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Proof of a Spectral Property related to the singularity formation for the L^2 critical nonlinear Schrödinger equation

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Abstract

We give a proof of a Spectral Property related to the description of the singularity formation for the L^2 critical nonlinear Schrödinger equation $\mathrm{i}u_t + \Delta u + u|u|^{\frac{4}{N}} = 0$ in dimensions $N = 2, 3, 4$.

Assuming this property, the rigorous mathematical analysis developed in a recent series of papers by Merle and Raphaël provides a complete description of the collapse dynamics for a suitable class of initial data. In particular, this implies in dimension $N = 2$ the existence of a large class of solutions blowing up with the log–log speed $|u(t)|_{H^1} \sim \sqrt{\frac{\log|\log(T-t)|}{T-t}}$ where $T > 0$ is the blow up time.

This Spectral Property is equivalent to the coercivity of some Schrödinger type operators. An analytic proof is given in [F. Merle, P. Raphaël, Blow up dynamic and upper bound on the blow up rate for critical nonlinear Schrödinger equation, Ann. of Math. 161 (1) (2005) 157–222] in dimension $N = 1$ and in this paper, we give a computer assisted proof in dimensions $N = 2, 3, 4$. We propose in particular a rigorous mathematical frame to reduce the check of this type of coercivity property to accessible and robust numerical results.

Keywords: Nonlinear Schrödinger equation; Self-similar solution; Singularity; Collapse; Log–log law

1. Introduction

1.1. Setting of the problem

This paper is devoted to the proof of a Spectral Property which is at the heart of the description of the singularity formation for the L^2 critical nonlinear Schrödinger equation

$$(\mathbf{NLS}) \begin{cases} \mathrm{i}u_t = -\Delta u - |u|^{\frac{4}{N}} u, & (t,x) \in [0,T) \times \mathbf{R}^N \\ u(0,x) = u_0(x), & u_0 : \mathbf{R}^N \to \mathbf{C} \end{cases} \tag{1}$$

with $u_0 \in H^1 = H^1(\mathbf{R}^N)$ in dimension $N \geq 1$. Let us briefly recall the main known facts about (1) and refer to [14] and references therein for a more complete introduction to the problem.

From a result of Ginibre and Velo [7], (1) is locally well-posed in H^1 and thus, for $u_0 \in H^1$, there exists $0 < T \leq +\infty$ such that $u(t) \in C([0,T), H^1)$ and either $T = +\infty$, and we say the solution is global, or $T < +\infty$ and then $\limsup_{t\uparrow T} |\nabla u(t)|_{L^2} = +\infty$, and we say the solution blows up in finite time.

Eq. (1) is an infinite dimensional Hamiltonian system with the following conservation laws in the energy space H^1:

$$L^2\text{-norm: } \int |u(t,x)|^2 = \int |u_0(x)|^2;$$

$$\text{Energy: } E(u(t,x)) = \frac{1}{2}\int |\nabla u(t,x)|^2 - \frac{1}{2+\frac{4}{N}} \int |u(t,x)|^{2+\frac{4}{N}} = E(u_0);$$

$$\text{Momentum: } \mathrm{Im}\left(\int \nabla u \overline{u}(t,x)\right) = \mathrm{Im}\left(\int \nabla u_0 \overline{u_0}(x)\right).$$

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It is classical from the conservation of the energy and the L^2-norm that the power nonlinearity in (1) is the smallest one for which blow up may occur. Indeed, recall the Gagliardo–Nirenberg inequality:

$$\int |u|^{p+1} \leq C \left(\int |u|^2\right)^{\frac{p+1}{2}-\alpha_p} \left(\int |\nabla u|^2\right)^{\alpha_p} \quad \text{with } \alpha_p = \frac{N(p-1)}{4};$$

then for $p < 1 + \frac{4}{N}$, $\alpha_p < 1$ and thus the conservation of the energy and the L^2 norm imply a uniform bound on the H^1 norm of the solution which is thus global and bounded from the Cauchy theory; see [7].

Eq. (1) admits a number of symmetries in the energy space H^1: if $u(t,x)$ is a solution to (1) then $\forall(\lambda_0, t_0, x_0, \beta_0, \gamma_0) \in \mathbf{R}^+_* \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}$, so is

$$v(t,x) = \lambda_0^{\frac{N}{2}} u(t+t_0, \lambda_0 x + x_0 - \beta_0 t) \mathrm{e}^{\mathrm{i}\frac{\beta_0}{2}\cdot(x - \frac{\beta_0}{2}t)} \mathrm{e}^{\mathrm{i}\gamma_0}.$$

A last symmetry is not in the energy space H^1 but in the virial space Σ, the pseudo-conformal transformation: if $u(t,x)$ solves (1), then so does

$$v(t,x) = \frac{1}{|t|^{\frac{N}{2}}} \overline{u}\left(\frac{1}{t}, \frac{x}{t}\right) \mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}}.$$

Special solutions play a fundamental role for the description of the dynamics of (1). They are the so-called solitary waves of the form $u(t,x) = \mathrm{e}^{\mathrm{i}\omega t} W_\omega(x)$, $\omega > 0$, where W_ω solves

$$\Delta W_\omega + W_\omega |W_\omega|^{\frac{4}{N}} = \omega W_\omega. \tag{2}$$

Eq. (2) is a standard nonlinear elliptic equation, and from [1,6,8], there is a unique positive solution up to translation $Q_\omega(x)$. Q_ω is in addition radially symmetric. Letting $Q = Q_{\omega=1}$, then $Q_\omega(x) = \omega^{\frac{N}{4}} Q(\omega^{\frac{1}{2}} x)$ from scaling property. Note that in dimension $N = 1$, the Q equation is explicitly integrable and

$$Q(x) = \left(\frac{3}{\mathrm{ch}^2(2x)}\right)^{\frac{1}{4}}. \tag{3}$$

For $|u_0|_{L^2} < |Q|_{L^2}$, the solution is global in H^1 from the conservation of the energy, the L^2-norm and the Gagliardo–Nirenberg inequality as exhibited by Weinstein in [26]:

$$\forall u \in H^1, \quad E(u) \geq \frac{1}{2}\left(\int |\nabla u|^2\right)\left(1 - \left(\frac{\int |u|^2}{\int Q^2}\right)^{\frac{2}{N}}\right).$$

In addition, this condition is sharp: for $|u_0|_{L^2} \geq |Q|_{L^2}$, blow up may occur. Indeed, the pseudo-conformal transformation applied to the stationary solution $\mathrm{e}^{\mathrm{i}t} Q(x)$ yields an explicit solution

$$S(t,x) = \frac{1}{|t|^{\frac{N}{2}}} Q\left(\frac{x}{t}\right) \mathrm{e}^{-\mathrm{i}\frac{|x|^2}{4t} + \frac{\mathrm{i}}{t}}$$

which blows up at $T = 0$ with $|S(t)|_{L^2} = |Q|_{L^2}$. Note that the blow up speed for $S(t)$ is:

$$|\nabla S(t)|_{L^2} \sim \frac{1}{|t|}.$$

Moreover, from [13], $S(t)$ is the unique minimal mass finite time blow up solution up to the symmetries.

Most results concerning the blow up dynamics of (1) now concern the perturbative situation when

$$u_0 \in B_{\alpha^*} = \left\{u_0 \in H^1 \text{ with } \int Q^2 \leq \int |u_0|^2 < \int Q^2 + \alpha^*\right\},$$

for some small constant $\alpha^* > 0$. In this setting, the variational characterization of the ground state Q as a blow up profile implies that finite time blow up solutions to (1) admit near the blow up time a geometrical decomposition

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{N}{2}}}(Q + \varepsilon)\left(t, \frac{x - x(t)}{\lambda(t)}\right) \mathrm{e}^{\mathrm{i}\gamma(t)}, \tag{4}$$

where

$$|\varepsilon(t)|_{H^1} \ll 1 \tag{5}$$

and

$$\lambda(t) \sim \frac{1}{|\nabla u(t)|_{L^2}}.$$

Smallness estimate (5) allows a perturbative analysis for (1) and at least two different blow up behaviors are known to possibly occur:

- There exists in dimensions $N = 1, 2$ a family of solutions of type $S(t)$ by a result of Bourgain and Wang, [2], that is solutions with $|\nabla u(t)|_{L^2} \sim \frac{1}{T-t}$ near blow up time.
- On the other hand, it has been suspected since the 1970's that the blow up speed of generic initial data is different from the $S(t)$ one, which indeed is never observed numerically. Let us say that quite an amount of both formal and numerical works has been devoted to the derivation of the exact blow up law for (1) and that different laws have been proposed, in particular by Zakharov [28]. Then in the 1980's, a combination of refined numerical simulations and formal asymptotic expansions led Fraiman [5], and independently LeMesurier, Landman, Papanicolaou, Sulem and Sulem [9,10], to propose in dimension $N = 2$ the log–log law $|\nabla u(t)|_{L^2} \sim \left(\frac{\log|\log(T-t)|}{T-t}\right)^{\frac{1}{2}}$ as the generic blow up speed. Note, however that the log–log law regime is reached only at exceedingly huge ($\gg 10^{200}$) focusing levels; see for example [4]. Further formal arguments to explain the log–log correction to self similar blow up may also be found in Fibich and Papanicolaou [4], Dyachenko et al. [3] and Pelinovsky [20]. We refer to the monograph [25] and references therein for a complete introduction to the history of the problem. Then in 2001, Perelman in [21] establishes rigorously the existence in dimension $N = 1$ of an even log–log solution and its stability in some space $E \subset H^1$.

