Physica D 240 (2011) 1119–1122

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Singular solutions of the subcritical nonlinear Schrödinger equation

Gadi Fibich

School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel

ARTICLE INFO

Article history:
Received 29 March 2010
Received in revised form
3 April 2011
Accepted 5 April 2011
Available online 12 April 2011
Communicated by J. Garnier

Keywords:
Singularity
Subcritical
Nonlinear Schrödinger equation

ABSTRACT

We show that the subcritical d-dimensional nonlinear Schrödinger equation $i\psi_t + \Delta\psi + |\psi|^{2\sigma}\psi = 0$, where $1 < \sigma d < 2$, admits smooth solutions that become singular in L^p for $p^* < p \le \infty$, where $p^* := \frac{\sigma d}{\sigma d - 1}$. Since $\lim_{\sigma d \to 2-} p^* = 2$, these solutions can collapse at any $2 < p \le \infty$, and in particular for $p = 2\sigma + 2$.

1. Introduction

The focusing nonlinear Schrödinger equation (NLS)

$$i\psi_t(t, \mathbf{x}) + \Delta\psi + |\psi|^{2\sigma}\psi = 0, \quad \psi(0, \mathbf{x}) = \psi_0(\mathbf{x}), \tag{1}$$

where $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $\Delta = \sum_{j=1}^{d} \partial_{x_j x_j}$ is the Laplacian, has been the subject of intensive study, due to its role in various areas of physics, such as nonlinear optics and Bose–Einstein condensates (BEC). The NLS is called subcritical, critical, and supercritical if $\sigma d < 2, \sigma d = 2$, and $\sigma d > 2$, respectively. It is well-known that in the critical and supercritical cases, the NLS (1) possesses solutions that become singular in a finite time in L^p for some finite p [1]. In this study we show that the subcritical NLS also admits solutions that become singular in L^p for some finite p.

NLS theory was originally developed for solutions that are in $H^1(\mathbb{R}^d)$. In this case, the initial condition $\psi_0 \in H^1$, and the NLS solution is said to become singular at $t = T_c$ if $\psi(t) \in H^1$ for $0 \le t < T_c$, and $\lim_{t\to T_c} \|\psi(t)\|_{H^1} = \infty$. In 1983, Weinstein proved that all H^1 solutions of the subcritical NLS exist globally:

Theorem 1 ([2]). *Let ψ be a solution of the NLS (1), let $0 < \sigma d < 2$, and let $\psi_0 \in H^1$. Then, ψ exists globally in H^1.*

In recent years, there has been a lot of work on the NLS with lower regularity than H^1. In this study we show that if we do not restrict ourselves to H^1 solutions, then the subcritical NLS also admits singular solutions. Here, by singular we mean that there exists some $2 < p < \infty$ such that $\|\psi\|_p$ becomes infinite in a finite time.[1] Our main result is as follows:

Theorem 2. *Let*

$$p^* < p \le \infty, \quad p^* := \frac{\sigma d}{\sigma d - 1}. \tag{2}$$

Then, the subcritical NLS with $1 < \sigma d < 2$ admits classical solutions that become singular at a finite time T_c in L^p, i.e.,

$$\|\psi(t)\|_p < \infty, \quad 0 \le t < T_c,$$

and

$$\lim_{t\to T_c} \|\psi(t)\|_p = \infty.$$

Theorem 2 follows from the following theorem:

Theorem 3. *Let p be in the range (2), let $1 < \sigma d < 2$, let $a > 0$ be a positive constant, and let $Q(\rho)$ be the solution of*

$$\Delta Q(\rho) - Q + ia\left(\frac{1}{\sigma}Q + \rho Q'\right) + |Q|^{2\sigma}Q = 0, \quad 0 < \rho < \infty, \tag{3a}$$

subject to

$$0 \ne Q(0) \in \mathbb{C}, \qquad Q'(0) = 0. \tag{3b}$$

[1] In the case of H^1 solutions of the NLS, blow-up of the H^1 norm implies blow-up of the L^p norms for $2\sigma + 2 \le p \le \infty$; see Appendix A.

