Appendix A: Proof of Propositions 2 and 7

In this section we formulate and solve the optimization problem when the firm incurs variable costs $C(x)$ and consumers have stochastic demands. The deterministic case (Proposition 2) is the special case when $X_x \equiv x$. The expected firm revenue is

$$\bar{\pi}_{\mathrm{c}}(x,p,T,F) = \mathbb{E}[\pi(X_x,p,T,F) - C(X_x)] = \begin{cases} F - \bar{C}(x), & \text{if} \quad M(x) \le T, \\ F - \bar{C}(x) + p\int_T^{M(x)} (y-T)\, g_x(y)\, dy, & \text{if} \quad M(x) > T, \end{cases}$$

where $\bar{C}(x) := \mathbb{E}[C(X_x)]$. Consequently, the firm optimization problem is $(p^{\mathrm{opt}}, T^{\mathrm{opt}}, F^{\mathrm{opt}}) = \arg\max_{p,T,F\ge 0} \bar{\Pi}_{\mathrm{c}}(p,T,F)$, where

$$\bar{\Pi}_{\mathrm{c}}(p,T,F) := \begin{cases} \pi_{\mathrm{c}}(x_{\bar{\mathrm{U}}}^{\mathrm{opt}}(p,T,F),p,T,F), & \text{if} \quad \bar{U}^{\mathrm{opt}}(p,T,F) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 7. *Suppose that $\bar{V}(x) - \bar{C}(x)$ has a unique global maximum which is attained at*

$$x_{\bar{\mathrm{V}},\mathrm{c}}^{\max} := \arg\max_{x\ge 0}\{\bar{V}(x) - \bar{C}(x)\}. \tag{47}$$

Assume also that $\bar{V}(x)$ is concave, and that $M(x)$ and $\bar{C}(x)$ are monotonically increasing in x.[3] *Then the optimal firm plan is*

$$F^{\mathrm{opt}} = \bar{V}(x_{\bar{\mathrm{V}},\mathrm{c}}^{\max}), \qquad T^{\mathrm{opt}} = M(x_{\bar{\mathrm{V}},\mathrm{c}}^{\max}), \qquad p^{\mathrm{opt}} \ge p_{\mathrm{c}}, \tag{48}$$

where

$$p_{\mathrm{c}} := \max_{x \ge x_{\bar{\mathrm{V}},\mathrm{c}}^{\max}+\delta} \left\{ \frac{\bar{V}(x) - \bar{V}(x_{\bar{\mathrm{V}},\mathrm{c}}^{\max})}{\int_{M(x_{\bar{\mathrm{V}},\mathrm{c}}^{\max})}^{M(x)} (y - M(x_{\bar{\mathrm{V}},\mathrm{c}}^{\max}))g_x(y)\,dy} \right\} \tag{49}$$

and δ is the smallest time increment that is billed by the firm. In addition,

1. *The minimal overage price satisfies $0 < p_{\mathrm{c}} < \infty$.*
2. *The consumer plans to talk $x_{\bar{\mathrm{V}},\mathrm{c}}^{\max}$ minutes, where $0 < x_{\bar{\mathrm{V}},\mathrm{c}}^{\max} < x_{\bar{\mathrm{V}}}^{\max}$.*
3. *The consumer expected utility is zero.*
4. *The firm expected revenue is $\bar{V}(x_{\bar{\mathrm{V}},\mathrm{c}}^{\max}) - \bar{C}(x_{\bar{\mathrm{V}},\mathrm{c}}^{\max})$.*

Proof We first note that $0 < p_{\mathrm{c}} < \infty$, because the numerator is continuous and bounded from above and below, and the denominator is continuous and bounded from below.

Suppose the consumer joins the plan (p,T,F) and uses $x_{\bar{\mathrm{U}}}^{\mathrm{opt}} := x_{\bar{\mathrm{U}}}^{\mathrm{opt}}(p,T,F)$ minutes. By (44), $0 \le \bar{U}^{\mathrm{opt}}(p,T,F) = \bar{V}(x_{\bar{\mathrm{U}}}^{\mathrm{opt}}) - \bar{\Pi}(p,T,F) - \bar{S}(x_{\bar{\mathrm{U}}}^{\mathrm{opt}})$. Subtracting $C(x)$ from both sides yields $\bar{\Pi}_c(p,T,F) \le \bar{V}(x_{\bar{\mathrm{U}}}^{\mathrm{opt}}) - \bar{S}(x_{\bar{\mathrm{U}}}^{\mathrm{opt}},p,T) - \bar{C}(x_{\bar{\mathrm{U}}}^{\mathrm{opt}})$. Therefore, from (4) and (47) we have that

$$\bar{\Pi}_c(p,T,F) \le \bar{V}(x_{\bar{\mathrm{V}},\mathrm{c}}^{\max}) - \bar{C}(x_{\bar{\mathrm{V}},\mathrm{c}}^{\max}). \tag{50}$$