The situation has been clarified in the series of papers [14–19] which provide a detailed insight into the singularity formation. The analysis relies on a refined studies of the dispersive effects of (1) in the vicinity of the ground state Q. At the heart of the proof lies a Spectral Property which corresponds to positivity properties of some explicit Schrödinger operators and implies the existence of new Lyapounov type functionals for (1). We note $y = (y_i)_{1\le i\le N}$ the space variable and $r = |y|$ the radial coordinate.

Spectral Property: *Let $N \ge 1$. Consider the two real Schrödinger operators*

$$L_1 = -\Delta + V_1, \qquad L_2 = -\Delta + V_2 \tag{6}$$

where

$$V_1(r) = \frac{2}{N}\left(\frac{4}{N}+1\right) Q^{\frac{4}{N}-1} r Q', \qquad V_2(r) = \frac{2}{N} Q^{\frac{4}{N}-1} r Q' \tag{7}$$

and the real valued quadratic form for $\varepsilon = \varepsilon_1 + \mathrm{i}\varepsilon_2 \in H^1$:

$$\begin{aligned} H(\varepsilon,\varepsilon) &= H_1(\varepsilon_1,\varepsilon_1) + H_2(\varepsilon_2,\varepsilon_2) \\ &= (L_1\varepsilon_1,\varepsilon_1) + (L_2\varepsilon_2,\varepsilon_2). \end{aligned} \tag{8}$$

Let

$$Q_1 = \frac{N}{2}Q + y\cdot\nabla Q, \qquad Q_2 = \frac{N}{2}Q_1 + y\cdot\nabla Q_1.$$

Then there exists a universal constant $\delta_0 > 0$ such that $\forall \varepsilon \in H^1$, if $(\varepsilon_1, Q) = (\varepsilon_1, Q_1) = (\varepsilon_1, y_i Q)_{1\le i\le N} = (\varepsilon_2, Q_1) = (\varepsilon_2, Q_2) = (\varepsilon_2, \partial_{y_i} Q)_{1\le i\le N} = 0$, then

$$H(\varepsilon,\varepsilon) \ge \delta_0 \left(\int |\nabla\varepsilon|^2 + \int |\varepsilon|^2 \mathrm{e}^{-|y|}\right).$$

We then have:

Theorem 1 (*Dynamics of (1), [14–18,22]*). *Let $N = 1$ or $N \ge 2$ assuming the Spectral Property holds true. There exist universal constants $\alpha^* > 0$, $C^* > 0$ such that the following holds true. For $u_0 \in B_{\alpha^*}$, let $u(t)$ be the corresponding solution to (1) with $[0,T)$ its maximum time interval existence on the right in H^1.*

(i) *Description of the singularity: Assume that $u(t)$ blows up in finite time, i.e. $0 < T < +\infty$, then there exist parameters $(\lambda(t), x(t), \gamma(t)) \in \mathbf{R}^*_+ \times \mathbf{R}^N \times \mathbf{R}$ and an asymptotic profile $u^* \in L^2$ such that*

$$u(t) - \frac{1}{\lambda(t)^{\frac{N}{2}}} Q\left(\frac{x - x(t)}{\lambda(t)}\right) \mathrm{e}^{\mathrm{i}\gamma(t)} \to u^* \quad \text{in } L^2 \text{ as } t \to T.$$

Moreover, the blow up point is finite in the sense that

$$x(t) \to x(T) \in \mathbf{R}^N \quad \text{as } t \to T.$$

(ii) *Estimates on the blow up speed: We have either*

$$\lim_{t\to T} \frac{|\nabla u(t)|_{L^2}}{|\nabla Q|_{L^2}} \left(\frac{T-t}{\log|\log(T-t)|}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2\pi}}, \tag{9}$$

or

$$|\nabla u(t)|_{L^2} \ge \frac{C(u_0)}{(T-t)},$$

for t close enough to T.

(iii) *Sufficient condition for log–log blow up: If $E(u_0) \le 0$ and $\int Q^2 < \int |u_0|^2 < \int Q^2 + \alpha^*$, then $u(t)$ blows up in finite time with the log–log speed (9). More generally, the set of initial data $u_0 \in B_{\alpha^*}$ such that the corresponding solution $u(t)$ to (1) blows up in finite time $0 < T < +\infty$ with the log–log speed (9) is open in H^1.*

1.2. The Spectral Property

The analysis developed for the proof of Theorem 1 is an N-dimensional H^1 theory. The only part of the proof which is not complete in any dimension is the Spectral Property. It has been proved in [14] for dimension $N = 1$ by using the explicit formula (3) for the ground state in this case. It was also remarked in [14,16] that the linear operators L_1, L_2 given by (6) are related through a remarkable commutation formula to the standard linear operators close to the ground state, as studied by Weinstein in [26] and given by

$$L_+ = -\Delta + 1 - \left(1 + \frac{4}{N}\right) Q^{\frac{4}{N}}, \qquad L_- = -\Delta + 1 - Q^{\frac{4}{N}}.$$

Indeed, for a given function f, let

$$f_1 = \frac{N}{2} f + y\cdot\nabla f = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\lambda^{\frac{N}{2}} f(\lambda y)\right]_{\lambda=1},$$

then

$$\begin{aligned} L_1 f &= \frac{1}{2}\left[L_+(f_1) - (L_+ f)_1\right], \\ L_2 f &= \frac{1}{2}\left[L_-(f_1) - (L_- f)_1\right]. \end{aligned} \tag{10}$$

At this stage, we do not know how to use the spectral structure of (L_+, L_-) which is known, see [26], and the commutation formula to give an analytic proof of the Spectral Property.

The aim of this paper is to give a computer assisted proof of this Spectral Property in dimensions $N = 2, 3, 4$.

Theorem 2. *The Spectral Property holds true in dimensions $N = 2, 3, 4$.*

Let us briefly recall more precisely the role of the Spectral Property in the proof of Theorem 1.

In the study of (NLS) type problems, the study of coercivity properties of explicit quadratic forms involving the ground state solution Q naturally appears when one considers the question of the stability or instability of these solutions. These quadratic forms are then related to the asymptotic form of the Hamiltonian near Q and their coercivity properties can be derived from the variational formulation of Q. See for example Weinstein [27].

The quadratic forms L_1, L_2, are of different nature and more related to the asymptotic stability (that is the convergence locally in space after renormalization to the ground state at blow up time). Indeed, this property leads to a notion of irreversibility as t goes to the blow up time, even if the equation is reversible.

Unfortunately, at this level of the analysis, we are not able to relate this property to the variational characterization of Q. In particular, L_1 and L_2 appear just as commutators involving L_+, L_- and differentiation with respect to the scaling of the equation, see (10), and from the computations in [14], the orthogonality conditions chosen in the Spectral Property are related to instability directions of the dynamics.

More precisely, let $u(t) \in B_{\alpha^*}$ be a small super critical mass blow up solution to (1), then it admits a geometrical decomposition (4) near blow up time. Such a decomposition is not unique, and each modulation parameter $\lambda(t), x(t), \gamma(t)$ may be used to ensure suitable orthogonality conditions on $\varepsilon(t)$. The choice of these orthogonality conditions is governed by two constraints: using some nonlinear degeneracy properties of the flow of (1) around Q, a specific choice of orthogonality conditions allows some decoupling in the finite dimensional dynamic governing the geometrical parameters $\lambda(t), x(t), \gamma(t)$ at the heart in particular of the estimates on the blow up speed; this choice should be compatible with the time averaged control of the infinite dimensional part of the solution, i.e. $\varepsilon(t)$, and here the positivity property provided by the Spectral Property is at the heart of the proof of these fine dispersive estimates — see in particular the proof of Proposition 1 in [14]. In short, the Spectral Property implies a non-degeneracy property of the flow of (1) around the solitary wave.

Observe that the Spectral Property should be decomposed into two parts. On the one hand, the potentials V_1, V_2 given by (7) are built on the ground state Q which is smooth and exponentially decreasing at infinity, and thus from standard linear theory, the quadratic form H given by (8) admits a finite number of negative eigenvalues. The Spectral Property first implies counting the number of these eigenvalues exactly. Note that in the frame of the proof of Theorem 1, we need this number of eigenvalues not to be too large with respect to the number of symmetries in order to ensure sufficiently many orthogonality conditions.

On the other hand, the Spectral Property also amounts to saying that the explicit choice of orthogonality conditions we gave and which is not the exact set of negative bound states of H is enough to ensure its coercivity. This explicit choice was first proposed in [14] as the result of a nonlinear game and indeed allows one to exhibit a suitable decomposition of the solution u to (1) for which the proof of Theorem 1 applies.

Remark 1. Let us say a word about what happens in dimensions $N = 5, 6$. The Spectral Property as stated is false for $N = 5, 6$. What goes wrong is not the number of negative eigenvalues which is the same as for $N = 4$, but the explicit choice of orthogonality conditions which is no longer enough to ensure the positivity.

