Letter

Phase transition in the susceptible-infected model on hypernetworks

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We derive the master equations for the Susceptible-Infected (SI) model on general hypernetworks with N-body interactions. We solve these equations exactly for infinite d-regular hypernetworks, and obtain an explicit solution for the expected infection level as a function of time. The solution shows that the epidemic spreads out to the entire population as $t \to \infty$ if and only if the initial infection level exceeds a positive threshold value. This phase transition is a high-order interaction effect, which is absent with pairwise interactions.

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Introduction. Spreading models on complex networks have been used to understand how diseases and information propagate through populations [1,2]. Traditional models represent all the interactions as pairwise contacts between individuals, neglecting the fact that many real-world interactions involve groups larger than two, such as meetings, social gatherings, or coauthorship networks [3]. In recent years, there has been a growing interest in modeling high-order interactions using hypergraphs and simplicial complexes, which can capture multibody interactions more accurately [4,5].

The Susceptible-Infected-Susceptible (SIS) model and the Susceptible-Infected-Recovered (SIR) model on networks with pairwise interactions can exhibit nontrivial final states and phase transitions that are characterized by critical infection and recovery rates that distinguish disease-free from endemic states [6,7]. Moreover, in the SIS and SIR models on hypernetworks, high-order interactions can lead to critical phenomena that are absent in traditional pairwise interactions [8–10].

In contrast, the Susceptible-Infected (SI) model on connected networks [11] and on hypernetworks [12] has so far shown a simpler spreading dynamics where, for any initial infection level, the epidemic spreads out to the entire network. This can be attributed to the fact that the SI model only allows for a single unidirectional transition. For example, in Ref. [11], the authors obtained an exact explicit expression for the expected infection level as a function of time in the SI model on infinite d-regular networks, which shows that for any positive initial infection level, the entire population becomes infected as $t \to \infty$.

In this work, we extend the calculations in Ref. [11] to the SI model on hypernetworks with high-order interactions, and obtain an exact explicit solution for the expected infection level as a function of time. We show analytically and numerically that the solution undergoes a phase transition at a positive initial infection level I_c^0. Specifically, the epidemic spreads to only a fraction of the population if $0 < I^0 \leqslant I_c^0$, but to the entire population if $I_c^0 < I^0 \leqslant 1$, where I^0 denotes the initial infection level. To our knowledge, this is the first demonstration that even in the SI model, higher-order interactions can lead to critical phenomena, which is absent with pairwise interactions.

The Si model on N-body hypernetworks. Consider M individuals/nodes $\mathcal{M} := \{1, \dots, M\}$. The state of individual j at time t is

$$X_j(t) = \begin{cases} 1, & \text{if } j \text{ is infected at time } t, \\ 0, & \text{otherwise,} \end{cases} \quad j \in \mathcal{M}. \tag{1a}$$

The initial states at $t = 0$ are stochastic, so that

$$X_j(0) = X_j^0 \in \{0, 1\}, \quad j \in \mathcal{M}, \tag{1b}$$

and

$$\mathbb{P}\big(X_j^0 = 1\big) = I^0, \quad \mathbb{P}\big(X_j^0 = 0\big) = 1 - I^0, \quad j \in \mathcal{M}, \tag{1c}$$

where $0 < I^0 < 1$, and the random variables $\{X_j^0\}_{j \in \mathcal{M}}$ are independent.

The nodes belong to a directed hypergraph, where every hyperedge links $N - 1$ "tail" nodes to a single "head" node. A hyperedge can transmit the infection to the head node only if all the $N - 1$ tail nodes are infected. If j is susceptible, its infection time is piecewise exponentially distributed with the infection rate

$$\lambda_j(t) = \sum_{\boldsymbol{k} \subset \mathcal{M}} q_{\boldsymbol{k} \to j} \prod_{i=1}^{N-1} X_{k_i}(t), \quad \boldsymbol{k} := \{k_i\}_{i=1}^{N-1}, \quad j \in \mathcal{M}. \tag{1d}$$

Here, $q_{\boldsymbol{k} \to j} \geqslant 0$ is the infection rate of j due to the set of $N - 1$ nodes $\boldsymbol{k}$, provided that all the nodes in $\boldsymbol{k}$ are infected. In addition, $q_{\boldsymbol{k} \to j} > 0$ if and only if $j \notin \boldsymbol{k}$ and the directional hyperdge $\boldsymbol{k} \to j$ exists. Once j becomes infected, it remains so at later times.