E-mail addresses: fibich@tau.ac.il, fibich@math.tau.ac.il.

0167-2789/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.physd.2011.04.004

Let

$$\psi_Q^{\text{explicit}}(t,r) = \frac{1}{L^{1/\sigma}(t)} Q(\rho) e^{i\tau(t)}, \tag{4a}$$

where

$$r = |\mathbf{x}|, \qquad L(t) = \sqrt{2a(T_c - t)}, \tag{4b}$$

and

$$\rho = \frac{r}{L(t)}, \qquad \tau = \int_0^t \frac{1}{L^2(s)}\, ds = \frac{1}{2a} \log \frac{T_c}{T_c - t}. \tag{4c}$$

Then, ψ_Q^{explicit} is an explicit solution of the subcritical NLS that becomes singular in L^p as $t \longrightarrow T_c$.

Remark. Although Q, and hence also ψ_Q^{explicit}, is not in H^1, it is smooth, and it decays to zero as $|\mathbf{x}| \longrightarrow \infty$; see Lemma 2.

Since

$$\lim_{\sigma d \to 2-} p^* = 2+,$$

then for any $2 < p < \infty$, there exists a singular solution of a subcritical NLS that becomes singular in L^p. In particular, if σd is sufficiently close to 2 from below, then ψ_Q^{explicit} becomes singular in $L^{2\sigma+2}$.

Remark. The linear Schrödinger equation $i\psi_t + \Delta\psi = 0$ admits the fundamental solution $\psi = \frac{1}{(4\pi i t)^{d/2}} e^{i|\mathbf{x}|^2/4t}$, which becomes singular in finite time in L^∞. Additional solutions of linear and subcritical nonlinear Schrödinger equations that become singular in finite time in L^∞ are given by Cordero-Soto et al. [3]. Unlike ψ_Q^{explicit}, however, these solutions do not become singular in L^p for any finite p.

2. Proof of Theorem 3

We begin with the following result.

Lemma 1. *Let ψ_Q^{explicit} be defined as in Theorem 3. Then, ψ_Q^{explicit} is an explicit solution of the NLS (1).*

Proof. Substituting ψ_Q^{explicit} in the NLS (1) and carrying out the differentiation proves the result. □

The result of Lemma 1 was used by Zakharov [4], and subsequently by others (see [1] and the references therein), in the study of singular H^1 solutions of the supercritical NLS. These solutions undergo a quasi-self-similar collapse, in which ψ_Q^{explicit} is the *asymptotic* blow-up profile of the collapsing core of the solution. Here, in contrast, ψ_Q^{explicit} is an explicit, "truly" self-similar solution of the subcritical NLS.

We now establish the decay at infinity of all solutions of Eq. (3):

Lemma 2. *Let $a > 0$ and that $1 < \sigma d < 2$. Then, for any $Q(0) \in \mathbb{C}$, the solution of Eq. (3) exists, is unique, and decays to zero as $\rho \longrightarrow \infty$, so that*

$$|Q| = O(\rho^{-d+1/\sigma}), \quad \rho \longrightarrow \infty.$$

Therefore, Q is in L^p for any $p^* < p \le \infty$.

Proof. The proof is nearly identical to the proof of Johnson and Pan in the supercritical case [5]; see Appendix B. □

Lemma 3. *For any $p^* < p \le \infty$, ψ_Q^{explicit} becomes singular in L^p as $t \longrightarrow T_c$.*

Proof. Since

$$\|\psi_Q^{\text{explicit}}(t)\|_p = \frac{\|Q\|_p}{L^{1/\sigma}(t)},$$

and $\|Q\|_p < \infty$, the result follows. □

This concludes the proof of Theorem 3.

