} X_0 \\ X_1 \end{pmatrix} = \begin{pmatrix} Y_0 + A_0\big(B^{(1)}\big) + A_0\big(D^{(1)}\big) \\ A_1\big(D^{(1)}\big) \end{pmatrix},$$
$$\begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix} = \begin{pmatrix} A_1\big(\Delta^{(0)}\big) + A_1\big(D^{(0)}\big) \\ X_1 + \mathbb{1}_{[\Delta^{(0)}>0]} + A_1\big(\Delta^{(0)}\big) + A_1\big(D^{(0)}\big) \end{pmatrix}, \tag{3.1}$$

where $A_i(t)$ is the number of (Poisson) arrivals to queue i during a time interval of length t.

Proof.

X_1, the number of jobs in queue 1 when queue 0 is polled, is equal to the number of arrivals to queue 1 during $D^{(1)}$, the switching time from queue 1 to queue 0. Hence, $X_1 = A_1(D^{(1)})$.

X_0, the number of jobs in queue 0 when it is polled, equals the sum of Y_0 (number of jobs in queue 0 when queue 1 is polled) plus the new arrivals to queue 0 during the time $B^{(1)} + D^{(1)}$ when the server is 'under the gravity' of queue 1. Hence, $X_0 = Y_0 + A_0(B^{(1)}) + A_0(D^{(1)})$.

Y_0, the number of jobs in queue 0 when queue 1 is polled, equals the number of arrivals to queue 0 during $\Delta^{(0)}$ plus the arrivals during the switching time, $D^{(0)}$, from queue 0 to queue 1.

Finally, Y_1 equals X_1, the number of jobs at queue 1 when queue 0 is polled, plus 1 (if $\Delta^{(0)} > 0$), plus the number of arrivals to queue 1 during $\Delta^{(0)} + D^{(0)}$. □

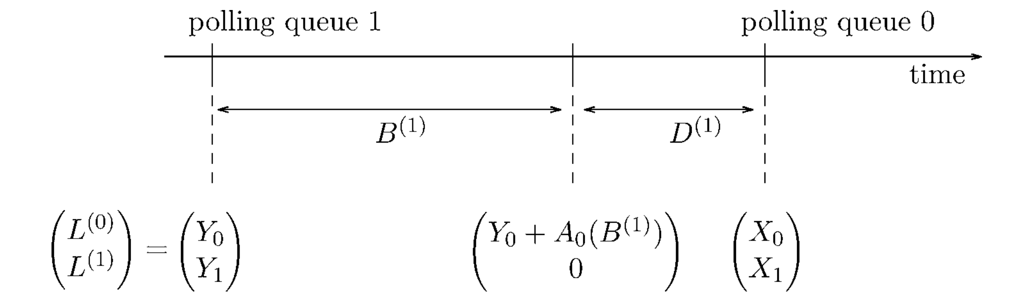
Note that we can also write

$$Y_1 = X_1 + A_1\big(B^{(0)}\big) + A_1\big(D^{(0)}\big).$$

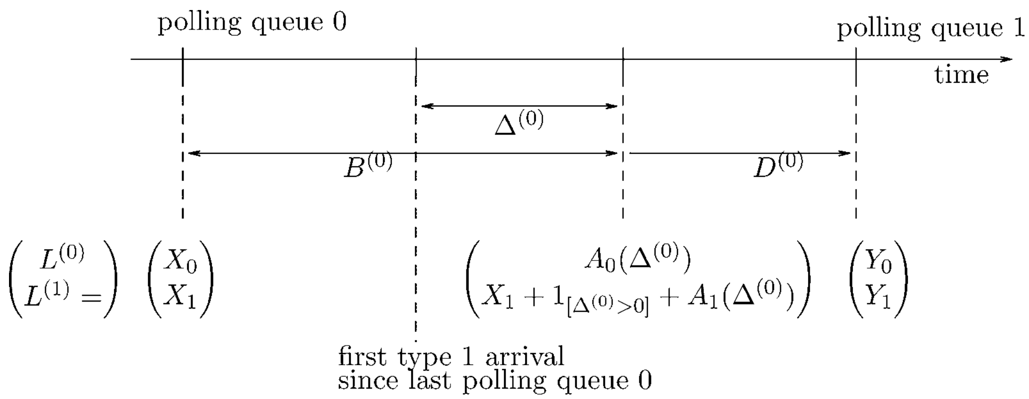
Moreover, observe that

$$\begin{aligned} &\text{(i)} \quad X_0 = A_0\big(\Delta^{(0)} + D^{(0)} + B^{(1)} + D^{(1)}\big), \\ &\phantom{\text{(i)}} \quad Y_1 = A_1\big(D^{(1)}\big) + \mathbb{1}_{[\Delta^{(0)}>0]} + A_1\big(\Delta^{(0)} + D^{(0)}\big). \\ &\text{(ii)} \quad C^{(0)} = B^{(0)} + D^{(0)} + B^{(1)} + D^{(1)}, \\ &\phantom{\text{(ii)}} \quad C^{(1)} = B^{(1)} + D^{(1)} + B^{(0)} + D^{(0)} = C^{(0)}. \end{aligned}$$

Figures 1(a) and (b) illustrate the evolution of the system, where $L^{(i)}$ = queue size in channel i.



(a)



(b)

Figure 1.

3.3. System state at polling instants

We now derive the joint Probability Generating Function (PGF) of $\mathbf{X}$ and $\mathbf{Y}$:

Define

$$G_{\mathbf{X}}(z_0, z_1) \triangleq E\big[z_0^{X_0}\, z_1^{X_1}\big], \qquad G_{\mathbf{Y}}(z_0, z_1) \triangleq E\big[z_0^{Y_0}\, z_1^{Y_1}\big].$$

Step 1. Applying relationship (3.1) we write

$$\begin{aligned} G_{\mathbf{X}}(z_0, z_1) &= E\big[z_0^{Y_0 + A_0(B^{(1)}) + A_0(D^{(1)})}\, z_1^{A_1(D^{(1)})}\big] \\ &= E\big[z_0^{Y_0 + A_0(B^{(1)})}\big] E\big[z_0^{A_0(D^{(1)})}\, z_1^{A_1(D^{(1)})}\big]. \end{aligned}$$

Now, setting $w_i = \lambda_i(1 - z_i)$ and observing that $B^{(1)} = \theta_{Y_1}^{(1)}$, we have

$$\begin{aligned} E\big[z_0^{Y_0 + A_0(B^{(1)})}\big] &= E\big[E\big[z_0^{Y_0 + A_0(B^{(1)})} \mid \mathbf{Y}\big]\big] = E\big[z_0^{Y_0} E\big[z_0^{A_0(B^{(1)})} \mid Y_1\big]\big] \\ &= E\big[z_0^{Y_0} E\big[E\big[z_0^{A_0(B^{(1)})} \mid B^{(1)}\big] \mid Y_1\big]\big] = E\big[z_0^{Y_0} E\big[\mathrm{e}^{-w_0 B^{(1)}} \mid Y_1\big]\big] \\ &= E\big[z_0^{Y_0} E\big[\mathrm{e}^{-w_0 \theta_{Y_1}^{(1)}} \mid Y_1\big]\big] = E\big[z_0^{Y_0} \widetilde{\theta}_1(w_0)^{Y_1}\big] = G_{\mathbf{Y}}\big(z_0, \widetilde{\theta}_1(w_0)\big). \end{aligned}$$

Also,

$$\begin{aligned}E\big[z_0^{A_0(D^{(1)})}\, z_1^{A_1(D^{(1)})}\big] &= E\big[E\big[z_0^{A_0(D^{(1)})}\, z_1^{A_1(D^{(1)})} \mid D^{(1)}\big]\big]\\ &= E\big[E\big[z_0^{A_0(D^{(1)})} \mid D^{(1)}\big]E\big[z_1^{A_1(D^{(1)})} \mid D^{(1)}\big]\big]\\ &= E\big[\mathrm{e}^{-w_0 D^{(1)}}\, \mathrm{e}^{-w_1 D^{(1)}}\big] = \widetilde{D}^{(1)}(w_0+w_1).\end{aligned}$$

Thus, we obtain

$$G_{\mathbf{X}}(z_0, z_1) = G_{\mathbf{Y}}\big(z_0, \widetilde{\theta}_1(w_0)\big)\widetilde{D}^{(1)}(w_0+w_1). \tag{3.2}$$

Step 2. Again, by (3.1)

$$\begin{aligned}G_{\mathbf{Y}}(z_0, z_1) &= E\big[z_0^{A_0(\Delta^{(0)})+A_0(D^{(0)})}\, z_1^{X_1+\mathbb{1}_{[\Delta^{(0)}>0]}+A_1(\Delta^{(0)})+A_1(D^{(0)})}\big]\\ &= E\big[z_0^{A_0(\Delta^{(0)})}\, z_1^{X_1+\mathbb{1}_{[\Delta^{(0)}>0]}+A_1(\Delta^{(0)})}\big]E\big[z_0^{A_0(D^{(0)})}\, z_1^{A_1(D^{(0)})}\big].\end{aligned}$$

Similarly to step 1, $E[z_0^{A_0(D^{(0)})}\, z_1^{A_1(D^{(0)})}] = \widetilde{D}^{(0)}(w_0+w_1)$.