The quantity of most interest is the expected infection level,

$$[I](t) := \frac{1}{M} \sum_{j=1}^{M} [I_j](t), \quad [I_j](t) := \mathbb{E}[X_j](t), \tag{2}$$

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where $[I_j]$ is the infection probability of node j. To compute $[I](t)$, let $S_\Omega(t)$ denote the event that all the nodes in $\Omega \subset \mathcal{M}$ are susceptible at time t, and let $[S_\Omega](t) := \mathbb{P}(S_\Omega(t))$. The stochastic dynamics of Eq. (1) can be modeled by the master equations

$$\frac{d[S_\Omega]}{dt} = -\sum_{\boldsymbol{k}\subset\Omega^c,\ |\boldsymbol{k}|=N-1} q_{\boldsymbol{k}\to\Omega}[S_\Omega \cap I_{\boldsymbol{k}}], \tag{3a}$$

where $\Omega^c := \mathcal{M} \setminus \Omega$, $q_{\boldsymbol{k}\to\Omega} := \sum_{m\in\Omega} q_{\boldsymbol{k}\to m}$ is the infection rate of the nodes in Ω due to the $N-1$ in $\boldsymbol{k}$, and $[S_\Omega \cap I_{\boldsymbol{k}}]$ is the probability that all the nodes in Ω are susceptible and all the nodes in $\boldsymbol{k}$ are infected. Using the inclusion–exclusion principle, $[S_\Omega \cap I_{\boldsymbol{k}}]$ can be expressed as

$$[S_\Omega \cap I_{\boldsymbol{k}}] = \sum_{i=0}^{N-1}(-1)^i \sum_{\boldsymbol{n}\subseteq\boldsymbol{k},\ |\boldsymbol{n}|=i} [S_{\Omega\cup\boldsymbol{n}}]. \tag{3b}$$

Combining Eqs. (3a) and (3b), we obtain

$$\frac{d[S_\Omega]}{dt} = -\sum_{\boldsymbol{k}\subset\Omega^c,\ |\boldsymbol{k}|=N-1} q_{\boldsymbol{k}\to\Omega} \sum_{i=0}^{N-1}(-1)^i \sum_{\boldsymbol{n}\subseteq\boldsymbol{k},\ |\boldsymbol{n}|=i} [S_{\Omega\cup\boldsymbol{n}}]. \tag{4a}$$

The initial conditions are [see Eq. (1c)]

$$[S_\Omega](0) = (1 - I^0)^{|\Omega|}. \tag{4b}$$

The master equations [Eq. (4)] are a closed system of $2^M - 1$ equations for $\{[S_\Omega]\}_{\emptyset\neq\Omega\subset\mathcal{M}}$. These equations are valid for every N-body hypernetwork (i.e., for any choice of $\{q_{\boldsymbol{k}\to j}\}$). The master equations are *exact*, as they are derived *without making any approximation*. For $N = 3$, the system [Eq. (4)] reduces to the one derived in Ref. [12].

d-regular N-body hypernetworks. An undirected N-body hypergraph H is called *d-regular* if every node has a hyperdegree d. Let $\boldsymbol{E} = (e_{\boldsymbol{k},j})$ be the adjacency tensor of H, and let all the hyperedges have weight $\frac{q}{d}$. The corresponding d-regular N-body hypernetwork is given by

$$q_{\boldsymbol{k}\to j} = \frac{q}{d} e_{\boldsymbol{k},j}, \quad j \in \mathcal{M}, \quad \boldsymbol{k} \subset \mathcal{M}, \quad |\boldsymbol{k}| = N - 1. \tag{5}$$

As $M \to \infty$, the master equations [Eqs. (4) and (5)] for infinite d-regular N-body hypernetwork have the exact explicit solution (see Supplementary Material [SM] [13]),

$$[I_{\substack{\text{d-reg}\\ \text{N-body}}}](t) = 1 - (1 - I^0)u^d\left(\frac{q}{d}t\right), \tag{6a}$$

where $\frac{u(\cdot)}{1-I^0}$ is the susceptible probability of a degree-one node in an otherwise infinite d-regular N-body hypernetwork,

$$\frac{du}{d\tau} = F_{\substack{\text{d-reg}\\ \text{N-body}}}(u, I^0), \quad u(0) = 1, \tag{6b}$$

and

$$F_{\substack{\text{d-reg}\\ \text{N-body}}} := 1 - u - (1 - (1 - I^0)u^{d-1})^{N-1}. \tag{6c}$$

On two-body networks ($N = 2$), the explicit solution [Eq. (6)] reduces to the one obtained in Ref. [11].