More precisely, the set of orthogonality conditions given for H_2 is still good, but not for H_1; see Remark 4. Now in dimension $N = 5$, we claim that the Spectral Property holds true with the orthogonality condition $(\varepsilon_1, Q_1) = 0$ replaced by

$$(\varepsilon_1, |y|^2 Q) = 0;$$

see again Remark 4. Using this new orthogonality condition and Lemma 8 in [16] in the analysis of the series of papers [14, 17], we can prove Theorem 1. This fact is based on the specific choice $|y|^2 Q$. From this, we can say that Theorem 1 holds true for $N = 5$ but not Theorem 2 as stated. In dimension $N = 6$, the choice $(\varepsilon_1, |y|^2 Q) = 0$ is not enough to have the coercivity of H_1 and Theorem 1 is open for $N = 6$.

The question of what happens in higher dimensions is wide open. Analytically, one could hope to obtain the explicit form of the ground state Q as the dimension N goes to infinity and thus to reduce the proof of the Spectral Property to that of an explicit limit problem. Let us nevertheless insist on the fact that the influence of the dimension on the blow up dynamics may sometimes be quite spectacular. A typical striking example is for the supercritical nonlinear heat equation $u_t = \Delta u + u^p$, where, for p large and $N \geq 11$, a new blow up regime appears; see Matano and Merle [12]. Whether or not similar phenomena should be expected for (1) is completely unknown.

More generally, we expect that similar spectral properties like the one we study here will be at the heart of the description of the dynamics of other nonlinear PDEs. Note that in the recent history of mathematical physics, some deep problems have been solved by reducing them to check coercivity estimates for certain Schrödinger operators; see for example Seco [24], de la Llave [11]. The aim of this paper is to provide a general simple and efficient frame to check this kind of property numerically, including the fact that we need to deal with orthogonality conditions which are a priori not adapted to the Schrödinger operator.

The paper is organized as follows. In Section 2, we prove the Spectral Property when restricted to the subset H^1_r of the H^1 distributions with radial symmetry. The general case is treated in Section 3. We briefly sketch in Section 4 an alternative proof of the main result.

We give a definition of the index of a bilinear form. Let B denote a bilinear form on a vector space V; we define the index of B on V as:

$$\mathrm{ind}_V(B) = \min\{k \in \mathbb{N}/\text{there exists a sub-space } P \text{ of codimension } k \text{ such that } B_{|P} \text{ is positive}\}.$$

2. The radial case

In this section, we let $N \geq 2$ and $V(r)$ denote either $V_1(r)$ or $V_2(r)$ which are given by (7). Note that $V(r)$ is a locally smooth potential with exponential decay at infinity:

$$|V(r)| \leq C\mathrm{e}^{-Cr}.$$

Let $L = -\Delta + V$ and $H(u, u) = (Lu, u)$ be the corresponding quadratic form; then from standard spectral theory, see [23], L admits a finite number of negative eigenvalues and has a continuous spectrum $[0, +\infty)$. Our aim in this section is to prove the spectral property numerically when restricted to the subset H^1_r of radial distributions.

2.1. Computation of the index

An efficient numerical way of estimating the index of H on H^1_r is based on the following classical Lemma whose proof is standard and similar to that of Theorem XIII.8 p. 90 in [23].

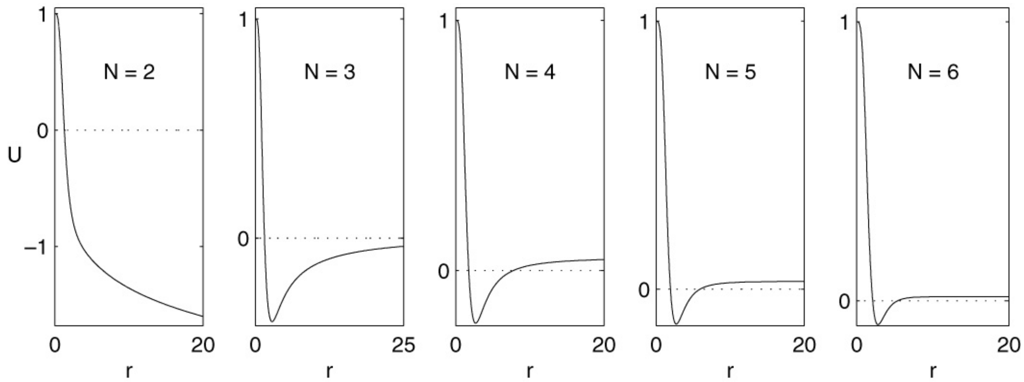

Fig. 1. $U(r)$.

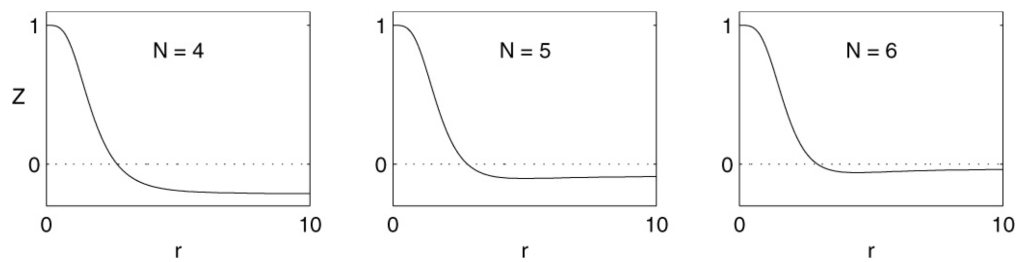

Fig. 2. $Z(r)$.

Lemma 1 (*Estimate on the Number of Eigenvalues*). *Let U be the solution to*

$$\begin{cases} LU = -U'' - \dfrac{N-1}{r}U' + V(r)U = 0, \\ U(0) = 1, \quad U'(0) = 0, \end{cases} \tag{11}$$

then the number $N(U)$ of zeros of U is finite and

$$\mathrm{ind}_{H^1_r} H = N(u).$$

Remark 2. The fact that the solution U to (11) admits a finite number of zeros relies on a simple ODE analysis which implies in particular that no zero can occur when the potential has become too small. This smallness can be explicitly estimated. Indeed, let r_0 be such that $Q(r_0) \leq \frac{1}{2}$, then for $r \geq r_0$, one has a precise exponential estimate on Q and thus on the potential. From that, one can explicitly estimate the size of a r_1 such that U does not vanish for $r \geq r_1$. This is of fundamental importance for the numerics as the exponential decay of V ensures that we need to compute U on a not too large interval only.

Let H_1, H_2 be given by (8). We first claim:

$$H_2(Q, Q) = 0.$$

Indeed, the Q equation is $\Delta Q - Q + Q^{1+\frac{4}{N}} = 0$ or equivalently $L_- Q = 0$. Injecting this into (10) and using the self-adjointness of L_- give:

$$\begin{aligned} H_2(Q, Q) &= (L_2(Q), Q) = \frac{1}{2}(L_-(Q_1) - (L_-(Q))_1, Q) \\ &= \frac{1}{2}(L_-(Q_1), Q) = \frac{1}{2}(Q_1, L_-(Q)) = 0. \end{aligned}$$

We then conclude:

$$H_1 < H_2, \quad H_2(Q, Q) = 0 \quad \text{and thus} \\ \mathrm{ind}_{H^1_r} H_1 \geq \mathrm{ind}_{H^1_r} H_2 \geq 1. \tag{12}$$

We now compute numerically the solutions to (11) with $V = V_1, V_2$.

More precisely, we let $U(r)$ be the solution to

$$-\Delta U(r) + \frac{2}{N}\left(\frac{4}{N} + 1\right) r R^{\frac{4}{N}-1} R_r U = 0, \\ U(0) = 1, U_r(0) = 0.$$

The solution is shown in Fig. 1 in dimensions $N \in [2, 6]$.

Observe that there is only one zero for $N = 2, 3$ and thus (12) implies:

$$\mathrm{ind}_{H^1_r} H_1 = \mathrm{ind}_{H^1_r} H_2 = 1 \quad \text{for } N = 2, 3.$$

They are two zeros for $N = 4, 5, 6$ from which

$$\mathrm{ind}_{H^1_r} H_1 = 2 \quad \text{for } N = 4, 5, 6.$$

We then compute numerically Z, the solution to

$$-\Delta Z + \frac{2}{N} r R^{\frac{4}{N}-1} R_r Z = 0, \qquad Z(0) = 1, Z_r(0) = 0.$$

The solution is shown in Fig. 2 for $N = 4, 5, 6$ and Z displays only one zero from which

$$\mathrm{ind}_{H^1_r} H_2 = 1 \quad \text{for } N = 4, 5, 6.$$

Let us observe that we may have performed the same numerical computations with the slightly perturbed quadratic form

$$\overline{H}(u,u) = (\overline{L}u,u) = (Lu,u) - \delta_0 \int |u|^2 \mathrm{e}^{-|y|}, \tag{13}$$

for some small enough universal constant $\delta_0 > 0$. We let $\overline{L} = -\Delta + \overline{V}$. We may summarize the results of this subsection as follows:

Proposition 1 (*Estimate on the Index of the Quadratic Forms on H^1_r*).

$\mathrm{ind}_{H^1_r} \overline{H}_1 = 1$ *for* $N = 2, 3$,

$\mathrm{ind}_{H^1_r} \overline{H}_1 = 2$ *for* $N \in [4, 6]$,

$\mathrm{ind}_{H^1_r} \overline{H}_2 = 1$ *for* $N \in [2, 6]$.