In addition, using $z_1^{\mathbb{1}_{[\Delta^{(0)}>0]}} = z_1 + (1-z_1)\cdot \mathbb{1}_{[\Delta^{(0)}=0]}$, we get

$$\begin{aligned}&E\big[z_0^{A_0(\Delta^{0})}\, z_1^{X_1+\mathbb{1}_{[\Delta^{(0)}>0]}+A_1(\Delta^{(0)})}\big]\\ &\quad= E\big[E\big[E\big[z_0^{A_0(\Delta^{(0)})}\, z_1^{X_1+\mathbb{1}_{[\Delta^{(0)}>0]}+A_1(\Delta^{(0)})} \mid \mathbf{X}, \Delta^{(0)}\big] \,\big|\, \mathbf{X}\big]\big]\\ &\quad= E\big[z_1^{X_1}E\big[z_1^{\mathbb{1}_{[\Delta^{(0)}>0]}}E\big[z_0^{A_0(\Delta^{(0)})}\, z_1^{A_1(\Delta^{(0)})} \mid \Delta^{(0)}\big] \,\big|\, \mathbf{X}\big]\big]\\ &\quad= E\big[z_1^{X_1}E\big[z_1^{\mathbb{1}_{[\Delta^{(0)}>0]}}E\big[z_0^{A_0(\Delta^{(0)})} \mid \Delta^{(0)}\big]E\big[z_1^{A_1(\Delta^{(0)})} \mid \Delta^{(0)}\big] \,\big|\, \mathbf{X}\big]\big]\\ &\quad= E\big[z_1^{X_1}E\big[z_1^{\mathbb{1}_{[\Delta^{(0)}>0]}}\mathrm{e}^{-w_0\Delta^{(0)}}\mathrm{e}^{-w_1\Delta^{(0)}} \mid \mathbf{X}\big]\big]\\ &\quad= E\big[z_1^{X_1}E\big[\big(z_1+(1-z_1)\mathbb{1}_{[\Delta^{(0)}=0]}\big)\mathrm{e}^{-(w_0+w_1)\Delta^{(0)}} \mid \mathbf{X}\big]\big]\\ &\quad= z_1E\big[z_1^{X_1}E\big[\mathrm{e}^{-(w_0+w_1)\Delta^{(0)}} \mid \mathbf{X}\big]\big] + (1-z_1)E\big[z_1^{X_1}E\big[\mathbb{1}_{[\Delta^{(0)}=0]} \mid \mathbf{X}\big]\big]\\ &\quad= z_1E\big[\big(z_1^{X_1}\widetilde{\Delta}^{(0)}=0\big)(w_0+w_1)\big] + (1-z_1)E\big[z_1^{X_1}P\big[\Delta^{(0)}=0 \mid X_0\big]\big],\end{aligned}$$

where $\widetilde{\Delta}_{X_0}^{(0)}(\cdot)$ is the LST of $\Delta^{(0)}$ when there are X_0 jobs in channel 0 at polling instant of queue 0.

Hence,

$$\begin{aligned}G_{\mathbf{Y}}(z_0, z_1) &= \big(z_1E\big[z_1^{X_1}\widetilde{\Delta}_{X_0}^{(0)}(w_0+w_1)\big] + (1-z_1)E\big[z_1^{X_1}P\big[\Delta^{(0)}=0 \mid X_0\big]\big]\big)\\ &\quad\times \widetilde{D}^{(0)}(w_0+w_1).\end{aligned} \tag{3.3}$$

Step 3. Adapting the analysis of the RTG regime to the secondary queue, it readily follows that the Timer's exponential duration is the inter-arrival time to queue 1, namely, T_1, and the intermission interval is distributed like $D^{(0)}+B^{(1)}+D^{(1)}$. The only modification needed is to set $\mu = \lambda_1$. Thus, by corollary 2.3,

$$\widetilde{\Delta}_{X_0}^{(0)}(w_0+w_1) = \big(1-\delta_0(w_0+w_1)\big)\big(\widetilde{\theta}_0(\lambda_1)\big)^{X_0} + \delta_0(w_0+w_1)\big(\widetilde{V}_0(w_0+w_1)\big)^{X_0},$$

where

$$\delta_0(w_0 + w_1) = \frac{\lambda_1}{\lambda_1 - (w_0 + w_1) + \lambda_0(1 - \widetilde{V}_0(w_0 + w_1))}.$$

Therefore,

$$\begin{aligned} E\big[z_1^{X_1}\widetilde{\Delta}_{X_0}^{(0)}(w_0 + w_1)\big] = {} & \big(1 - \delta_0(w_0 + w_1)\big)G_{\mathbf{X}}\big(\widetilde{\theta}_0(\lambda_1), z_1\big) \\ & + \delta_0(w_0 + w_1)G_{\mathbf{X}}\big(\widetilde{V}_0(w_0 + w_1), z_1\big). \end{aligned} \tag{3.4}$$

Let $\theta_{X_0}^{(0)}$ be the duration of a busy period in queue 0 starting with X_0 jobs. We then have

$$\begin{aligned} P\big[\Delta^{(0)} = 0 \mid X_0\big] &= P\big[T_1 > \theta_{X_0}^{(0)} \mid X_0\big] = E\big[P\big[T_1 > \theta_{X_0}^{(0)} \mid \theta_{X_0}^{(0)}\big] \big| X_0\big] \\ &= E\big[\mathrm{e}^{-\lambda_1\theta_{X_0}^{(0)}} \mid X_0\big] = \big(\widetilde{\theta}^{(0)}(\lambda_1)\big)^{X_0}. \end{aligned}$$

Clearly, because of the memoryless property of T_1, $\Delta^{(0)} = 0$ only if each regular $M/G/1$-type busy period in queue 0 terminates before time T_1. The above implies that

$$E\big[z_1^{X_1}P\big[\Delta^{(0)} = 0 \mid X_0\big]\big] = G_{\mathbf{X}}\big(\widetilde{\theta}_0(\lambda_1), z_1\big). \tag{3.5}$$

Substituting (3.4) and (3.5) in equation (3.3) yields

$$\begin{aligned} G_{\mathbf{Y}}(z_0, z_1) = {} & \big\{z_1\big(1 - \delta_0(w_0 + w_1)\big)G_{\mathbf{X}}\big(\widetilde{\theta}_0(\lambda_1), z_1\big) \\ & + z_1\delta_0(w_0 + w_1)G_{\mathbf{X}}\big(\widetilde{V}_0(w_0 + w_1), z_1\big) \\ & + (1 - z_1)G_{\mathbf{X}}\big(\widetilde{\theta}_0(\lambda_1), z_1\big)\big\}\widetilde{D}^{(0)}(w_0 + w_1). \end{aligned}$$

Thus, finally

$$\begin{aligned} G_{\mathbf{Y}}(z_0, z_1) = {} & \big(\big(1 - z_1\delta_0(w_0 + w_1)\big)G_{\mathbf{X}}\big(\widetilde{\theta}_0(\lambda_1), z_1\big) \\ & + z_1\delta_0(w_0 + w_1)G_{\mathbf{X}}\big(\widetilde{V}_0(w_0 + w_1), z_1\big)\big)\widetilde{D}^{(0)}(w_0 + w_1). \end{aligned} \tag{3.6}$$