Figure 1 confirms the excellent agreement between numerical simulations of the SI model [Eqs. (1) and (5)] on d-regular networks and hypernetworks, and the exact explicit solution [Eq. (6)]. All the numerical experiments were carried out on hypergraphs of size $M = 10^5$. We generated 10 independent quenched hypergraphs, evolved the SI model on each of them 10 times, and present ensemble-averaged results over the 100 runs [14]. When $N = 2$, the final infection level $I^\infty := \lim_{t\to\infty}[I_{\substack{\text{d-reg}\\ \text{N-body}}}](t)$ is equal to one (i.e., the infection spreads to the entire network) for $I^0 = 0.1, 0.25, 0.4$. When $N = 3$ or 4, however, $I^\infty = 1$ only if I^0 is sufficiently large. Indeed, plotting I^∞ as a function of I^0 reveals a *jump discontinuity* at $I^0 = I^0_c$ (see Fig 2), where $I^0_c = 0$ on networks ($N = 2$), and $I^0_c > 0$ on hypernetworks ($N \geqslant 3$).

To prove these numerical observations, we note that the critical (equilibrium) points of Eq. (6b) are obtained by equating $F_{\substack{\text{d-reg}\\ \text{N-body}}}$ to zero. Since $u(0) = 1$ and $\frac{du}{d\tau}(0) = F_{\substack{\text{d-reg}\\ \text{N-body}}}(1, I^0) < 0$, $u(\tau)$ is monotonically decreasing toward the first critical point below 1, which we shall denote by

$$u^\infty(I^0) := \max_{u<1}\left\{u \mid F_{\substack{\text{d-reg}\\ \text{N-body}}}(u, I^0) = 0\right\}. \tag{7a}$$

Thus, $u^\infty := \lim_{\tau\to\infty} u(\tau)$, and so by Eq. (6a),

$$I^\infty = 1 - (1 - I^0)(u^\infty)^d. \tag{7b}$$

Specifically, on two-body networks ($N = 2$),

$$F_{\substack{\text{d-reg}\\ \text{2-body}}} := u((1 - I^0)u^{d-2} - 1) < 0, \quad 0 < u < 1.$$

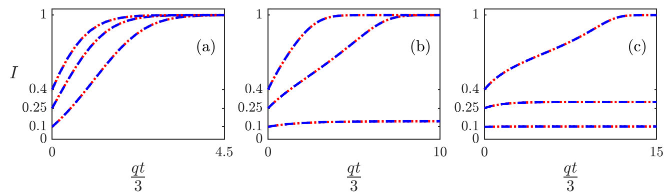

FIG. 1. The infection level on 3-regular N-body hypernetworks as a function of time, for $I^0 = 0.1, 0.25$, and 0.4. The numerical solution of the Susceptible-Infected model [Eqs. (1) and (5)] (red dots) is indistinguishable from the explicit solution [Eq. (6)] (blue dashes). (a) $N = 2$. (b) $N = 3$. (c) $N = 4$.

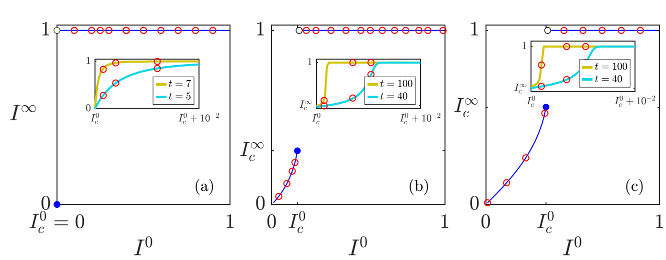

FIG. 2. The final infection level as a function of the initial infection level in the Susceptible-Infected model on 3-regular N-body hypernetworks. The red circles are numerical simulations of the Susceptible-Infected model [Eqs. (1) and (5)]; The blue line is the explicit solution [Eq. (7)]. The insets show the infection levels at $t = t_1$ and $t = t_2$ for initial infection levels slightly above the critical threshold I^0_c. (a) $N = 2$, $I^0_c = 0$, $t_1 = 5$, $t_2 = 7$. (b) $N = 3$, $I^0_c = \frac{5}{32}$, $t_1 = 40$, $t_2 = 100$. (c) $N = 4$, $I^0_c = \frac{1}{108}(6 + 7\sqrt{21})$, $t_1 = 40$, $t_2 = 100$.