3. The Q equation in the subcritical case

As in [6], the far-field asymptotics of Q can be calculated using the WKB method:

Lemma 4. *Let $Q(\rho)$ be a solution of Eq. (3), where $1 < \sigma d < 2$. Then,*

$$Q(\rho) \sim c_1 Q_1(\rho) + c_2 Q_2(\rho), \quad \rho \longrightarrow \infty, \tag{5}$$

where c_1 and c_2 are complex numbers, and

$$Q_1 \sim \rho^{-i/a - 1/\sigma}, \qquad Q_2 \sim e^{-ia\rho^2/2} \rho^{i/a - d + 1/\sigma}.$$

Proof. See Appendix C. □

Corollary 1. *If $1 < \sigma d < 2$, then*

$$Q_1 \in L^2(\mathbb{R}^d), \qquad \nabla Q_1 \in L^2(\mathbb{R}^d),$$

and

$$Q_2 \notin L^2(\mathbb{R}^d), \qquad \nabla Q_2 \notin L^2(\mathbb{R}^d).$$

In addition, $Q_1 \in L^p(\mathbb{R}^d)$ for any $2 \le p \le \infty$, and

$$Q_2 \in L^p(\mathbb{R}^d), \qquad \frac{\sigma d}{\sigma d - 1} < p \le \infty.$$

Proof. This follows from Lemma 4. □

In the supercritical case, a key role is played by the zero-Hamiltonian solutions of the Q equation, which behave as $c_1 Q_1$ at large ρ [1]. We now show that there are no such solutions in the subcritical case:

Lemma 5. *When $1 < \sigma d < 2$, there are no nontrivial solutions of the Q Eq. (3), such that $c_2 = 0$, i.e., that*

$$Q(\rho) \sim c_1 Q_1(\rho), \quad \rho \longrightarrow \infty.$$

Proof. By negation. Assume that there is such a Q. In this case, it follows from Corollary 1 that $Q \in H^1$. Hence, ψ_Q^{explicit} is a solution of the subcritical NLS that becomes singular in H^1, which is in contradiction with Theorem 1. □

Fig. 1 shows two numerical solutions of Eq. (3). As expected (see Lemma 4),

$$|Q| \sim |c_1 \rho^{-i/a - 1/\sigma} + c_2 e^{-ia\rho^2/2} \rho^{i/a - d + 1/\sigma}|$$

decreases to zero as $\rho \longrightarrow \infty$, while undergoing faster and faster oscillations. The "cleaner picture" in Fig. 1 A has to do with the fact that the values of a and $Q(0)$ were chosen so as to minimize the value of c_2.[2,3]

[2] These values of a and $Q(0)$ were calculated using the shooting algorithm of Budd et al. [7] for calculating the zero-Hamiltonian solutions in the supercritical case.

[3] The value of c_2 cannot be equal to zero; see Lemma 5.

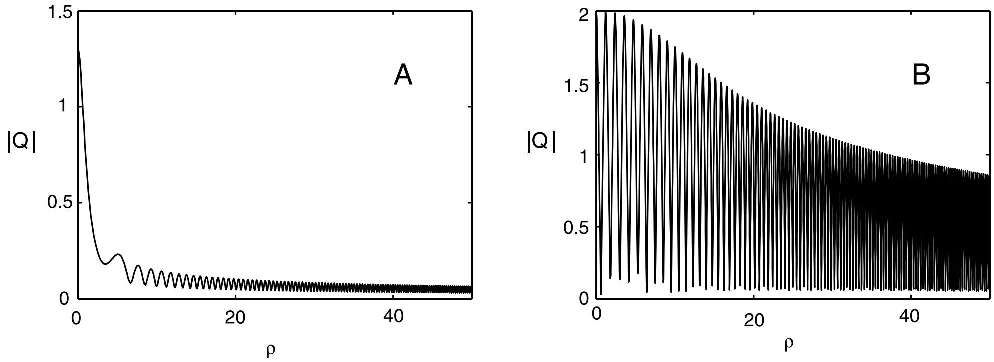

Fig. 1. Solutions of Eq. (3) with $d = 1$, $\sigma = 1.9$, $a = 0.5145$, and with A: $Q(0) = 1.2953$, and B: $Q(0) = 3$.