Next, we show that if a firm plan satisfies (48), then

$$x_{\bar{\mathrm{U}}}^{\mathrm{opt}}(p,T,F) = x_{\bar{\mathrm{V}},\mathrm{c}}^{\max}. \tag{51}$$

To see that, it is enough to show that $\bar{U}(x,p,T,F) \le \bar{U}\left(x_{\bar{\mathrm{V}},\mathrm{c}}^{\max},p,T,F\right)$ for $x \ne x_{\bar{\mathrm{V}},\mathrm{c}}^{\max}$.

1. If $x < x^{\max}_{\bar{\mathrm{V}},\mathrm{c}}$, since $M(x)$ is monotonically increasing, then $M(x) < M(x^{\max}_{\bar{\mathrm{V}},\mathrm{c}}) = T$. Therefore, the expected consumer utility is $\bar{U}(x,p,T,F) = \bar{V}(x) - F$. Since $\bar{V}(x)$ is concave and attains its maximum at $x^{\max}_{\bar{\mathrm{V}}}$, see (37), $\bar{V}(x)$ is monotonically increasing in x for $0 \le x \le x^{\max}_{\bar{\mathrm{V}}}$. In addition, since $\bar{V}(x)$ is concave and $\bar{C}(x)$ is monotonically increasing, then $x^{\max}_{\bar{\mathrm{V}},\mathrm{c}} < x^{\max}_{\bar{\mathrm{V}}}$. Hence, $V(x) < V(x^{\max}_{\bar{\mathrm{V}},\mathrm{c}})$, and so $\bar{U}(x,p,T,F) < \bar{V}(x^{\max}_{\bar{\mathrm{V}},\mathrm{c}}) - F = \bar{U}\big(x^{\max}_{\bar{\mathrm{V}},\mathrm{c}},p,T,F\big)$.

2. If $x > x^{\max}_{\bar{\mathrm{V}},\mathrm{c}}$, since $M(x)$ is monotonically increasing, then $M(x) > M(x^{\max}_{\bar{\mathrm{V}},\mathrm{c}}) = T$. Therefore, the consumer exceeds T with a positive probability. Hence, $\bar{\pi}(x,p,T,F) = F + p\int_T^{M(x)} (y-T)\, g_x(y)\, dy$. Consequently,

$$\bar{U}(x,p,T,F) = \bar{V}(x) - \bar{S}(x,p,T) - \bar{\pi}(x,p,T,F) = \bar{V}(x) - \bar{S}(x,p,T) - F - p\int_T^{M(x)} (y-T)\, g_x(y)\, dy.$$

Since $\bar{S}(x,p,T) > 0$, see (4), $\bar{U}(x,p,T,F) < \bar{V}(x) - F - p\int_T^{M(x)} (y-T)\, g_x(y)\, dy < \bar{V}(x^{\max}_{\bar{\mathrm{V}},\mathrm{c}}) - F = \bar{U}(x^{\max}_{\bar{\mathrm{V}},\mathrm{c}},p,T,F)$, where the second inequality follows from (48) and (49).