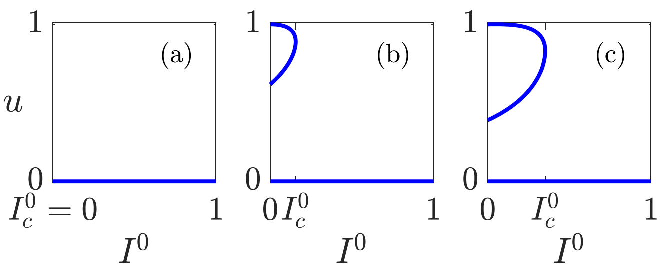

FIG. 3. The critical points $F_{\substack{3-\text{reg}\\ \text{N-body}}}(u, I^0) = 0$ (blue solid line) in $[0, 1]^2$. (a) $N = 2$, $I_c^0 = 0$. (b) $N = 3$, $I_c^0 = \frac{5}{32}$. (c) $N = 4$, $I_c^0 = \frac{1}{108}(6 + 7\sqrt{21})$.

Therefore $u^\infty \equiv 0$ for any $0 < I^0 < 1$ [see Fig 3(a)], and so

$$I^\infty = 0, \quad \text{if } I^0 = 0,$$
$$I^\infty = 1, \quad \text{if } 0 < I^0 \leqslant 1.$$

Hence, $I_c^0 = 0$ [see Fig 2(a)].

On N-body hypernetworks with $N \geqslant 3$, however, $F_{\substack{\text{d}-\text{reg}\\ \text{N-body}}} = uP(u)$, where $P(u)$ is a polynomial of degree $(N-1)(d-1)-1$ that has two real roots in $(0, 1)$ for $0 < I^0 < I_c^0$, a double root at I_c^0, and no real roots in $(0, 1)$ for $I_c^0 < I^0 < 1$ [see SM [13] and Fig 3(b) and 3(c)]. Therefore,

$$I_c^0 \leqslant I^\infty < 1, \quad \text{if } 0 \leqslant I^0 \leqslant I_c^0,$$
$$I^\infty = 1, \quad \text{if } I_c^0 < I^0 \leqslant 1.$$

In particular, $0 < I_c^0 < 1$. The dynamical system [Eq. (6)] thus admits a *saddle-node bifurcation* that leads to a phase transition at I_c^0 [Fig. 3(b) and 3(c)]. This results in a critical slowdown of the dynamics for I^0 slightly above I_c^0 (see insets of Fig 2), so that the time that the solution $[I_{\substack{\text{d}-\text{reg}\\ \text{N-body}}}](t)$ "lingers" around I_c^∞ scales as (see SM [13])

$$T_{\text{slowdown}} \sim \frac{1}{\sqrt{I^0 - I_c^0}}, \quad 0 < I^0 - I_c^0 \ll 1. \tag{8}$$

Intuitively, on infinite d-regular networks ($N = 2$), any two nodes are connected by a finite path, with probability one [15]. Hence, a single infected node at $t = 0$ is sufficient for the epidemic to spread out to the entire network as $t \to \infty$, and so $I_c^0 = 0$. Similarly, on infinite d-regular N-body hypernetworks, any two nodes are connected by a finite hyperpath with probability one [16]. Maintaining the propagation of an infection along a hyperpath, however, requires more than just for the first hyperedge to propagate the infection. Indeed, assume that $N - 1$ nodes within a hyperedge are initially infected. Then the hyperedge propagates the infection to its N^{th} node. To further propagate the infection to additional hyperedges, $N - 2$ nodes of the new hyperedge should be infected, in addition to the infected node from the previous hyperedge. Therefore, the initial infection level I^0 should be sufficiently large for the infection to be able to spread out to the entire hypernetwork.

The explicit expression [Eq. (7)] for the final infection level I^∞ has the following interpretation. As $t \to \infty$, an arbitrary node j is susceptible (with probability $1 - I^\infty$) if and only if j was not infected initially (with probability $1 - I^0$) and if none

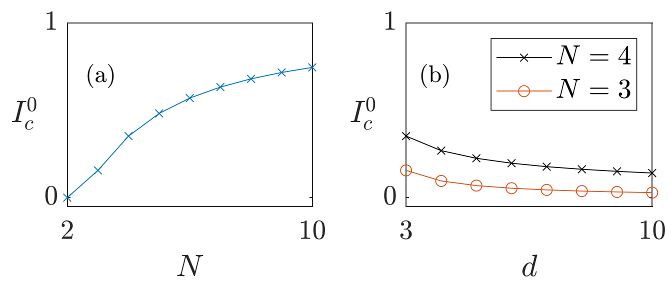

FIG. 4. (a) The initial infection level threshold I_c^0 as a function of N for $d = 3$. (b) I_c^0 as a function of d for $N = 3$, see Eq. (10), and $N = 4$, see Eq. (11).