4. Final remarks

In this study we showed that if we do not limit ourselves to H^1 solutions, then the subcritical NLS admits solutions that become singular at a finite time in L^p for some finite p. This finding raises several questions, which are currently open. One question is whether the explicit singular solutions are stable. This question is hard to study numerically, because of the slow decay, coupled with the ever faster oscillations, of the solution at infinity. Another open question is that of whether the subcritical NLS admits singular solutions that are not self-similar.[4] The answers to these questions will determine whether singularity formation in the subcritical NLS will remain as an anecdote, or lead to a new line of research.

Acknowledgment

This research was partially supported by the Israel Science Foundation (ISF grant No. 123/08).

Appendix A. Blow-up of $\|\psi\|_p$

The NLS conserves the power (mass, L^2 norm) and the Hamiltonian, i.e.,

$$\|\psi\|_2^2 \equiv \|\psi_0\|_2^2, \qquad H(t) := \|\nabla\psi\|_2^2 - \frac{1}{\sigma+1}\|\psi\|_{2\sigma+2}^{2\sigma+2} \equiv H(0).$$

Therefore, when $\|\psi\|_{H^1}$ becomes infinite, then so does $\|\nabla\psi\|_2$; hence $\|\psi\|_{2\sigma+2}$. Therefore, since $\|\psi\|_2$ is conserved, it follows from the interpolation inequality for L^p norms that $\|\psi\|_p$ also becomes infinite for $2\sigma + 2 \le p \le \infty$.

Appendix B. Proof of Lemma 2

As in the proof of Johnson and Pan in the supercritical case [5], let

$$Q(\rho) = u(\rho)e^{-ia\rho^2/4}, \qquad u(\rho) = u_1(\rho) + iu_2(\rho),$$

where u_1 and u_2 are real functions, that

$$v_j(\rho) = \rho^{(d-1)/2}u_j(\rho), \quad j = 1, 2,$$

that

$$t = \rho^2, \quad x_j(t) = v_j(\rho), \qquad y_j(t) = \frac{dx_j}{dt},$$

that

$$f_j(t) = t^{1/4}x_j(t), \qquad g_j(t) = \frac{df_j}{dt},$$

and that

$$H(t) = \frac{1}{2}(g_1^2 + g_2^2) + \frac{1}{8}\left(\lambda - \frac{1}{t} - \frac{e}{t^2}\right)(f_1^2 + f_2^2) + \frac{1}{4(2\sigma+2)}t^{-\beta}(f_1^2 + f_2^2)^{\sigma+1},$$

where

$$\lambda = \frac{a^2}{4}, \qquad \beta = 1 + \frac{\sigma d}{2}, \qquad e = \frac{1}{4}d(d-4).$$

Then,

$$H'(t) = \frac{B}{4t}(f_1g_2 - f_2g_1) + \frac{1}{8t^2}\left(1 + \frac{2e}{t}\right)(f_1^2 + f_2^2) - \frac{\beta}{4(2\sigma+2)}t^{-\beta-1}(f_1^2 + f_2^2)^{\sigma+1},$$

where

$$B = a\left(\frac{d}{2} - \frac{1}{\sigma}\right) < 0.$$

In addition, from the Cauchy–Schwartz inequality [5],

$$|f_1g_2| \le \frac{1}{2}\left[\frac{\sqrt{\lambda}}{2}f_1^2 + \frac{2}{\sqrt{\lambda}}g_2^2\right],$$

$$|f_2g_1| \le \frac{1}{2}\left[\frac{\sqrt{\lambda}}{2}f_2^2 + \frac{2}{\sqrt{\lambda}}g_1^2\right].$$