2.2. *L is invertible*

We claim as a consequence of Proposition 1 that the operator L is invertible modulo the suitable boundary conditions at infinity:

Proposition 2 (*Invertibility of L_1, L_2*). *Let $N \in [2, 6]$ and $f \in C^0_{\mathrm{loc}}(\mathbf{R}^N)$ with radial symmetry and $|f(r)| \le \mathrm{e}^{-Cr}$; then there exists a unique radial solution to*

$$Lu = f \quad \text{with } (1 + r^{N-2})u \in L^\infty. \tag{14}$$

Proof of Proposition 2. There are two standard approaches to prove this kind of result: either direct ODE techniques, which we detail below following the analysis in dimension $N = 1$ in [14]; or variational techniques based on the Lax–Milgram theorem. This last approach requires changing the functional space in which we work in the whole paper and is briefly presented in Section 4.

(A) Uniqueness: Let u with $Lu = 0$ and $(1 + r^{N-2})u \in L^\infty$. From standard elliptic theory and the smoothness of f, u is locally smooth. From $u'(0) = 0$, we have

$$u'(r) = \frac{1}{r^{N-1}} \int_0^r \tau^{N-1} V(\tau) u(\tau) \mathrm{d}\tau$$

which together with (14) implies

$$\lim_{r\to+\infty} r u'(r) \to 0 \quad \text{for } N = 2, \qquad |u'(r)| \le \frac{C}{r^{N-1}} \quad \text{for } N \in [3, 6]. \tag{15}$$

We then multiply the equation by u and integrate by parts; the boundary term goes away from (15) and thus

$$\int |\nabla u|^2 < +\infty \quad \text{and} \quad H(u,u) = 0. \tag{16}$$

Let now $A > 1$ and $u_A = \chi_A u$ where

$$\chi_A(r) = \chi\left(\frac{r}{A}\right),$$

$$\chi(r) = \begin{cases} 1 & \text{for } r \le 1, \\ 0 & \text{for } r \ge 2, \end{cases} \quad \text{for } N \in [3, 6], \tag{17}$$

$$\chi_A(r) = \begin{cases} 1 & \text{for } r \le A, \\ 2\left(\dfrac{\log A}{\log r} - \dfrac{1}{2}\right) & \text{for } A \le r \le A^2, \\ 0 & \text{for } r \ge A^2, \end{cases} \quad \text{for } N = 2. \tag{18}$$

We claim:

$$H(u_A, u_A) \to H(u,u) \quad \text{as } A \to +\infty. \tag{19}$$

Indeed, we compute

$$\int |\nabla u_A|^2 = \int \chi_A^2 |\nabla u|^2 + \int |\nabla \chi_A|^2 |u|^2 + 2\int (\chi_A \nabla u)\cdot(\nabla \chi_A u).$$

We estimate from (14) and (15) in dimensions $N \ge 3$:

$$\int |\nabla \chi_A|^2 |u|^2 + 2\left|\int (\chi_A \nabla u)\cdot(\nabla \chi_A u)\right| \le C\frac{A^N}{A^2 A^{2(N-2)}} + C\frac{A^N}{A A^{N-2} A^{N-1}} \le \frac{C}{A^{N-2}}$$

and in dimension $N = 2$:

$$\int |\nabla \chi_A|^2 |u|^2 + 2\left|\int (\chi_A \nabla u)\cdot(\nabla \chi_A u)\right| \le C(\log A)^2 \int_A^{A^2} \frac{\mathrm{d}r}{r(\log r)^4} + o(\log A)\int_A^{A^2} \frac{\mathrm{d}r}{r(\log r)^2} \le o(1) \to 0 \quad \text{as } A \to +\infty,$$

and (19) follows from the decay of $\overline{V}$.

Let $N = 2, 3$. Let ψ be the first eigenvector of L with eigenvalue $\lambda < 0$, $|\psi|_{L^2} = 1$, and consider $V_A = \mathrm{span}(\psi, u_A)$ and the matrix $M_A = \mathrm{mat}_{V_A} \overline{H}$; then $\overline{H}(u_A, u_A) \to \overline{H}(u,u) = -\delta_0 \int |u|^2 \mathrm{e}^{-|y|}$ as $A \to +\infty$ from (16) and (19), $(\overline{L}u_A, \psi) \to -\delta_0(\mathrm{e}^{-|y|}u, \psi)$ and $\overline{H}(\psi,\psi) \le -|\lambda|$, and thus M_A is definite negative for A large enough provided $\delta_0 > 0$ is small enough. From Proposition 1 for $N = 2, 3$, we have $\mathrm{ind}_{H^1_r}\overline{H} = 1$ and thus $\dim V_A = 1$ and $u_A = \mu_A \psi$. From $Lu = 0$ and $L\psi = \lambda\psi$, we conclude $\lambda\mu_A = (u_A, \lambda\psi) = (Lu_A, \psi) \to 0$ as $A \to +\infty$ from which $u = 0$. We argue similarly for $N \in [4, 6]$.

(B) Existence: By fixed point argument, we construct $(\phi_i)_{i=1,2}$ solutions to the linear homogeneous equation $L\phi_i = 0$ by solving on $[A, +\infty)$, $A > 0$ large enough, the integral equation

$$\text{for } N = 2, \begin{cases} \phi_1(r) = \log r + \displaystyle\int_r^{+\infty} \frac{\mathrm{d}\tau}{\tau^{N-1}} \times \int_\tau^{+\infty} \sigma^{N-1} V(\sigma)\phi_1(\sigma)\mathrm{d}\sigma \\ \phi_2(r) = 1 + \displaystyle\int_r^{+\infty} \frac{\mathrm{d}\tau}{\tau^{N-1}} \times \int_\tau^{+\infty} \sigma^{N-1} V(\sigma)\phi_2(\sigma)\mathrm{d}\sigma, \end{cases}$$

$$\text{for } N \in [3,6], \begin{cases} \phi_1(r) = 1 + \displaystyle\int_r^{+\infty} \frac{\mathrm{d}\tau}{\tau^{N-1}} \\ \qquad \times \displaystyle\int_\tau^{+\infty} \sigma^{N-1} V(\sigma)\phi_1(\sigma)\mathrm{d}\sigma \\ \phi_2(r) = \dfrac{1}{r^{N-2}} + \displaystyle\int_r^{+\infty} \frac{\mathrm{d}\tau}{\tau^{N-1}} \\ \qquad \times \displaystyle\int_\tau^{+\infty} \sigma^{N-1} V(\sigma)\phi_2(\sigma)\mathrm{d}\sigma. \end{cases}$$

These solutions may be continued on $(0, +\infty)$ from linear theory. Let ϕ_{rad} be the solution to

$$\begin{cases} L\phi_{\mathrm{rad}} = 0 \\ \phi_{\mathrm{rad}}(0) = 1, \qquad \phi'_{\mathrm{rad}}(0) = 0, \end{cases}$$

then $\phi_{\mathrm{rad}} \in \mathrm{Vect}(\phi_1, \phi_2)$ and from the uniqueness result, ϕ_{rad} and ϕ_2 are linearly independent, from which there holds for some $W_0 \neq 0$:

$$W(\phi_{\mathrm{rad}}, \phi_2) = \phi'_{\mathrm{rad}}\phi_2 - \phi_{\mathrm{rad}}\phi'_2 = \frac{W_0}{r^{N-1}}.$$

Last, we compute:

$$\phi_2(r) = \phi_2(1) + \frac{\phi'_2(1)}{\log r} + \int_1^r \frac{\mathrm{d}\tau}{\tau^{N-1}} \times \int_\tau^1 \sigma^{N-1} V(\sigma)\phi_2(\sigma)\mathrm{d}\sigma \quad \text{for } N = 2,$$

$$\phi_2(r) = \phi_2(1) - \frac{\phi'_2(1)}{(N-2)r^{N-2}} + \int_1^r \frac{\mathrm{d}\tau}{\tau^{N-1}} \times \int_\tau^1 \sigma^{N-1} V(\sigma)\phi_2(\sigma)\mathrm{d}\sigma \quad \text{for } N \in [3,6],$$

from which

$$|\phi'_2(r)| \le \frac{C}{r^{N-1}}, \quad |\phi_2(r)| \le \frac{C}{r^{N-2}} + C|\log r| \quad \text{as } r \to 0.$$

From direct verification and using the decay property of f at infinity,

$$u(r) = -\frac{1}{W_0}\left(\phi_2(r)\int_0^r \tau^{N-1}\phi_{\mathrm{rad}}(\tau) f(\tau)\mathrm{d}\tau + \phi_{\mathrm{rad}}(r)\int_r^{+\infty} \tau^{N-1}\phi_2(\tau) f(\tau)\mathrm{d}\tau\right)$$

is a smooth solution on $[0, +\infty)$ to $Lu = f$ with $u'(0) = 0$ and $(1 + r^{N-2})u \in L^\infty$. This concludes the proof of Proposition 2. □

2.3. *Numerical check of the orthogonality conditions*

Let us observe that Proposition 2 also holds for $\overline{H}$ given by (13) for $\delta_0 > 0$ small enough. We now proceed to the numerical computations which will ensure that the set of orthogonality conditions selected for the Spectral Conjecture is indeed enough to ensure the coercivity properties of H_1, H_2.