of its d hyperedges $\{\boldsymbol{e}_i\}_{i=1}^d$ transmitted the infection to j. Let θ denote the probability that a hyperedge $\boldsymbol{e_i}$ that contains j has not transmitted the infection to j as $t \to \infty$. Then

$$1 - I^\infty = (1 - I^0)\theta^d. \tag{9a}$$

Thus, θ is the probability that a hyperedge $\boldsymbol{e_i}$ that contains j has at most $N - 2$ infected nodes as $t \to \infty$, conditioned on the event that j is susceptible as $t \to 0$. Let ϕ denote the probability that a node $k \in \boldsymbol{e_i} \setminus \{j\}$ is infected as $t \to \infty$. Then

$$\theta = 1 - \phi^{N-1}. \tag{9b}$$

The node k is susceptible as $t \to \infty$ if and only if it was not infected initially, and if none of its other $d - 1$ hyperedges transmitted the infection to it. Therefore,

$$1 - \phi = (1 - I^0)\theta^{d-1}. \tag{9c}$$

Substituting (25) in (24) and rearranging (23) gives

$$I^\infty = 1 - (1 - I^0)\theta^d, \quad \theta = 1 - (1 - (1 - I^0)\theta^{d-1})^{N-1},$$

which, after replacing θ with u^∞, gives Eqs. (6c) and (7b).

The initial infection level threshold I_c^0 increases as N increases and d is held fixed [Fig. 4(a)]. This is because each hyperedge requires more infected nodes to propagate the infection. Similarly, I_c^0 decreases as d increases and N is held fixed, since the hypernetwork becomes more connected [Fig. 4(b)]. Indeed, for $N = 3$, we can derive the explicit expression (see SM [13])

$$I_c^0 = 1 - (d-2)^{2-d}(d-1)^{1-d}\left(d - \tfrac{3}{2}\right)^{2d-3}. \tag{10}$$

The corresponding final infection level is [see Eq. (6a)]

$$I_c^\infty = \frac{4d^2 - 10d + 5}{(2d-3)^3}.$$

Similarly, for $N = 4$ we have

$$I_c^0 = 1 - u_c^{-(d-1)}(1 - (1 - u_c)^{\frac{1}{3}}), \tag{11a}$$

where

$$u_c = \frac{3(d-1)(18d(d-3) - \sqrt{12d - 15} + 39)}{2(3d-4)^3}. \tag{11b}$$

Furthermore, for any $N \geqslant 3$ we have that (see SM [13])

$$I_c^0 \sim \frac{N-2}{(N-1)^{\frac{N-1}{N-2}}} d^{-\frac{1}{N-2}}, \quad d \to \infty. \tag{12}$$

As expected, the three explicit expressions, Eqs. (10), (11), and (12), are decreasing in d.

The final infection level I^∞ of the stochastic SI model [Eqs. (1) and (5)] on N-body hypernetworks is equal to the size of the final active set in the deterministic bootstrap percolation model [17]. Our dynamical system formulation provides an alternative approach for obtaining the bootstrap percolation threshold, which is different from those used in graph theory. In Ref. [17], Morrison and Noel derived the asymptotic limit of the percolation threshold in infinite d-regular N-body hypergraphs that undergo a thinning process, whereby each hyperedge is independently kept with a probability that diminishes to zero as $d \to \infty$. Remarkably, although their analysis does not cover the case of d-regular hypergraphs that do not undergo a thinning process, our asymptotic limit [Eq. (12)] as $d \to \infty$ precisely matches theirs. Note, however, that in Ref. [17], they did not obtain the exact critical infection level for finite d, they did not derive the explicit expression [Eq. (6)] for the infection level as a function of time, and they did not show that $I^\infty < 1$ when $I^0 = I_c^0$.

Conclusion. In conclusion, while previous work on the SI model on hypernetworks with N-body interactions showed a dynamics that is qualitatively similar to that on networks [12], this work shows that high-order interactions can lead to a dramatic change in the dynamics. This change is manifested by a phase transition at a *positive* threshold of the initial infection level, which is absent in networks with pairwise interactions. We expect that high-order interactions will lead to critical dynamics in other N-body hypernetworks, such as sparse Erdős–Rényi hypernetworks. From a methodological perspective, this paper differs from most studies of the spreading dynamics on hypernetworks by deriving an *exact* expression for the infection level as a function of time, which is obtained by solving the master equations without making any approximation.

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Data availability. The data that support the findings of this article are openly available [18].

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