Denoting $g(z_0, z_1) \triangleq z_1\delta_0(w_0 + w_1)$ we summarize steps 1–3 by the following implicit set of equations:

Proposition 3.2.

$$\begin{aligned} G_{\mathbf{X}}(z_0, z_1) &= G_{\mathbf{Y}}\big(z_0, \widetilde{\theta}_1(w_0)\big)\widetilde{D}^{(1)}(w_0 + w_1), \\ G_{\mathbf{Y}}(z_0, z_1) &= \big[\big(1 - g(z_0, z_1)\big)G_{\mathbf{X}}\big(\widetilde{\theta}_0(\lambda_1), z_1\big) \\ &\quad + g(z_0, z_1)G_{\mathbf{X}}\big(\widetilde{V}_0(w_0 + w_1), z_1\big)\big]\widetilde{D}^{(0)}(w_0 + w_1), \end{aligned}$$

where

$$w_i = \lambda_i(1 - z_i) \quad \text{and} \quad g(z_0, z_1) = \frac{\lambda_1 z_1}{\lambda_1 z_1 + \lambda_0(z_0 - \widetilde{V}_0(w_0 + w_1))}.$$

Now clearly, the PGF of the number of jobs in channel 0 and in channel 1 (upon their polling instants), $\mathbf{X}_0$ and $\mathbf{Y}_1$, is given by, respectively,

$$\widehat{X}_0(z) = G_{\mathbf{X}}(z,1), \qquad \widehat{Y}_1(z) = G_{\mathbf{Y}}(1,z).$$

3.4. *Performance measures*

We denote: $d_i = E[D^{(i)}],\ i = 0,1,\ d = d_1 + d_2$, and $\gamma_0 = (1 - \widehat{X}_0(\widetilde{\theta}^{(0)}(\lambda_1)))/\lambda_1$. From the observation following proposition 3.1 we readily have

$$E\big[C^{(0)}\big] = E\big[C^{(1)}\big] = E\big[B^{(0)}\big] + E\big[B^{(1)}\big] + d \triangleq E[C]. \tag{3.7}$$

From theorem 2.7 (applied to the secondary queue)

$$E\big[B^{(0)}\big] = E[X_0]E[V_0] + \rho_0\gamma_0.$$

Furthermore, since channel 1 is exhaustive,

$$E\big[B^{(1)}\big] = E\big[\theta^{(1)}_{Y_1}\big] = \frac{E[Y_1]E[V_1]}{1-\rho_1}.$$

Therefore, by the observation following proposition 3.1 and the fact that $E[B^{(0)} - \Delta^{(0)}] = \gamma_0$ we get

$$\begin{aligned} E\big[B^{(0)}\big] &= E\big[A_0\big(\Delta^{(0)} + D^{(0)} + B^{(1)} + D^{(1)}\big)\big]E[V_0] + \rho_0\gamma_0 \\ &= \rho_0\big(E\big[\Delta^{(0)}\big] + E\big[B^{(1)}\big] + d\big) + \rho_0\gamma_0 = \rho_0\big(E\big[B^{(1)}\big] + d + E\big[B^{(0)}\big]\big) \\ &= \rho_0 E[C]. \end{aligned} \tag{3.8}$$

In step 3 above we showed that $P[\Delta^{(0)} = 0 \mid X_0] = \widetilde{\theta}^{(0)}(\lambda_1)^{X_0}$. Hence, $P[\Delta^{(0)} > 0] = 1 - P[\Delta^{(0)} = 0] = 1 - E[P[\Delta^{(0)} = 0 \mid X_0]] = 1 - \widehat{X}_0(\widetilde{\theta}^{(0)}(\lambda_1)) = \lambda_1\gamma_0$.

Now, since queue 1 is served exhaustively,

$$\begin{aligned} E\big[B^{(1)}\big] &= \frac{E[A_1(D^{(1)}) + \mathbb{1}_{[\Delta^{(0)}>0]} + A_1(\Delta^{(0)} + D^{(0)})]E[V_1]}{1-\rho_1} \\ &= \frac{\rho_1}{1-\rho_1}\bigg(E\big[D^{(1)}\big] + \frac{1}{\lambda_1}P\big[\Delta^{(0)} > 0\big] + E\big[\Delta^{(0)}\big] + E\big[D^{(0)}\big]\bigg) \\ &= \frac{\rho_1}{1-\rho_1}\big(E\big[\Delta^{(0)}\big] + d + \gamma_0\big) = \frac{\rho_1}{1-\rho_1}\big(d + E\big[B^{(0)}\big]\big) \\ &= \frac{\rho_1}{1-\rho_1}E[C] - \frac{\rho_1}{1-\rho_1}E\big[B^{(1)}\big]. \end{aligned}$$

Thus,

$$E\big[B^{(1)}\big] = \rho_1 E[C]. \tag{3.9}$$

In addition, $E[C] \triangleq E[B^{(0)}] + E[B^{(1)}] + d = \rho_0 E[C] + \rho_1 E[C] + d$, and hence

$$(1-\rho)E[C] = d. \tag{3.10}$$

Note that the expression for the mean cycle time, $E[C] = d/(1-\rho)$, is the same expression obtained for many other work-conserving polling systems (see [11,19,21]).

We summarize:

Proposition 3.3.

(i) $E[C^{(0)}] = E[C^{(1)}] = d/(1-\rho) \triangleq E[C]$.

(ii) The busy-time fraction of channel i, $P_{\text{busy}}^{(i)} \triangleq E[B^{(i)}]/E[C]$, is ρ_i.

(iii) The server's busy fraction $P_{\text{busy}} \triangleq (E[B^{(0)}] + E[B^{(1)}])/E[C]$ is ρ.

Furthermore,

$$\rho_0 \frac{d}{1-\rho} = \rho_0 E[C] = E\big[B^{(0)}\big] = E[X_0]E[V_0] + \rho_0\gamma_0,$$

implying $E[X_0] = \lambda_0 d/(1-\rho) - \lambda_0\gamma_0 = \lambda_0 E[C] - \lambda_0\gamma_0$,

$$\rho_1 \frac{d}{1-\rho} = \rho_1 E[C] = E\big[B^{(1)}\big] = \frac{E[Y_1]E[V_1]}{1-\rho_1},$$

implying $E[Y_1] = (1-\rho_1)/(1-\rho)\lambda_1 d$.

By the law of motion (3.1) and the expression for $E[\Delta^{(0)}]$ in theorem 2.7, we have

$$\begin{aligned}
E[X_1] &= E\big[A_1\big(D^{(1)}\big)\big] = \lambda_1 d_1,\\
E[Y_0] &= E\big[A_0\big(\Delta^{(0)} + D^{(0)}\big)\big] = \lambda_0 E\big[\Delta^{(0)}\big] + \lambda_0 d_0\\
&= \lambda_0\big(E[X_0]E[V_0] - (1-\rho_0)\gamma_0\big) + \lambda_0 d_0\\
&= \rho_0 \frac{\lambda_0 d}{1-\rho} - \rho_0\lambda_0\gamma_0 - \lambda_0\gamma_0 + \lambda_0\rho_0\gamma_0 + \lambda_0 d_0 = \lambda_0\left(\frac{\rho_0 d}{1-\rho} + d_0 - \gamma_0\right).
\end{aligned}$$

We conclude:

Proposition 3.4.

$$\begin{aligned}
E[X_0] &= \lambda_0\left(\frac{d}{1-\rho} - \gamma_0\right), & E[Y_0] &= \lambda_0\left(\frac{\rho_0}{1-\rho}d + d_0 - \gamma_0\right),\\
E[X_1] &= \lambda_1 d_1, & E[Y_1] &= \lambda_1 \frac{1-\rho_1}{1-\rho} d.
\end{aligned}$$

Finally, by the above computations, theorems 2.7 and 2.8 and the fact that channel 1 is exhaustive, we obtain:

Proposition 3.5.