Lemma 2 (*Numerical Computations for $N = 2, 3, 4, 5, 6$*).
(i) *Case of $\overline{H}_1$: Let $\overline{U}_0$ be the radial solution to*

$$\overline{L}_1\overline{U}_0 = Q \quad \textit{with } (1 + r^{N-2})\overline{U}_0 \in L^\infty, \tag{20}$$

then

$$(\overline{U}_0, Q) < 0 \quad \textit{for } N = 2, 3. \tag{21}$$

For $N = 4, 5, 6$, let $\overline{U}_3$ be the radial solution to

$$\overline{L}_1\overline{U}_3 = Q_1 \quad \textit{with } (1 + r^{N-2})\overline{U}_3 \in L^\infty, \tag{22}$$

and let

$$\overline{K}_1 = (\overline{U}_0, Q), \quad \overline{K}_2 = (\overline{U}_3, Q_1), \quad \overline{K}_3 = (\overline{U}_0, Q_1), \tag{23}$$

then

$$\overline{K}_1 < 0 \quad \textit{and} \quad \overline{K}_1\overline{K}_2 - \overline{K}_3^2 \begin{cases} > 0 & \textit{for } N = 4, \\ < 0 & \textit{for } N = 5, 6. \end{cases} \tag{24}$$

(ii) *Case of $\overline{H}_2$: Let $\overline{U}_1, \overline{U}_2$ be the radial solutions to*

$$\overline{L}_2\overline{U}_1 = Q_1, \quad \overline{L}_2\overline{U}_2 = Q_2 \quad \textit{with } (1 + r^{N-2})\overline{U}_1 \in L^\infty, \; (1 + r^{N-2})\overline{U}_2 \in L^\infty, \tag{25}$$

and let

$$\overline{J}_1 = (\overline{U}_1, Q_1), \quad \overline{J}_2 = (\overline{U}_2, Q_2), \quad \overline{J}_3 = (\overline{U}_2, Q_1), \tag{26}$$

then

$$\frac{1}{\overline{J}_2}(\overline{J}_3^2 - \overline{J}_1\overline{J}_2) > 0. \tag{27}$$

Numerically, we compute the $(U_i)_{0 \le i \le 3}$ on H instead of $\overline{H}$ using a shooting method on $U_i(0)$ and restrict ourselves to $N \in [2, 6]$. The exact numerical values are given below and imply Lemma 2 for $\overline{H}$ with $\delta_0 > 0$ small enough.

We first compute U_0, the radial solution to $L_1U_0 = Q$ with $(1 + r^{N-2})U_0 \in L^\infty$, and estimate that $(U_0, Q) < 0$ for $N \in [2, 11]$. The exact results are given in Table 1.

Table 1
(U_0, Q)

Dimension	2	3	4	5	6
(U_0, Q)	−0.65	−2.1	−8	−40	−190

Next, in dimensions $N = 4, 5, 6$, we compute U_3, the radial solution to $L_1U_3 = Q_1$ with $r^{N-2}U_3 \in L^\infty$, and estimate the inner products $K_1 = (U_0, Q)$, $K_2 = (U_3, Q_1)$, $K_3 = (U_0, Q_1)$. Let $\tilde{K}_3 = (U_3, Q)$; then analytically:

$$\tilde{K}_3 = (U_3, L_1U_0) = (L_1U_3, U_0) = (Q_1, U_0) = K_3,$$

and thus the computation of $K_3 - \tilde{K}_3$ is a good measurement of the accuracy of the numerics. The results of Table 2 prove (24) and show in particular that the sign of $\overline{K}_1\overline{K}_2 - \overline{K}_3^2$ changes for $N \ge 5$.

Eventually, we compute U_1, U_2, the radial solutions to $L_2U_1 = Q_1$, $L_2U_2 = Q_2$ with $(1 + r^{N-2})U_1$, $(1 + r^{N-2})U_2 \in L^\infty$, and estimate $J_1 = (U_1, Q_1)$, $J_2 = (U_2, Q_2)$, $J_3 = (U_2, Q_1)$. We also test the accuracy of the numerics by computing $\tilde{J}_3 = (U_2, Q_1) = J_3$. Results are given in Table 3 and show that (27) is always fulfilled.

Remark 3. Note that it can be checked that the boundary condition (14) is equivalent to assuming that the solutions u to

Table 2
$(K_i)_{1\le i\le 3}$

Dimension	K_1	K_2	K_3	$K_1K_2 - K_3^2$	$K_3 - \tilde{K}_3$
4	−8.0178	−31.8102	4.6472	233.4524	1e−7
5	−40.3903	12.4921	35.5946	−1.7715e+003	−5.4214e−006
6	−190.7594	5432.4	854.9890	−1.7673e+006	−5.9598e−004

Table 3
$(J_i)_{1\le i\le 3}$

Dimension	J_1	J_2	J_3	$J_3^2 - J_1J_2$	$J_3 - \tilde{J}_3$
2	0.9969	12.4211	−4.4095	7.0616	3e−006
3	5.9141	100.36881	−28.0500	193.2117	−6e−006
4	147	3404	−728	28 933	1e−004
5	−128	−4321	609	−1.8293e+005	6e−004
6	−131.623	−1e+004	169.0371	−1.4e+006	8e−004

Table 4
$(M_i)_{1\le i\le 3}$

Dimension	M_1	M_2	M_3	$M_1M_2 - M_3^2$	$M_3 - \tilde{M}_3$	$\langle U_0, L(R)\rangle - \langle R, R\rangle$
4	−8.0	−639	−26	4.4e+3	−4.8e−7	2e−5
5	−40	−2.3e+3	−166	6.4e+4	5.7e−5	1e−4
6	−190	2.0e+3	−2.25e+3	−5.4e+006	9.6e−4	8e−4

$Lu = f$ belongs to the Lax–Milgram space (36) of Section 4. In particular, it suffices to find a solution to $Lu = f$ with $\int |\nabla u|^2 < +\infty$ that is a boundary condition $\partial_r u \to 0$ as $r \to +\infty$, and this is precisely what is done numerically to get the results of Lemma 2.

Remark 4. (24) means that the choice of orthogonality conditions $(\varepsilon_1, Q) = (\varepsilon_1, Q_1) = 0$ is not enough in dimensions $N = 5, 6$ to ensure the coercivity of H_1. Nevertheless, the index of H_1 being no greater than two from Proposition 1, one may expect that another set of two orthogonality conditions will allow one to both ensure the coercivity of H_1 and have the proof of Theorem 1 go through. As a matter of fact and from direct verification, if the Spectral Property holds true for H_1 with the new set of orthogonality conditions $(\varepsilon_1, Q) = (\varepsilon_1, |y|^2 Q) = 0$, then the proof of Theorem 1 goes through. To check whether the replacement of Q_1 by $|y|^2 Q$ is a better bet, we compute U_4, the solution to $L_1 U_4 = |y|^2 Q$ with $(1 + r^{N-2}) U_4 \in L^\infty$ in dimensions $N = 5, 6$. Let $M_1 = (U_0, Q)$, $M_2 = (U_4, |y|^2 Q)$, $M_3 = (U_0, |y|^2 Q)$ and $\tilde{M}_3 = (Q, U_4) = M_3$ to test the numerics. Recall from Table 1 that $M_1 = (U_0, Q) < 0$ and thus as for (24), it remains to check that

$$M_1 M_2 - M_3^2 > 0.$$

The results of Table 4 show that indeed this holds true in dimension $N = 5$, and thus the proof of Theorem 1 is also complete for $N = 5$, but fails in dimension $N = 6$.

2.4. *Proof of the spectral property for* $N = 2, 3, 4$ *on* H_r^1

We now are in position to conclude the proof of the Spectral Property in dimensions $N = 2, 3, 4$ when restricted to radial data. We indeed claim that under the orthogonality conditions of the Spectral Property, $\overline{H}_1$ and $\overline{H}_2$ are positive, which from (13) concludes the proof. □

Step 1. $\overline{H}_1$ in dimensions $N = 2, 3$.

We start with $\overline{H}_1$ in dimensions $N = 2, 3$. We adapt the proof of Lemma 13 in [14] and argue in three steps.

(α) Let $\overline{U}_0$ given by (20). For $A > 1$, let the cut off function χ_A be given by (17) and (18) and set $(\overline{U}_0)_A = \chi_A \overline{U}_0$, $(P_0)_A = \mathrm{Vect}((\overline{U}_0)_A)$. Let us introduce the norm

$$\|f\| = \left(\int |\nabla f|^2 + \int |V_1||f|^2\right)^{\frac{1}{2}},$$

then arguing as for the proof of (19), we have

$$\left|\overline{H}_1((U_0)_A, (U_0)_A) - \overline{H}(U_0, U_0)\right| + \|(U_0)_A - U_0\| \to 0 \quad \text{as } A \to +\infty. \tag{28}$$

From (21), $\overline{H}(U_0, U_0) = (U_0, Q) < 0$ and thus the quadratic form is non-degenerate on $(P_0)_A$ for A large enough. This implies from standard argument:

$$H_r^1 = (P_0)_A \oplus (P_0)_A^{\perp}.$$

(β) From Proposition 1, the index of $\overline{H}_1$ on H_r^1 is 1 and thus $\overline{H}_1 \ge 0$ on $(P_0)_A^{\perp}$. Indeed, by contradiction, assume that there is a non-zero $Z \in (P_0)_A^{\perp}$ with $\overline{H}_1(Z, Z) < 0$ and consider the plane $\mathrm{Vect}((U_0)_A, Z)$; then from $Z \in (P_0)_A^{\perp}$, the quadratic form is definite negative on $\mathrm{Vect}((U_0)_A, Z)$ which is of dimension 2. Let now P be of codimension 1; then $P \cap \mathrm{Vect}((U_0)_A, Z) \ne \emptyset$ and thus $(\overline{H}_1)_{|P}$ is not positive. This means that $\mathrm{ind}_{H_r^1} \overline{H}_1 \ge 2$ and contradicts Proposition 1.