Since $\beta > 0$ and $B < 0$,

$$\begin{aligned}
H'(t) &\le \frac{|B|}{4t}(|f_1g_2| + |f_2g_1|) + \frac{1}{8t^2}\left(1 + \frac{2e}{t}\right)(f_1^2 + f_2^2) \\
&\le \frac{|B|}{8t}\left(\frac{\sqrt{\lambda}}{2}(f_1^2 + f_2^2) + \frac{2}{\sqrt{\lambda}}(g_1^2 + g_2^2)\right) \\
&\quad + \frac{1}{8t^2}\left(1 + \frac{2e}{t}\right)(f_1^2 + f_2^2) \\
&= \frac{|B|}{2\sqrt{\lambda}t}\left(\frac{\lambda}{8}(f_1^2 + f_2^2) + \frac{1}{2}(g_1^2 + g_2^2)\right) \\
&\quad + \frac{1}{8t^2}\left(1 + \frac{2e}{t}\right)(f_1^2 + f_2^2) \\
&\le \frac{|B|}{2\sqrt{\lambda}t}\left(1 + O\left(\frac{1}{t}\right)\right)H(t).
\end{aligned}$$

Since $H(t) > 0$ for large t,

$$\frac{H'}{H} \le \frac{|B|}{2\sqrt{\lambda}t} + O\left(\frac{1}{t^2}\right).$$

[4] Cordero-Soto et al. [3] derived solutions of subcritical nonlinear Schrödinger equations that become singular in finite time in L^∞, which are not self-similar. These solutions, however, do not become singular in L^p for any finite p.

Therefore, as in [5], there exists a constant $c > 0$, such that

$$H(t) \le c(1 + t^{2|\alpha|}), \quad 0 \le t < \infty,$$

where

$$2|\alpha| = \frac{|B|}{2\sqrt{\lambda}} = \frac{1}{\sigma} - \frac{d}{2}.$$

Hence,

$$|f_j(t)| \le c(1 + t^{|\alpha|}),$$
$$|x_j(t)| \le ct^{-1/4}(1 + t^{|\alpha|}),$$
$$|v_j(\rho)| \le c\rho^{-1/2}(1 + \rho^{2|\alpha|}),$$

and

$$|u_j(\rho)| \le c\rho^{-d/2}(1 + \rho^{2|\alpha|}).$$

Therefore,

$$|u| = O(\rho^{-d+1/\sigma}), \quad \rho \longrightarrow \infty.$$

Remark. The only difference from the original proof of Johnson and Pan is that in the supercritical case, $B > 0$. Therefore, we take the absolute value of B, instead of B, in the bounds for H'.

Appendix C. Proof of Lemma 4

Let

$$Q(\rho) = e^{-\frac{1}{2}\int\left(\frac{d-1}{\rho} + ia\rho\right)} Z(\rho) = e^{-ia\rho^2/4}\rho^{-(d-1)/2} Z(\rho).$$

Therefore, the equation for Z is given by

$$Z''(\rho) + \left(\frac{a^2}{4}\rho^2 - 1 - ia\frac{d\sigma - 2}{2\sigma} - \frac{(d-1)(d-3)}{4\rho^2} + |Q|^{2\sigma}\right) Z = 0. \quad \text{(C.1)}$$

Since by Lemma 2, $\lim_{\rho\to\infty} Q = 0$, let us look for an asymptotic solution of the form

$$Z = e^{w(\rho)}, \qquad w(\rho) \sim w_0(\rho) + w_1(\rho) + \cdots.$$

The equation for $\{w_i(t)\}$ is given by

$$(w_0'' + w_1'' + \cdots) + (w_0' + w_1' + \cdots)^2 + \left(\frac{a^2}{4}\rho^2 - 1 - ia\frac{d\sigma - 2}{2\sigma} - \frac{(d-1)(d-3)}{4\rho^2} + |Q|^{2\sigma}\right) = 0. \quad \text{(C.2)}$$