$$E\big[B^{(0)}\big] = \rho_0 \frac{d}{1-\rho}, \qquad E\big[N^{(0)}\big] = \lambda_0 \frac{d}{1-\rho}, \qquad E\big[\Delta^{(0)}\big] = \rho_0 \frac{d}{1-\rho} - \gamma_0,$$
$$E\big[B^{(1)}\big] = \rho_1 \frac{d}{1-\rho}, \qquad E\big[N^{(1)}\big] = \lambda_1 \frac{d}{1-\rho}, \qquad E\big[\Delta^{(1)}\big] = 0.$$

3.5. Busy and idle intervals

In this section we compute

(1) The joint LST of the busy and idle intervals in channel 0 during $C^{(0)}$:

$$H_0(t,s) \triangleq E\big[\mathrm{e}^{-tB^{(0)}-sI^{(0)}}\big], \quad \text{where } I^{(0)} \triangleq D^{(0)} + B^{(1)} + D^{(1)}.$$

(2) The joint LST of the busy and idle intervals in channel 1 during $C^{(1)}$:

$$H_1(t,s) \triangleq E\big[\mathrm{e}^{-tB^{(1)}-sI^{(1)}}\big], \quad \text{where } I^{(1)} \triangleq D^{(1)} + B^{(0)} + D^{(0)}.$$

(3) The LST of $C^{(0)}$ and $C^{(1)}$.

(4) The joint LST of the server's busy and idle intervals.

Step 1. To ease the computation of $H_0(\cdot,\cdot)$ we use the illustration in figure 2.

$$\begin{aligned}
H_0(t,s) &\triangleq E\big[\mathrm{e}^{-tB^{(0)}-sI^{(0)}}\big] \\
&= E\big[\mathrm{e}^{-tB^{(0)}-s(D^{(0)}+B^{(1)}+D^{(1)})}\big] = E\big[\mathrm{e}^{-tB^{(0)}-s(D^{(0)}+B^{(1)})}\big]\widetilde{D}^{(1)}(s) \\
&= E\big[E\big[\mathrm{e}^{-tB^{(0)}-s(D^{(0)}+B^{(1)})} \mid \mathbf{X}, B^{(0)}, \Delta^{(0)}, D^{(0)}\big]\big]\widetilde{D}^{(1)}(s) \\
&= E\big[\mathrm{e}^{-tB^{(0)}-sD^{(0)}} E\big[\mathrm{e}^{-sB^{(1)}} \mid X_1, \Delta^{(0)}, D^{(0)}\big]\big]\widetilde{D}^{(1)}(s).
\end{aligned}$$

Now,

$$\begin{aligned}
&E\big[\mathrm{e}^{-sB^{(1)}} \mid X_1, \Delta^{(0)}, D^{(0)}\big] \\
&\quad = E\big[E\big[\mathrm{e}^{-sB^{(1)}} \mid Y_1\big] \,\big|\, X_1, \Delta^{(0)}, D^{(0)}\big]
\end{aligned}$$

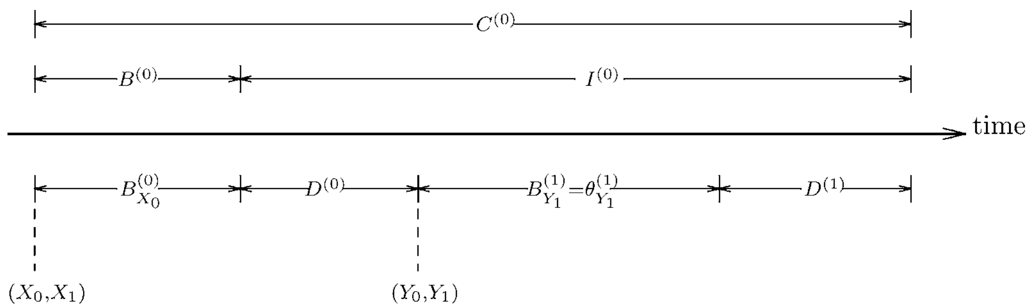


Figure 2.

$$\begin{aligned}&= E\big[E\big[\mathrm{e}^{-s\theta^{(1)}_{Y_1}} \mid Y_1\big] \,\big|\, X_1, \Delta^{(0)}, D^{(0)}\big]\\ &= E\big[\widetilde{\theta}_1(s)^{Y_1} \mid X_1, \Delta^{(0)}, D^{(0)}\big].\end{aligned}$$

Setting $z_1 = \widetilde{\theta}_1(s)$, $w_1 = \lambda_1(1 - z_1)$ and using the law of motion (3.1) we get

$$\begin{aligned}&E\big[z_1^{Y_1} \mid X_1, \Delta^{(0)}, D^{(0)}\big]\\ &\quad= E\big[z_1^{X_1+\mathbb{1}_{[\Delta^{(0)}>0]}+A_1(\Delta^{(0)}+D^{(0)})} \mid X_1, \Delta^{(0)}, D^{(0)}\big]\\ &\quad= z_1^{X_1+\mathbb{1}_{[\Delta^{(0)}>0]}} E\big[z_1^{A_1(\Delta^{(0)}+D^{(0)})} \mid \Delta^{(0)}, D^{(0)}\big] = z_1^{X_1+\mathbb{1}_{[\Delta^{(0)}>0]}} \mathrm{e}^{-w_1(\Delta^{(0)}+D^{(0)})}\\ &\quad= \big[z_1 + (1 - z_1)\mathbb{1}_{[\Delta^{(0)}=0]}\big] z_1^{X_1} \mathrm{e}^{-w_1(\Delta^{(0)}+D^{(0)}}\\ &\quad= z_1 z_1^{X_1} \mathrm{e}^{-w_1(\Delta^{(0)}+D^{(0)})} + (1 - z_1) z_1^{X_1} \mathrm{e}^{-w_1 D^{(0)}} \mathbb{1}_{[\Delta^{(0)}=0]}.\end{aligned}$$

So, we obtain

$$\begin{aligned}H_0(t, s) &= E\big[\mathrm{e}^{-tB^{(0)}-sD^{(0)}} z_1 z^{X_1} \mathrm{e}^{-w_1(\Delta^{(0)}+D^{(0)})}\big]\widetilde{D}^{(1)}(s)\\ &\quad+ E\big[\mathrm{e}^{-tB^{(0)}-sD^{(0)}}(1 - z_1) z_1^{X_1} \mathrm{e}^{-w_1 D^{(0)}} \mathbb{1}_{[\Delta^{(0)}=0]}\big]\widetilde{D}^{(1)}(s)\\ &= z_1 E\big[z_1^{X_1} \mathrm{e}^{-tB^{(0)}-w_1\Delta^{(0)}} \mathrm{e}^{-(s+w_1)D^{(0)}}\big]\widetilde{D}^{(1)}(s)\\ &\quad+ (1 - z_1) E\big[z_1^{X_1} \mathrm{e}^{-tB^{(0)}} \mathbb{1}_{[\Delta^{(0)}=0]} \mathrm{e}^{-(s+w_1)D^{(0)}}\big]\widetilde{D}^{(1)}(s).\end{aligned}$$

That is,

$$\begin{aligned}H_0(t, s) = &\big\{z_1 E\big[z_1^{X_1} \mathrm{e}^{-tB^{(0)}-w_1\Delta^{(0)}}\big] + (1 - z_1) E\big[z_1^{X_1} \mathrm{e}^{-tB^{(0)}} \mathbb{1}_{[\Delta^{(0)}=0]}\big]\big\}\\ &\times \widetilde{D}^{(0)}(s + w_1)\widetilde{D}^{(1)}(s).\end{aligned} \tag{3.11}$$