(γ) Let now $u \in H_r^1$ be non-zero such that $(u, Q) = 0$. For $A > 0$ large enough, decompose $u = \alpha_A (U_0)_A + u_A^{(2)}$ where $u_A^{(2)} \in$

$(P_0)_A^{\perp}$; then by definition, $\overline{H}_1(u,u) = \alpha_A^2 \overline{H}_1((U_0)_A,(U_0)_A) + \overline{H}_1(u_A^{(2)}, u_A^{(2)}) \geq \alpha_A^2 \overline{H}_1((U_0)_A,(U_0)_A)$ from (β). The conclusion now follows from (28) and

$$\alpha_A \to 0 \quad as\ A \to +\infty. \tag{29}$$

Indeed, from $(u,Q) = 0$, we have $\alpha_A((U_0)_A, Q) = -(u_A^{(2)}, \overline{L}_1 U_0) = -(u_A^{(2)}, \overline{L}_1(U_0 - (U_0)_A))$ from which

$$|\alpha_A| \leq C\|u_A^2\|\|U_0 - (U_0)_A\|. \tag{30}$$

Now $\|u_A^2\| \leq C(\|u\| + |\alpha_A|\|U_0\|)$ and thus (28) and (30) imply $\|u_A^2\| \leq C\|u\|$ and (29) follows.

Step 2. $\overline{H}_1$ in dimension $N = 4$.

In this case, the orthogonality against Q is no longer enough to ensure the coercivity of $\overline{H}_1$ and we use the second condition $(u, Q_1) = 0$. Let $\overline{U}_0$ be given by (20), $\overline{U}_3$ be given by (22), and $(\overline{K}_i)_{1\leq i\leq 3}$ be given by (23). For $A > 1$, let the cut off function χ_A be given by (17) and (18) and set $(\overline{U}_0)_A = \chi_A \overline{U}_0$, $(\overline{U}_3)_A = \chi_A \overline{U}_3$, $(P_3)_A = \mathrm{Vect}((\overline{U}_0)_A, (\overline{U}_3)_A)$, then

$$\begin{aligned}
&\begin{vmatrix} \overline{H}_1((\overline{U}_0)_A,(\overline{U}_0)_A) & \overline{H}_1((\overline{U}_0)_A,(\overline{U}_3)_A) \\ \overline{H}_1((\overline{U}_3)_A,(\overline{U}_0)_A) & \overline{H}_1((\overline{U}_3)_A,(\overline{U}_3)_A) \end{vmatrix} \\
&\quad = \overline{K}_1\overline{K}_2 - \overline{K}_3^2 + o(1) > 0
\end{aligned}$$

from (24) for $A > 0$ large enough. From $\overline{K}_1 < 0$, we conclude that for $A > 0$ large enough, the quadratic form $\overline{H}_1$ restricted to $(P_3)_A$ is definite negative and thus $H_r^1 = (P_3)_A \oplus (P_3)_A^{\perp}$. From Proposition 1, the index of $\overline{H}_1$ on H_r^1 is at most 2 and thus $\overline{H}_1 \geq 0$ on $(P_3)_A^{\perp}$. Arguing as for the proof of (γ) of step 1, we now easily conclude that for $u \in H_r^1$ with $(u,Q) = (u,Q_1) = 0$, $\overline{H}_1(u,u) \geq 0$.

This concludes the proof of the Spectral Property for H_1 restricted to radial data in dimensions $N = 2, 3, 4$.

Step 3. $\overline{H}_2$ in dimensions $N = 2, 3, 4$.

We now turn to $\overline{H}_2$ in dimensions $N = 2, 3, 4$. Let $\overline{U}_1$, $\overline{U}_2$ be given by (25) and $\overline{J}_1, \overline{J}_2, \overline{J}_3$ be given by (26), and let

$$Q_4 = Q_1 - \frac{\overline{J}_3}{\overline{J}_2} Q_2, \qquad \overline{U}_4 = \overline{U}_1 - \frac{\overline{J}_3}{\overline{J}_2}\overline{U}_2,$$

then $\overline{L}_2 U_4 = Q_4$ and

$$\begin{aligned}
(\overline{U}_4, Q_4) = \overline{H}_2(\overline{U}_4,\overline{U}_4) &= \overline{J}_1 - 2\frac{\overline{J}_3}{\overline{J}_2}\overline{J}_3 + \frac{\overline{J}_3^2}{\overline{J}_2^2}\overline{J}_2 \\
&= -\frac{1}{\overline{J}_2}(\overline{J}_3^2 - \overline{J}_1\overline{J}_2) < 0
\end{aligned}$$

from (27). We now follow the proof of step 1 and conclude from Proposition 1, i.e., $\mathrm{ind}_{H_r^1}\overline{H}_2 = 1$, that for all $u \in H_r^1$ with $(u, Q_4) = 0$, $\overline{H}_2(u,u) \geq 0$. Now $(u,Q_1) = (u,Q_2) = 0$ implies $(u,Q_4) = 0$ from $Q_4 \in \mathrm{Vect}(Q_1, Q_2)$. This concludes the proof of the Spectral Property for H_2 restricted to radial data in dimensions $N = 2, 3, 4$.

3. The non-radial case

This section is devoted to the proof of the Spectral Property in the non-radial case. Our main tools rely on the theory of Schrödinger operators in the so-called central case; see [23], section XIII B., i.e., the fact that the potentials V_1, V_2 have radial symmetry. This allows us to decompose H into the sum of the restricted quadratic forms obtained by projecting onto the basis of spherical harmonics.

More precisely, let Harm_k be the space of spherical harmonics of degree k and let $a_k = \dim \mathrm{Harm}_k$, explicitly $a_0 = 1$, $a_1 = N$ and $a_k = C_{N+k-1}^k - C_{N+k-3}^{k-2}$ for $k \geq 2$. For each $k \geq 0$, let $(Y_i^{(k)})_{1\leq i\leq a_k}$ be an orthonormal basis of Harm_k, then any function $u \in L^2(\mathbf{R}^N)$ has a unique expansion

$$\begin{aligned}
&u = \sum_{k=0}^{\infty}\sum_{i=1}^{a_k} c_{k,i} Y_i^{(k)}\left(\frac{x}{|x|}\right) \quad \text{with} \\
&\quad c_{k,i} = \int_{S^{N-1}} u(|x|\theta)\overline{Y_i^{(k)}}(\theta)\mathrm{d}\theta.
\end{aligned} \tag{31}$$

Moreover, the potential V being radial, we have

$$H(u,u) = \sum_{k=0}^{+\infty}\sum_{i=1}^{a_k} H^{(k)}(c_{k,i}, c_{k,i}) \tag{32}$$

where

$$\begin{aligned}
&H^{(k)}(w,w) = (L^{(k)}w, w), \\
&L^{(k)} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + V(r) + \frac{k(k+N-2)}{r^2}.
\end{aligned}$$

Note that the operator $L^{(k)}$ for $k > 0$ should be thought of as a differential operator on a weighted L^2 space on $(0,\infty)$ with boundary conditions $w(0) = 0$ and the corresponding quadratic form $H^{(k)}$ is well defined on the subset $\tilde{H}_r^1(\mathbf{R}^N)$ of radial distributions $w \in H_r^1(\mathbf{R}^N)$ such that

$$\int_{\mathbf{R}^N}\frac{|w|^2}{|x|^2} < +\infty.$$

3.1. Computation of the index

The computation of the index of $H^{(k)}$ on $H_r^1(\mathbf{R}^N)$ follows from the following classical Lemma; see Theorem XIII.8 p. 90 in [23].