A priori, the equation for the leading-order terms is

$$w_0'' + (w_0')^2 + \frac{a^2}{4}\rho^2 = 0.$$

The substitution $w_0 = c\rho^n$ shows that the orders of the terms in this equation are ρ^{n-2}, ρ^{2n-2}, and ρ^2, respectively. Since the only consistent way to balance the leading-order terms is if $n = 2$, the equation for the leading-order terms is given by

$$(w_0')^2 + \frac{a^2}{4}\rho^2 = 0.$$

Therefore,

$$w_0' = \pm\frac{ia}{2}\rho, \qquad w_0 = \pm\frac{ia}{4}\rho^2.$$

The balance of the next-order terms is given by

$$w_0'' + 2w_0'w_1' - 1 - ia\frac{d\sigma - 2}{2\sigma} = 0.$$

Substituting $w_0' = \pm ia\rho/2$ and rearranging gives

$$w_1' = \mp\frac{i}{a\rho} \pm \frac{d\sigma - 2}{2\sigma}\frac{1}{\rho} - \frac{1}{2\rho},$$
$$w_1 = \left(\mp\frac{i}{a} \pm \frac{d\sigma - 2}{2\sigma} - \frac{1}{2}\right)\log\rho.$$

We will now show that $w_2 = o(1)$. Therefore, we obtained the two solutions

$$w^{(1)}(\rho) = ia\frac{\rho^2}{4} + \left(-\frac{i}{a} - \frac{1-d}{2} - \frac{1}{\sigma}\right)\log\rho + o(1),$$
$$w^{(2)}(\rho) = -ia\frac{\rho^2}{4} + \left(\frac{i}{a} + \frac{-1-d}{2} + \frac{1}{\sigma}\right)\log\rho + o(1).$$

Substituting $Q_i(\rho) = e^{-ia\rho^2/4}\rho^{-(d-1)/2}e^{w^{(i)}(\rho)}$ leads to the result.

In order to confirm that $w_2 = o(1)$, we note that the equation for w_2 is given by

$$w_1'' + (w_1')^2 + 2w_0'w_2' - \frac{(d-1)(d-3)}{4\rho^2} + |Q|^{2\sigma} = 0. \quad \text{(C.3)}$$

In the case of Q_1, since $|Q_1|^{2\sigma} \sim \rho^{-2}$, substituting the expressions for w_0 and w_1 gives

$$w_2' = O\left(\frac{1}{\rho^3}\right), \qquad w_2 = O\left(\frac{1}{\rho^2}\right).$$

In the case of Q_2, since $|Q_2|^{2\sigma} \sim \rho^{-2\sigma d+2} \gg \rho^{-2}$, the leading-order equation for w_2 becomes

$$2w_0'w_2' + |Q_2|^{2\sigma} = 0.$$

Since $w_0' \sim \rho$, then $w_2' \sim \rho^{-2\sigma d+1}$ and $w_2 \sim \rho^{-2\sigma d+2} = o(1)$.

Finally, we note that this proof is rigorous, since solutions of linear ODEs always have their asymptotics obtained by WKB calculations, and the ODE (C.1) for Z is "linear", since it can be written as

$$Z''(\rho) + \left(\frac{a^2}{4}\rho^2 - 1 - ia\frac{d\sigma - 2}{2\sigma} - \frac{(d-1)(d-3)}{4\rho^2} + O(\rho^{-d+1/\sigma})\right)Z = 0.$$

References

[1] C. Sulem, P.L. Sulem, The Nonlinear Schrödinger Equation, Springer, New York, 1999.

[2] M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Comm. Math. Phys. 87 (1983) 567–576.

[3] R. Cordero-Soto, R.M. Lopez, E. Suaz, S.K. Suslov, Propagator of a charged particle with a spin uniform magnetic and perpendicular electric fields, Lett. Math. Phys. 84 (2008) 159–178.

[4] V.E. Zakharov, Handbook of Plasma Physics, vol. 2, Elsevier, New York, 1984.

[5] R. Johnson, X. Pan, On an elliptic equation related to the blow-up phenomenon in the nonlinear Schrödinger equation, Proc. Roy. Soc. Edinburgh Sect. A 123 (1993) 763–782.

[6] B.J. LeMesurier, G. Papanicolaou, C. Sulem, P.L. Sulem, Focusing and multi-focusing solutions of the nonlinear Schrödinger equation, Physica D 31 (1988) 78–102.

[7] C.J. Budd, S. Chen, R.D. Russel, New self-similar solutions of the nonlinear Schrödinger equation with moving mesh computations, J. Comput. Phys. 152 (1999) 756–789.