Step 2. Using theorem 2.1 we have

$$\begin{aligned}&E\big[z_1^{X_1} \mathrm{e}^{-tB^{(0)}-w_1\Delta^{(0)}}\big]\\ &\quad= E\big[E\big[z_1^{X_1} \mathrm{e}^{-tB^{(0)}-w_1\Delta^{(0)}} \mid \mathbf{X}\big]\big]\\ &\quad= E\big[z_1^{X_1} E\big[\mathrm{e}^{-tB^{(0)}-w_1\Delta^{(0)}} \mid X_0\big]\big] = E\big[z_1^{X_1} \varphi^{(0)}_{X_0}(t, w_1)\big]\\ &\quad= E\big[\big\{\big(1 - \alpha_0(t, w_1)\big)\big(\widetilde{\theta}^{(0)}(\lambda_1 + t)\big)^{X_0} + \alpha_0(t, w_1)\big(\widetilde{V}_0(t + w_1)\big)^{X_0}\big\} z_1^{X_1}\big]\\ &\quad= \big(1 - \alpha_0(t, w_1)\big) G_{\mathbf{X}}\big(\widetilde{\theta}^{(0)}(\lambda_1 + t), z_1\big) + \alpha_0(t, w_1) G_{\mathbf{X}}\big(\widetilde{V}_0(t + w_1), z_1\big),\end{aligned}$$

where

$$\alpha_0(t, w_1) = \frac{\lambda_1}{\lambda_1 - w_1 + \lambda_0(1 - \widetilde{V}_0(t + w_1))} = \frac{\lambda_1}{\lambda_1 z_1 + \lambda_0(1 - \widetilde{V}_0(t + w_1))}.$$

Also, since

$$\begin{aligned}&E\big[\mathrm{e}^{-tB^{(0)}} \mathbb{1}_{[\Delta^{(0)}=0]} \mid X_0\big]\\ &\quad= E\big[\mathrm{e}^{-t\theta^{(0)}_{X_0}} \mathbb{1}_{[T_1 \geqslant \theta^{(0)}_{X_0}]} \mid X_0\big]\end{aligned}$$

$$= E\big[\mathrm{e}^{-t\theta^{(0)}_{X_0}} P\big[T_1 \geqslant \theta^{(0)}_{X_0} \mid \theta^{(0)}_{X_0}\big] \mid X_0\big]$$
$$= E\big[\mathrm{e}^{-t\theta^{(0)}_{X_0}} \mathrm{e}^{-\lambda_1\theta^{(0)}_{X_0}} \mid X_0\big] = E\big[\mathrm{e}^{-(\lambda_1+t)\theta^{(0)}_{X_0}} \mid X_0\big] = \big(\widetilde{\theta}^{(0)}(\lambda_1+t)\big)^{X_0}$$

we get

$$\begin{aligned} E\big[z_1^{X_1}\mathrm{e}^{-tB^{(0)}}\mathbb{1}_{[\Delta^{(0)}=0]}\big] &= E\big[E\big[z_1^{X_1}\mathrm{e}^{-tB^{(0)}}\mathbb{1}_{[\Delta^{(0)}=0]} \mid \mathbf{X}\big]\big] \\ &= E\big[z_1^{X_1}E\big[\mathrm{e}^{-tB^{(0)}}\mathbb{1}_{[\Delta^{(0)}=0]} \mid X_0\big]\big] = E\big[z_1^{X_1}\big(\widetilde{\theta}^{(0)}(\lambda_1+t)\big)^{X_0}\big] \\ &= G_{\mathbf{X}}\big(\widetilde{\theta}^{(0)}(\lambda_1+t), z_1\big). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} H_0(t,s) = \big\{& z_1\big(1-\alpha_0(t,w_1)\big)G_{\mathbf{X}}\big(\widetilde{\theta}^{(0)}(\lambda_1+t), z_1\big) + z_1\alpha_0(t,w_1)G_{\mathbf{X}}\big(\widetilde{V}_0(t+w_1), z_1\big) \\ &+ (1-z_1)G_{\mathbf{X}}\big(\widetilde{\theta}^{(0)}(\lambda_1+t), z_1\big)\big\}\widetilde{D}^{(0)}(s+w_1)\widetilde{D}^{(1)}(s). \end{aligned} \tag{3.12}$$

Summarizing, we have:

Proposition 3.6.

$$\begin{aligned} H_0(t,s) = \big\{&\big(1-h(t,w_1)\big)G_{\mathbf{X}}\big(\widetilde{\theta}^{(0)}(t+\lambda_1), z_1\big) + h(t,w_1)G_{\mathbf{X}}\big(\widetilde{V}_0(t+w_1), z_1\big)\big\} \\ &\times \widetilde{D}^{(0)}(s+w_1)\widetilde{D}^{(1)}(s), \end{aligned} \tag{3.13}$$

where

$$z_1 = \widetilde{\theta}^{(1)}(s), \qquad w_1 = \lambda_1(1-z_1), \qquad h(t,w_1) = \frac{\lambda_1 z_1}{\lambda_1 z_1 + \lambda_0(1-\widetilde{V}_0(t+w_1))}.$$

Since $C^{(0)} = B^{(0)} + I^{(0)}$ we also have:

Corollary 3.7.

$$\widetilde{C}^{(0)}(t) = H_0(t,t).$$

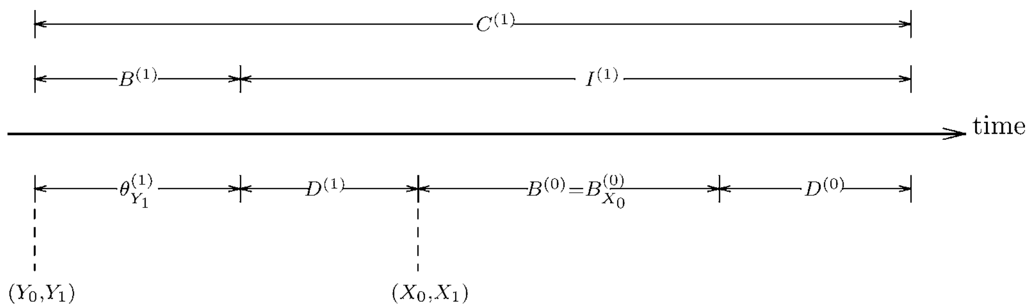


Figure 3.

Step 3. To ease the computation of $H_1(\cdot,\cdot)$ we use the illustration in figure 3. We first obtain

$$\begin{aligned}H_1(t,s) &= E\big[\mathrm{e}^{-tB^{(1)}-sI^{(1)}}\big] = E\big[\mathrm{e}^{-tB^{(1)}-s(D^{(1)}+B^{(0)}+D^{(0)})}\big]\\ &= E\big[\mathrm{e}^{-tB^{(1)}-s(D^{(1)}+B^{(0)})}\big]\widetilde{D}^{(0)}(s)\\ &= E\big[E\big[\mathrm{e}^{-tB^{(1)}-s(D^{(1)}+B^{(0)})} \mid \mathbf{Y}, B^{(1)}, D^{(1)}\big]\big]\widetilde{D}^{(0)}(s)\\ &= E\big[\mathrm{e}^{-tB^{(1)}-sD^{(1)}}E\big[\mathrm{e}^{-sB^{(0)}} \mid \mathbf{Y}, B^{(1)}, D^{(1)}\big]\big]\widetilde{D}^{(0)}(s).\end{aligned}$$

Then, by using corollary 2.2,

$$\begin{aligned}&E\big[\mathrm{e}^{-sB^{(0)}} \mid \mathbf{Y}, B^{(1)}, D^{(1)}\big]\\ &\quad= E\big[E\big[\mathrm{e}^{-sB^{(0)}} \mid X_0\big] \mid \mathbf{Y}, B^{(1)}, D^{(1)}\big] = E\big[\widetilde{B}^{(0)}_{X_0}(s) \mid \mathbf{Y}, B^{(1)}, D^{(1)}\big]\\ &\quad= E\big[\big[\big(1-\beta(s)\big)\big(\widetilde{\theta}^{(0)}(\lambda_1+s)\big)^{X_0} + \beta(s)\big(\widetilde{V}_0(s)\big)^{X_0}\big] \mid \mathbf{Y}, B^{(1)}, D^{(1)}\big]\\ &\quad= \big(1-\beta(s)\big)E\big[\widetilde{\theta}_0(\lambda_1+s)^{X_0} \mid \mathbf{Y}, B^{(1)}, D^{(1)}\big] + \beta(s)E\big[\widetilde{V}_0(s)^{X_0} \mid \mathbf{Y}, B^{(1)}, D^{(1)}\big],\end{aligned}$$

where $\beta(s) = \lambda_1/(\lambda_1 + \lambda_0(1-\widetilde{V}_0(s)))$.