Lemma 3 (*Estimate on the Index of $H^{(k)}$*). *For $k \geq 1$, let W be the solution to*

$$\begin{cases}
L^{(k)}W = -W'' - \dfrac{N-1}{r}W' + V(r)W \\
\qquad + \dfrac{k(k+N-2)}{r^2}W = 0, \\
W(0) = 0, \qquad \displaystyle\lim_{r\to 0}\frac{W(r)}{r^k} = 1,
\end{cases}$$

then the number $N_k(W)$ of zeros of W is finite and

$$\mathrm{ind}_{\tilde{H}_r^1} H^{(k)} = N_k(W).$$

Note that the number of k for which the quadratic form $H^{(k)}$ has negative directions is clearly finite. Moreover, $H^{(k')} \geq H^{(k)}$ for $k' \geq k$. Numerically, we compute W, the solution

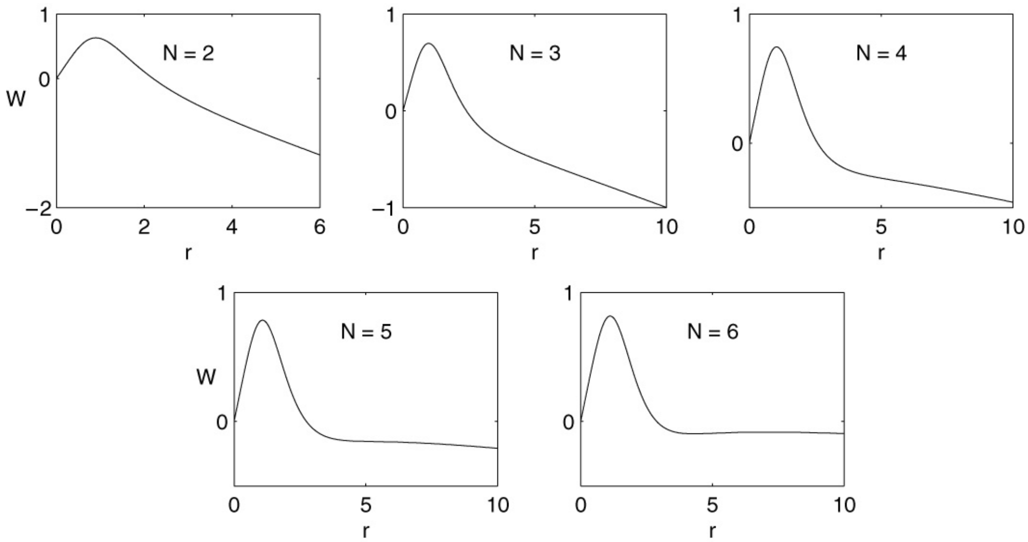

Fig. 3. $W(r)$ for $k = 1$, $i = 1$ and $N = 2, \dots, 6$.

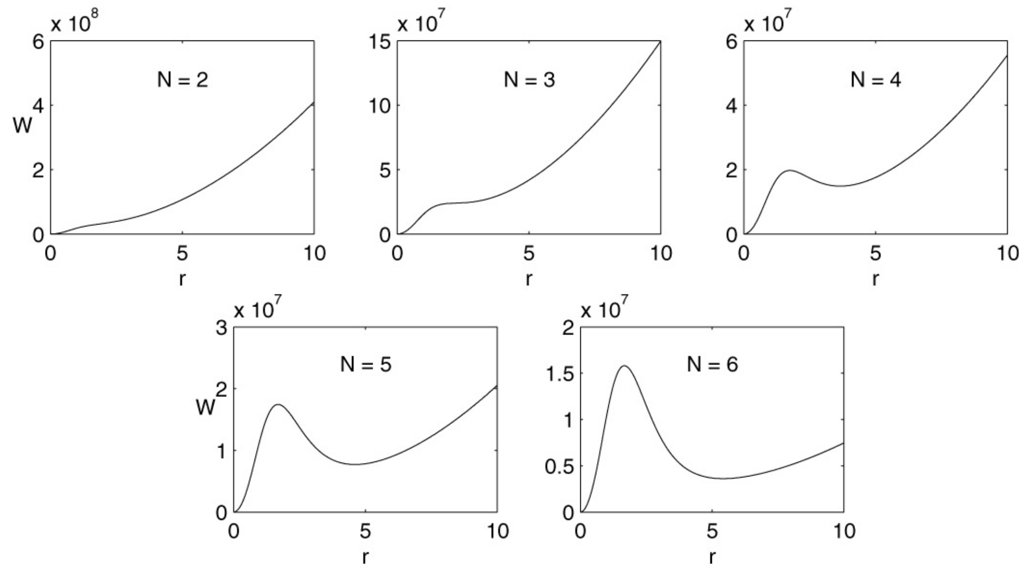

Fig. 4. $W(r)$ for $k = 2$, $i = 1$ and $N = 2, \dots, 6$.

to

$$-\Delta W + \frac{2}{N}\left(\frac{4}{N}+1\right)^{2-i} r Q' Q^{\frac{4}{N}-1} W + \frac{k(k+N-2)}{r^2} W = 0, \quad i = 1, 2,$$

where

$$W \sim r^k \quad \text{for } r \ll 1,$$

and $i = 1, 2$ corresponds to the potential V_1, V_2, respectively. To calculate this numerically, we use the fact that $W \sim r^k(1 + O(r^4))$ near $r = 0$ to overcome the singularity at $r = 0$. The solution is seen in Figs. 3–5 for $N = 2, \dots, 6$. In all dimensions, there is one zero for ($k = 1$; $i = 1$) and no zeros for ($k = 1$; $i = 2$), ($k = 2$; $i = 1, 2$).

Again, these computations could have been performed with $\overline{H}$ given by (13) instead of H for $\delta_0 > 0$ small enough, and the outcome is the following:

Proposition 3 (*Estimate on the Index of $\overline{H}^{(k)}$ on H_r^1*).

$$\operatorname{ind}_{\tilde{H}_r^1} \overline{H}_1^{(1)} = 1, \qquad \operatorname{ind}_{\tilde{H}_r^1} \overline{H}_1^{(2)} = 0, \qquad \operatorname{ind}_{\tilde{H}_r^1} \overline{H}_2^{(1)} = 0 \quad \text{for } N \in [2, 6].$$

3.2. $L^{(k)}$ is invertible

From Proposition 3, $L^{(k)}$ is invertible modulo the suitable boundary conditions at infinity:

Proposition 4 (*Invertibility of $L^{(k)}$*). *Let $N \in [2, 6]$, $k \geq 1$, $f \in C^0_{\mathrm{loc}}(\mathbf{R}^N)$ with radial symmetry and $|f(r)| \leq \mathrm{e}^{-Cr}$; then*

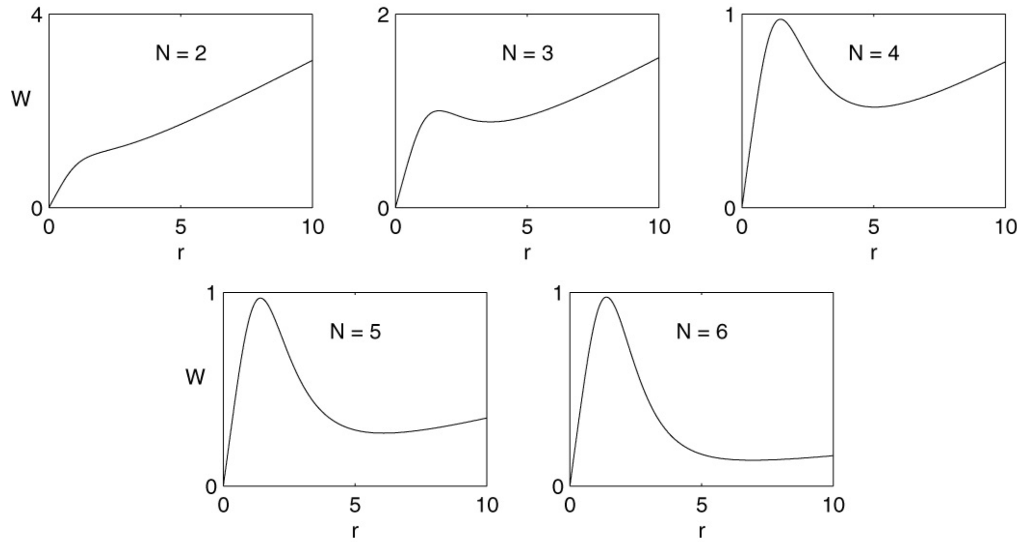

Fig. 5. $W(r)$ for $k = 1$, $i = 2$ and $N = 2, \dots, 6$.

there exists a unique radial solution to

$$\begin{cases} L^{(k)}u = f, \\ u(0) = 0 \quad \text{and} \quad (1 + r^{k+N-2})u \in L^\infty. \end{cases} \tag{33}$$

Proof of Proposition 2. The argument is similar to the one of Proposition 2 and we briefly sketch the proof.

(A) Uniqueness: Let u with $L^{(k)}u = 0$, $u(0) = 0$ and $(1 + r^{k+N-2})u \in L^\infty$. From standard elliptic theory and the smoothness of f, u is locally smooth on $(0, +\infty)$. From direct verification, $Z = \frac{u}{r^k}$ satisfies

$$\frac{1}{r^{2k+N-1}} \left(r^{2k+N-1} Z' \right)' = \frac{Vu}{r^k},$$

and thus

$$u'(r) = \frac{ku}{r} + C_1 r^{k+N-1} + \frac{1}{r^{k+N-1}} \int_1^r \tau^{k+N-1} V(\tau) u(\tau) \mathrm{d}\tau \tag{34}$$

for some constant C_1. From $r^{k+N-2}u \in L^\infty$ and the well localization of V, we conclude

$$|u'(r)| \le \frac{C}{r^{k+N-1}} \quad \text{as } r \to +\infty.$$

Let χ_A be the cut off function given by (17) in all dimensions; we let $u_A = \chi_A u$ and conclude as for the proof of (19) that

$$H(u_A, u_A) \to H(u, u) \quad \text{as } A \to +\infty$$

from which

$$\int |\nabla u|^2 < +\infty \quad \text{and} \quad H(u, u) = 0.$$

Arguing as for the proof of Proposition 2, we conclude from Proposition 3 that $u = 0$, which concludes the proof of the uniqueness part.