Using (3.1) once more, and taking $z = \widetilde{\theta}^{(0)}(\lambda_1+s)$,

$$E\big[z^{X_0} \mid \mathbf{Y}, B^{(1)}, D^{(1)}\big] = E\big[z^{Y_0+A_0(B^{(1)}+D^{(1)})} \mid \mathbf{Y}, B^{(1)}, D^{(1)}\big] = z^{Y_0}\mathrm{e}^{-w(B^{(1)}+D^{(1)})},$$

where $w = \lambda_0(1-z)$. Hence,

$$\begin{aligned}&E\big[\mathrm{e}^{-tB^{(1)}-sD^{(1)}}z^{Y_0}\mathrm{e}^{-w(B^{(1)}+D^{(1)})}\big]\\ &\quad= E\big[z^{Y_0}\mathrm{e}^{-(t+w)B^{(1)}}\mathrm{e}^{-(s+w)D^{(1)}}\big] = E\big[z^{Y_0}\mathrm{e}^{-(t+w)B^{(1)}}\big]\widetilde{D}^{(1)}(s+w)\\ &\quad= E\big[E\big[z^{Y_0}\mathrm{e}^{-(t+w)B^{(1)}} \mid \mathbf{Y}\big]\big]\widetilde{D}^{(1)}(s+w) = E\big[z^{Y_0}E\big[\mathrm{e}^{-(t+w)\theta^{(1)}_{Y_1}} \mid Y_1\big]\big]\widetilde{D}^{(1)}(s+w)\\ &\quad= E\big[z^{Y_0}\big(\widetilde{\theta}^{(1)}(t+w)\big)^{Y_1}\big]\widetilde{D}^{(1)}(s+w) = G_{\mathbf{Y}}\big(z, \widetilde{\theta}^{(1)}(t+w)\big)\widetilde{D}^{(1)}(s+w).\end{aligned}$$

Therefore,

$$\begin{aligned}H_1(t,s) &= E\big[\mathrm{e}^{-tB^{(1)}-sD^{(1)}}\big\{\big(1-\beta(s)\big)E\big[z^{X_0} \mid \mathbf{Y}, B^{(1)}, D^{(1)}\big]\big|_{z=\widetilde{\theta}^{(0)}(\lambda_1+s)}\\ &\qquad + \beta(s)E\big[z^{X_0} \mid \mathbf{Y}, B^{(1)}, D^{(1)}\big]\big|_{z=\widetilde{V}_0(s)}\big\}\big]\widetilde{D}^{(0)}(s)\\ &= \big\{\big(1-\beta(s)\big)E\big[\mathrm{e}^{-tB^{(1)}-sD^{(1)}}z^{Y_0}\mathrm{e}^{-w(B^{(1)}+D^{(1)})}\big]\big|_{z=\widetilde{\theta}^{(0)}(\lambda_1+s),\, w=\lambda_0(1-z)}\\ &\qquad + \beta(s)E\big[\mathrm{e}^{-tB^{(1)}-sD^{(1)}}z^{Y_0}\mathrm{e}^{-w(B^{(1)}+D^{(1)})}\big]\big|_{z=\widetilde{V}_0(s),\, w=\lambda_0(1-z)}\big\}\widetilde{D}^{(0)}(s).\end{aligned}$$

Finally, we can state:

Proposition 3.8.

$$\begin{aligned}H_1(t,s) = \big\{&\big(1-\beta(s)\big)G_{\mathbf{Y}}\big(z_0, \widetilde{\theta}^{(1)}\big(t+\lambda_0(1-z_0)\big)\big)\widetilde{D}^{(1)}\big(s+\lambda_0(1-z_0)\big)\big|_{z_0=\widetilde{\theta}^{(0)}(\lambda_1+s)}\\ &+ \beta(s)G_{\mathbf{Y}}\big(z_0, \widetilde{\theta}^{(1)}\big(t+\lambda_0(1-z_0)\big)\big)\widetilde{D}^{(1)}\big(s+\lambda_0(1-z_0)\big)\big|_{z_0=\widetilde{V}(s)}\big\}\widetilde{D}^{(0)}(s),\end{aligned}$$

where $\beta(s) = \lambda_1/(\lambda_1 + \lambda_0(1-\widetilde{V}_0(s)))$.

Since $C^{(1)} = B^{(1)} + I^{(1)}$ we also have:

Corollary 3.9.

$$\widetilde{C}^{(1)}(t) = H_1(t,t).$$

To conclude, we compute the joint LST of the server's busy and idle intervals during $C^{(0)}$ *and* $C^{(1)}$:

$$Q_0(t,s) = E\big[\mathrm{e}^{-tB^{(0)}-sD^{(0)}-tB^{(1)}-sD^{(1)}}\big] \quad \text{(see figure 2)},$$
$$Q_1(t,s) = E\big[\mathrm{e}^{-tB^{(1)}-sD^{(1)}-tB^{(0)}-sD^{(0)}}\big] \quad \text{(see figure 3)}.$$

Since the computation of $Q_0(t,s)$ and $Q_1(t,s)$ is very similar to that of $H_0(t,s)$ and $H_1(t,s)$, respectively, we state without proof:

Proposition 3.10.

$$\begin{aligned}Q_0(t,s) = {} & \big\{\big(1-h(t,w_1)\big)G_{\mathbf{X}}\big(\widetilde{\theta}^{(0)}(t+\lambda_1), z_1\big) + h(t,w_1)G_{\mathbf{X}}\big(\widetilde{V}_0(t+w_1), z_1\big)\big\}\\ & \times \widetilde{D}^{(0)}(s+w_1)\widetilde{D}^{(1)}(s),\end{aligned}$$

where

$$z_1 = \widetilde{\theta}^{(1)}(t), \qquad w_1 = \lambda_1(1-z_1), \qquad h(t,w_1) = \frac{\lambda_1 z_1}{\lambda_1 z_1 + \lambda_0(1-\widetilde{V}_0(t+w_1))}.$$

Proposition 3.11.

$$\begin{aligned}Q_1(t,s) = {} & \big\{\big(1-\beta(t)\big)G_{\mathbf{Y}}\big(z_0, \widetilde{\theta}^{(1)}\big(t+\lambda_0(1-z_0)\big)\big)\widetilde{D}^{(1)}\big(s+\lambda_0(1-z_0)\big)\big|_{z_0=\widetilde{\theta}_0(\lambda_1+t)}\\ & + \beta(t)G_{\mathbf{Y}}\big(z_0, \widetilde{\theta}^{(1)}\big(t+\lambda_0(1-z_0)\big)\big)\widetilde{D}^{(1)}\big(s+\lambda_0(1-z_0)\big)\big|_{z_0=\widetilde{V}_0(t)}\big\}\widetilde{D}^{(0)}(s),\end{aligned}$$

where $\beta(t) = \lambda_1/(\lambda_1 + \lambda_0(1-\widetilde{V}_0(t)))$.

3.6. *Mean queue size and waiting time*

Let $L^{(i)}$ and $W^{(i)}$ be the queue size and waiting time in channel i ($i = 0, 1$) in steady state. We claim

Proposition 3.12.

$$E\big[L^{(0)}\big] = \rho_0\left(1+\frac{\lambda_0}{\lambda_1}\right) + \frac{1-\rho}{d}\left(\frac{1+\rho_0}{2\lambda_0}E\big[X_0(X_0-1)\big] - \frac{\lambda_0\gamma_0}{\lambda_1}\right),$$
$$E\big[L^{(1)}\big] = E[L_{M_1/G_1/1}] + \frac{1-\rho}{1-\rho_1}\frac{E[Y_1(Y_1-1)]}{2\lambda_1 d}.$$

Proof. Applying theorem 2.9 for queue 0 in isolation, while setting the Timer's parameter to be λ_1, the inter-arrival rate to queue 1, and by using $E[X_0] = \lambda_0 d/(1-\rho) - \lambda_0\gamma_0$

(proposition 3.4) along with $E[N^{(0)}] = \lambda_0 d/(1-\rho)$ (proposition 3.5), we obtain the expression for $E[L^{(0)}]$.

As was noted in section 2, the exhaustive service discipline is a limiting case of the general RTG model when $\mu \to 0$. In this case the probability generating function of the queue size $L^{(1)}$ in queue 1 is given by (see [10, equation (4.55)])

$$\widehat{L}^{(1)}(z) = \widehat{L}_{M/G/1}(z) \cdot \frac{1-\widehat{Y}_1(z)}{E[Y_1](1-z)},$$

where $L_{M/G/1}$ is the queue size in steady state of the regular $M/G/1$ queue with arrival rate λ_1 and service times V_1. Now, the expression for $E[L^{(1)}]$ is obtained from $E[L^{(1)}] = (\mathrm{d}/\mathrm{d}z)[\widehat{L}^{(1)}(z)]|_{z=1}$. □

Using Little's law, $E[W^{(i)}] = E[L^{(i)}]/\lambda_i$, we readily have:

Proposition 3.13.

$$E\big[W^{(0)}\big] = \rho_0\left(\frac{1}{\lambda_0} + \frac{1}{\lambda_1}\right) + \frac{1-\rho}{d}\left(\frac{1+\rho_0}{2\lambda_0^2}E\big[X_0(X_0-1)\big] - \frac{\gamma_0}{\lambda_1}\right),$$
$$E\big[W^{(1)}\big] = E[W_{M_1/G_1/1}] + \frac{1-\rho}{1-\rho_1}\frac{E[Y_1(Y_1-1)]}{2\lambda_1^2 d}.$$

Note that $E[X_0(X_0-1)]$ and $E[Y_1(Y_1-1)]$ can be obtained by solving the set of six linear equations:

$$E\big[X_0(X_0-1)\big] = \left.\frac{\partial^2 G_{\mathbf{X}}(\mathbf{z})}{\partial z_0^2}\right|_{\mathbf{z}=(1,1)}, \qquad E\big[Y_0(Y_0-1)\big] = \left.\frac{\partial^2 G_{\mathbf{Y}}(\mathbf{z})}{\partial z_0^2}\right|_{\mathbf{z}=(1,1)},$$
$$E[X_0X_1] = \left.\frac{\partial^2 G_{\mathbf{X}}(\mathbf{z})}{\partial z_0 \partial z_1}\right|_{\mathbf{z}=(1,1)}, \qquad E[Y_0Y_1] = \left.\frac{\partial^2 G_{\mathbf{Y}}(\mathbf{z})}{\partial z_0 \partial z_1}\right|_{\mathbf{z}=(1,1)},$$
$$E\big[X_1(X_1-1)\big] = \left.\frac{\partial^2 G_{\mathbf{X}}(\mathbf{z})}{\partial z_1^2}\right|_{\mathbf{z}=(1,1)}, \qquad E\big[Y_1(Y_1-1)\big] = \left.\frac{\partial^2 G_{\mathbf{Y}}(\mathbf{z})}{\partial z_1^2}\right|_{\mathbf{z}=(1,1)},$$

where $G_X(z)$ and $G_Y(z)$ are given in proposition 3.2.

3.7. Numerical calculations

The expressions for $E[W^{(0)}]$ and $E[W^{(1)}]$ given in proposition 3.13 depend on the values of $E[X_0(X_0-1)]$ and $E[Y_1(Y_1-1)]$, which can be obtained by twice differentiating $G_X(\cdot)$ and $G_Y(\cdot)$, and also depend on γ_0, which itself is a function of the PGF of X_0 (evaluated at $\tilde{\theta}^{(0)}(\lambda_1)$). To calculate those expressions we employ a numerical algorithm (summarized in appendix A), whose results are depicted in figures 4–6.

For specific calculations we assumed exponential service times with rates μ_1 and μ_2, and deterministic switch-over times d (common in communication systems).

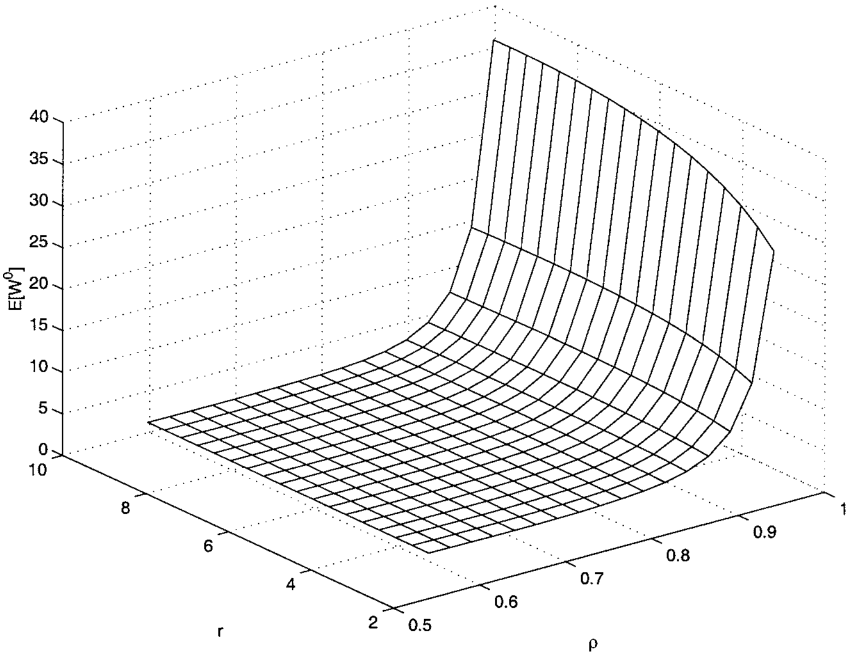


Figure 4. $E[W^{(0)}]$ as a function of r and ρ. Parameters used are $\mu_0 = \mu_1 = 2,\ d = 1,\ \lambda_0 = \mu_0(r/r+1)\rho$ and $\lambda_1 = \mu_0(1/r+1)\rho$.

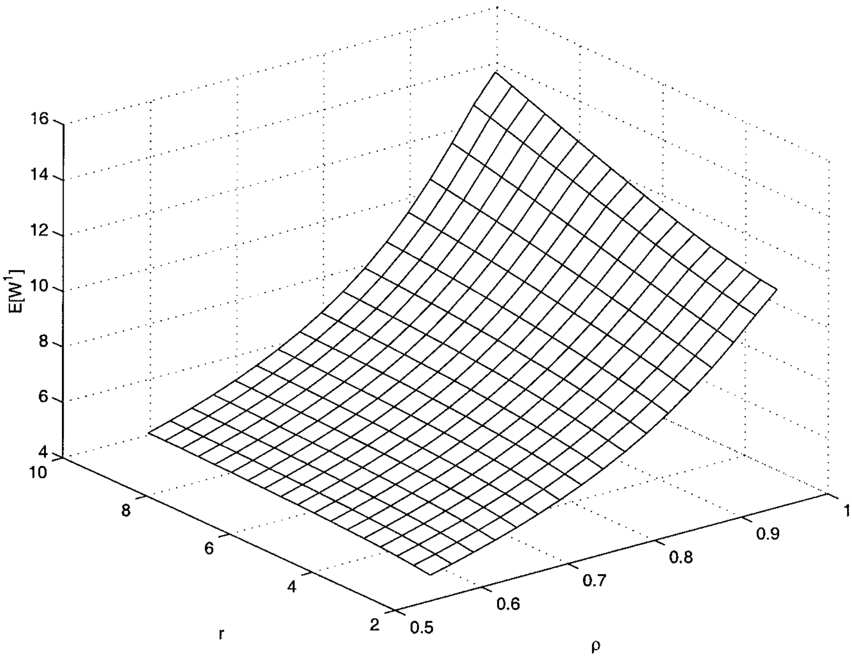


Figure 5. $E[W^{(1)}]$ as a function of r and ρ. Parameters are as in figure 4.