(B) Existence: By fixed point argument, we construct $(\phi_i)_{i=1,2}$ solutions to the linear homogeneous equation $L^{(k)}\phi_i = 0$ by solving on $[A, +\infty)$, $A > 0$ large enough, the integral equation

$$\begin{cases} \phi_1(r) = r^k \left(1 + \displaystyle\int_r^{+\infty} \frac{\mathrm{d}\tau}{\tau^{2k+N-1}} \times \int_\tau^{+\infty} \sigma^{k+N-1} V(\sigma) \phi_1(\sigma) \mathrm{d}\sigma \right), \\ \phi_2(r) = r^k \left(\dfrac{1}{r^{2k+N-2}} + \displaystyle\int_r^{+\infty} \frac{\mathrm{d}\tau}{\tau^{2k+N-1}} \times \int_\tau^{+\infty} \sigma^{k+N-1} V(\sigma) \phi_2(\sigma) \mathrm{d}\sigma \right). \end{cases}$$

These solutions may be continued on $(0, +\infty)$ from linear theory. Let ϕ_{rad} be the solution to

$$\begin{cases} L\phi_{\text{rad}} = 0 \\ \phi_{\text{rad}}(0) = 0, \quad \displaystyle\lim_{r \to 0} \frac{\phi_{\text{rad}}(r)}{r^k} = 1, \end{cases}$$

then $\phi_{\text{rad}} \in \text{Vect}(\phi_1, \phi_2)$ and from the uniqueness result, ϕ_{rad} and ϕ_2 are linearly independent, from which there holds for some $W_0 \neq 0$:

$$W(\phi_{\text{rad}}, \phi_2) = \phi'_{\text{rad}}\phi_2 - \phi_{\text{rad}}\phi'_2 = \frac{W_0}{r^{N-1}}.$$

Last, we easily conclude from the integral equation that

$$|\phi'_2(r)| \le \frac{C}{r^{k+N-1}}, \qquad |\phi_2(r)| \le \frac{C}{r^{k+N-2}} \quad \text{as } r \to 0.$$

From direct verification and using the decay property of f at infinity,

$$u(r) = -\frac{1}{W_0} \left(\phi_2(r) \int_0^r \tau^{N-1} \phi_{\text{rad}}(\tau) f(\tau) \mathrm{d}\tau + \phi_{\text{rad}}(r) \int_r^{+\infty} \tau^{N-1} \phi_2(\tau) f(\tau) \mathrm{d}\tau \right)$$

is a smooth solution on $[0, +\infty)$ to $Lu = f$ with $u(0) = 0$ and $(1 + r^{k+N-2})u \in L^\infty$. This concludes the proof of Proposition 4. □

Table 5
(T_0, rQ)

Dimension	2	3	4	5	6
(T_0, rQ)	−3.1	−11	−54	−337	−2512

3.3. Check of the orthogonality conditions

Proposition 4 also holds for $\overline{H}^{(k)}$ with $\delta_0 > 0$ small enough. Note that from Proposition 3, $\overline{H}_1^{(k)}$ is positive for $k \geq 2$ and $\overline{H}_2^{(k)}$ is positive for $k \geq 1$. The numerical check that the orthogonality conditions selected for the Spectral Conjecture are enough to ensure the coercivity $H_1^{(1)}$ is a consequence of the following:

Lemma 4 (*Numerical Computations for $N = 2, 3, 4, 5, 6$*). *Let $\overline{T}_0$ be the radial solution to*

$$\overline{L}_1^{(1)}\overline{T}_0 = rQ \quad \text{with } \overline{T}_0(0) = 0 \quad \text{and} \quad (1 + r^{N-1})\overline{T}_0 \in L^\infty,$$

then

$$(\overline{T}_0, rQ) < 0 \quad \text{for } N \in [2, 6]. \tag{35}$$

Again, the numerical computation is made directly on $H^{(1)}$ using a shooting method. We compute T_0, the solution to $L_1 T_0 + \frac{N-1}{r^2} T_0 = rQ$ with $T_0(0) = 0$ and $r^{N-1}T_0 \in L^\infty$, numerically. As pointed out in Remark 3, the boundary condition on T_0 is equivalent to assuming that it belongs to the Lax–Milgram space (37), so that numerically, it suffices to check a boundary condition $\partial_r T_0 \to 0$ as $r \to +\infty$. We also use the fact that $T_0(r) \sim cr(1+O(r^3))$ near $r = 0$ to overcome the singularity at $r = 0$. We then compute the inner product (T_0, rQ), and the results of Table 5 show that (35) holds for $N = 2, 3, 4$.

3.4. Proof of the spectral property for $N = 2, 3, 4$ on H^1

We now are in position to conclude the proof of the Spectral Property in dimensions $N = 2, 3, 4$.

We start with $\overline{H}_1$ and claim that for all $u \in H^1$ with $(u, Q) = (u, Q_1) = (u, yQ) = 0$, we have $\overline{H}_1(u, u) \geq 0$ which concludes the proof.

Indeed, decompose u into spherical harmonics according to (31) and recall (32):

$$\overline{H}(u, u) = \sum_{k=0}^{+\infty} \sum_{i=1}^{a_k} \overline{H}^{(k)}(c_{k,i}, c_{k,i}).$$

We claim that each term in the above sum is positive, which concludes the proof. Indeed, first observe from Proposition 3 that

$$\overline{H}_1^{(k)} \geq 0 \quad \text{for } k \geq 2.$$

From $(u, Q) = (u, Q_1) = 0$ and Q, Q_1 radial, the radial part of u satisfies

$$(c_{0,1}, Q) = (c_{0,1}, Q_1) = 0$$

and thus from Section 2.3 and $\overline{H}_1^{(0)} = \overline{H}_1$,

$$\overline{H}_1^{(0)}(c_{0,1}, c_{0,1}) \geq 0.$$

Last, observe from Q radial that $(u, yQ) = 0$ is equivalent to

$$(c_{1,i}, rQ) = 0, \quad 1 \leq i \leq N.$$

Now arguing as for the proof of step 1 of Section 2.3, $\text{ind}_{\tilde{H}_r^1}\overline{H}_1^{(1)} = 1$ from Proposition 3 and (35) imply that for every $w \in \tilde{H}_r^1$,

$$(w, rQ) = 0 \quad \text{implies } \overline{H}_1^{(1)}(w, w) \geq 0.$$

This concludes the proof of the Spectral Property for H_1 in dimension $N = 2$.

We argue similarly for $\overline{H}_2$. Let $u \in H^1$ with $(u, Q_1) = (u, Q_2) = 0$. Decompose u into spherical harmonics and write down formula (32). From Proposition 3, $\overline{H}_2^{(k)} \geq 0$ for $k \geq 1$. For the radial part, $(u, Q_1) = (u, Q_2) = 0$ implies $(c_{0,1}, Q_1) = (c_{0,1}, Q_2) = 0$ and $\overline{H}_2^{(0)}(c_{0,1}, c_{0,1}) = \overline{H}_2(c_{0,1}, c_{0,1}) \geq 0$ from Section 2.3. This yields $\overline{H}_2(u, u) \geq 0$ and concludes the proof of the Spectral Property for H_2 in dimensions $N = 2, 3, 4$. □

4. Alternative proof of the spectral property

We propose in this section a slightly different approach for the proof of the Spectral Property. The idea is to enlarge the functional space H^1 which does not contain the directions built in Propositions 2 and 4 and to work with

$$E_r = \left\{u \text{ radial}, \int |\nabla u|^2 + \int |u|^2 \mathrm{e}^{-\gamma_0|y|} < +\infty\right\} \tag{36}$$

for some small $\gamma_0 > 0$ small enough in the radial case and

$$\tilde{E}_r = \left\{u \text{ radial}, \int |\nabla u|^2 + \int \frac{|u|^2}{r^2} < +\infty\right\} \tag{37}$$

in the non-radial case. Let us focus on the radial situation only.

We now consider the quadratic forms H_1, H_2 not on H_r^1 but on E_r. Note that the inner products introduced in the Spectral Property still make sense thanks to the exponential decay property of Q.

The first step is to generalize Lemma 1 and prove that the number of zeros of the solution U to (11) indeed corresponds to the index of H_1 *on* E_r. This is achieved using localization arguments and the standard Hardy inequality to control the error terms:

$$\int \frac{|u|^2}{r^2} \leq C \int |\nabla u|^2 \quad \text{for } N \geq 3$$

and the modified two-dimensional version,

$$\int_{r\geq 1} \frac{|u|^2}{r^2 \log^2 r} \leq C \left(\int |\nabla u|^2 + \int |u|^2 \mathrm{e}^{-r}\right).$$

The second step is now to prove Proposition 2 using the Lax–Milgram theorem. This first requires projecting out the negative directions of L in order to recover a coercive operator on E_r. Then the Lax–Milgram theorem ensures the existence and uniqueness of a solution $u \in E_r$ of $Lu = f$. Using now ODE techniques, we prove using the exponential decay of f

that for a solution to $Lu = f$, it is equivalent to be in E_r or to satisfy the boundary condition $(1 + r^{N-2})u \in L^\infty$ which is the one which we check numerically. Here again, the case $N = 2$ requires a slightly more refined argument to overcome some logarithmic divergences.

Now as the directions build from Proposition 2 belong to E_r and the index of H on E_r is known, the proof of the Spectral Property given in Section 2.4 may be slightly simplified as it does not require any localization argument anymore.

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