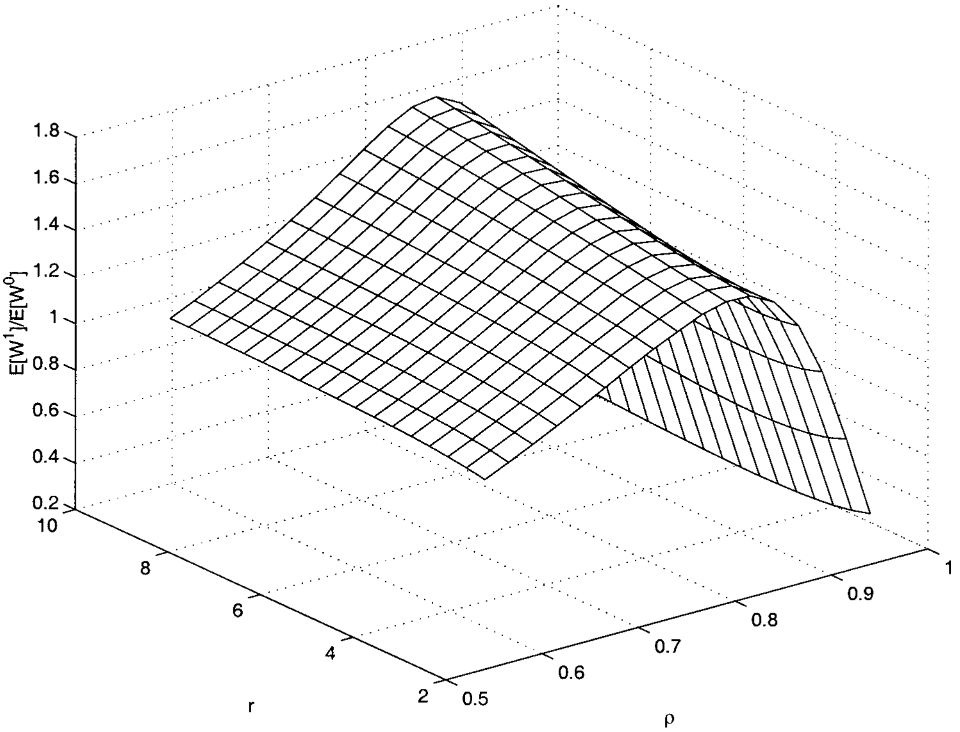


Figure 6. $E[W^{(1)}]/E([W^{(0)}]$ as a function of r and ρ. Parameters are as in figure 4.

We use $\mu_0 = \mu_1 = 2$, $d = 1$, and compute $E[W^{(0)}]$ and $E[W^{(1)}]$ as function of λ_1 and λ_2. The results are plotted as functions of the pair (r, ρ), where

$$\lambda_0 = 2\frac{r}{r+1}\rho, \qquad \lambda_1 = 2\frac{1}{r+1}\rho,$$

on the domain $2.2 \leqslant r \leqslant 9$; $0.55 \leqslant \rho \leqslant 0.95$.

Figures 4 and 5 depict $E[W^{(0)}]$ and $E[W^{(1)}]$ as a function of the pair (r, ρ), respectively. Figure 6 depicts the *ratio* $E[W^{(1)}]/E[W^{(0)}]$ as a function of that pair. As expected, both $E[W^{(0)}]$ and $E[W^{(1)}]$ increase in ρ, exhibiting exponential growth near $\rho = 1$. We further observe that for any given ρ, both $E[W^{(0)}]$ and $E[W^{(1)}]$ increase with r, showing the diverse effect of the increasing relative load of the secondary queue on the waiting times in both queues.

An interesting phenomenon is that the ratio $E[W^{(1)}]/E[W^{(0)}]$ increases initially as ρ grows, but the trend is *reversed* when ρ gets closer to 1. The reason for that is that the rate of exponential growth of $E[W^{(1)}]$ is *slower* than that of $E[W^{(0)}]$, showing the improving effect of the RTG regime with respect to the primary queue, which is the goal of that regime.

Appendix A. Numerical algorithm

The equations for G_X and G_Y in proposition 3.2 can be rewritten in a decoupled form as

$$G(z_0, z_1) = \sum_{k=1}^{2} A^{(k)}(z_0, z_1) \cdot G\big(\Phi^{(k)}(z_0, z_1)\big), \quad (z_0, z_1) \in [0, 1]^2. \tag{A.1}$$

When the service times are exponential with rates μ_0 and μ_1, we have for $G = G_X$,

$$\begin{aligned}
A^{(1)}(z_0, z_1) &= \big(1 - g\big(z_0, h(z_0)\big)\big) \cdot \exp\big\{d\big(2\lambda_0 z_0 + \lambda_1 z_1 + \lambda_1 h(z_0) - 2\lambda_0 - 2\lambda_1\big)\big\},\\
A^{(2)}(z_0, z_1) &= g\big(z_0, h(z_0)\big) \cdot \exp\big\{d\big(2\lambda_0 z_0 + \lambda_1 z_1 + \lambda_1 h(z_0) - 2\lambda_0 - 2\lambda_1\big)\big\},\\
\Phi^{(1)}(z_0, z_1) &= \big(\Theta_0(\lambda_1), h(z_0)\big),\\
\Phi^{(2)}(z_0, z_1) &= \big(v\big(z_0, h(z_0)\big), h(z_0)\big),
\end{aligned}$$

and for $G = G_Y$,

$$\begin{aligned}
A^{(1)}(z_0, z_1) &= \big(1 - g(z_0, z_1)\big) \cdot \exp\big\{d\big(\lambda_0 z_0 + 2\lambda_1 z_1 + \lambda_0 \Theta_0(\lambda_1) - 2\lambda_0 - 2\lambda_1\big)\big\},\\
\Phi^{(1)}(z_0, z_1) &= \big(\Theta_0(\lambda_1), h\big(\Theta_0(\lambda_1)\big)\big),\\
A^{(2)}(z_0, z_1) &= g(z_0, z_1) \cdot \exp\big\{d\big(\lambda_0 z_0 + 2\lambda_1 z_1 + \lambda_0 v(z_0, z_1) - 2\lambda_0 - 2\lambda_1\big)\big\},\\
\Phi^{(2)}(z_0, z_1) &= \big(v(z_0, z_1), h\big(v(z_0, z_1)\big)\big),
\end{aligned}$$

where

$$\begin{aligned}
\Theta_i(s) &= \frac{(\lambda_i + \mu_i + s) - \sqrt{(\lambda_i + \mu_i + s)^2 - 4\lambda_i \mu_i}}{2\lambda_i}, \quad i = 0, 1,\\
h(s) &= \Theta_1\big(\lambda_0(1 - s)\big),\\
v(z_0, z_1) &= \frac{\mu_0}{(\mu_0 + \lambda_0 + \lambda_1) - \lambda_0 z_0 - \lambda_1 z_1},\\
g(z_0, z_1) &= \frac{\lambda_1 z_1}{\lambda_1 z_1 + \lambda_0 z_0 - \lambda_0 v(z_0, z_1)}.
\end{aligned}$$

We introduce the grid $\{0/N, 1/N, \ldots, N/N\}^2$ and discretize (A.1) as

$$\begin{aligned}
G_{i,j} = \sum_{k=1}^{2} A_{i,j}^{(k)} \big[&\alpha_{1,ij}^{(k)} G_{m_{ij}^{(k)}, n_{ij}^{(k)}} + \alpha_{2,ij}^{(k)} G_{m_{ij}^{(k)}+1, n_{ij}^{(k)}}\\
&+ \alpha_{3,ij}^{(k)} G_{m_{ij}^{(k)}, n_{ij}^{(k)}+1} + \alpha_{4,ij}^{(k)} G_{m_{ij}^{(k)}+1, n_{ij}^{(k)}+1}\big],
\end{aligned} \tag{A.2}$$

where $i, j = 0, 1, \ldots, N$, $G_{ij} = G(i/N, j/N)$ and $A_{ij}^{(k)} = A^{(k)}(i/N, j/N)$. The value of $G(\Phi^{(k)}(i/N, j/N))$ is approximated using linear interpolation of the values of G at the four nearest grid-points.

Equations (A.2), together with the condition that $G(1, 1) = 1$, are linear and can be written in a matrix form as $AG = b$, where A is a sparse $(N + 1)^2 \times (N + 1)^2$ matrix.

Since solving this system requires $O(N^4)$ operations, one can only solve this system for moderate values of N. Therefore, the partial derivatives at $(1, 1)$ are calculated using high-order one-sided schemes.

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