Finally, we show that if the firm plan satisfies (48), then $\bar{\Pi}(p^{\mathrm{opt}},T^{\mathrm{opt}},F^{\mathrm{opt}}) = \bar{V}(x^{\max}_{\bar{\mathrm{V}},\mathrm{c}}) - \bar{C}(x^{\max}_{\bar{\mathrm{V}},\mathrm{c}})$. Indeed, suppose the firm sets $T^{\mathrm{opt}} = M(x^{\max}_{\bar{\mathrm{V}},\mathrm{c}})$. Then for any $p^{\mathrm{opt}} > p_{\mathrm{c}}$ the consumer will plan to use $x = x^{\max}_{\bar{\mathrm{V}},\mathrm{c}}$ minutes, see (51). Therefore, the consumer expected utility is $\bar{U}(x^{\max}_{\bar{\mathrm{V}},\mathrm{c}},p^{\mathrm{opt}},T^{\mathrm{opt}},F^{\mathrm{opt}}) = \bar{V}(x^{\max}_{\bar{\mathrm{V}},\mathrm{c}}) - F^{\mathrm{opt}} = 0$. Hence, he signs up to the plan. In this case, the firm revenue is $\bar{\Pi}_c(p^{\mathrm{opt}},T^{\mathrm{opt}},F^{\mathrm{opt}}) = F^{\mathrm{opt}} - \bar{C}(\bar{x}^{\max}_{\mathrm{V},\mathrm{c}}) = \bar{V}(x^{\max}_{\bar{\mathrm{V}},\mathrm{c}}) - \bar{C}(x^{\max}_{\bar{\mathrm{V}},\mathrm{c}})$. By (50), this is the maximal firm revenue.

Appendix B: Proof of Lemma 5

By Proposition 1, the optimal firm plan that maximizes the revenue from the light users satisfies $F^{\mathrm{opt}}_{\mathrm{L}} = V^{\max}_{\mathrm{L}}$, $p^{\mathrm{opt}}_{\mathrm{L}} \ge 0$ and $T^{\mathrm{opt}}_{\mathrm{L}} \ge x^{\max}_{\mathrm{V,L}}$. Since $V^{\max}_{\mathrm{H}} > V^{\max}_{\mathrm{L}} = F^{\mathrm{opt}}_{\mathrm{L}}$, the heavy users will sign up to the plan if $p^{\mathrm{opt}}_{\mathrm{L}}$ is sufficiently small. In order to extract overage payments from the heavy users, the firm should set $T^{\mathrm{opt}}_{\mathrm{L}} < x^{\max}_{\mathrm{V,H}}$. Since

$$\frac{\partial U_H}{\partial x} = V'_H(x) - p - \frac{\partial S_H}{\partial x} = v_H(x) - p - s_H(x,p), \tag{52}$$

if the firm sets p to be sufficiently small, then $\frac{\partial U_H}{\partial x} > 0$ at $x = T^{\mathrm{opt}}_{\mathrm{L}}$, since $v_H(x) > 0$ for $x < x^{\max}_{\mathrm{V,H}}$, and $p, s_H(x,p) \to 0$ as $p \to 0$. Hence, the heavy users will benefit from exceeding $T^{\mathrm{opt}}_{\mathrm{L}}$, and so the firm would gain additional revenues. Therefore, $x^{\max}_{\mathrm{V,L}} < x^{\mathrm{opt}}_{\mathrm{U,H}}$ and $p^{\mathrm{opt}}_{\mathrm{L}} > 0$. We now show that $T^{\mathrm{opt}}_{\mathrm{L}} = x^{\max}_{\mathrm{V,L}}$. Indeed, assume by negation that $T^{\mathrm{opt}}_{\mathrm{L}} > x^{\max}_{\mathrm{V,L}}$. Then if the firm slightly lowers $T^{\mathrm{opt}}_{\mathrm{L}}$ by $\Delta T \ll 1$, this will not affect the light users, since they can still talk $x^{\max}_{\mathrm{V,L}}$. Under this change the heavy users will still use the same number of minutes, since their usage $x^{\mathrm{opt}}_{\mathrm{U,H}}$ is determined from $\frac{\partial U_H}{\partial x} = 0$, see (7), and $\frac{\partial U_H}{\partial x}$ is independent of T, see (52) and (19). Therefore, they will pay $p\Delta T$ more to the firm, which is in contradiction to the optimality of the plan. Finally, by (52), $0 = \frac{\partial U_H}{\partial x}(x^{\mathrm{opt}}_{\mathrm{U,H}}) < V'_H(x^{\mathrm{opt}}_{\mathrm{U,H}})$. Therefore, $x^{\mathrm{opt}}_{\mathrm{U,H}} < x^{\max}_{\mathrm{V,H}}$.

Appendix C: Proof of Proposition 4

By Lemma 2, the only two plans that extract the maximal revenue from the light and heavy users are $F_1 = V_{\rm L}^{\max}$, $T_1 \geq x_{\rm V,L}^{\max}$, $p_1 \geq 0$, and $F_2 = V_{\rm H}^{\max}$, $T_2 \geq x_{\rm V,H}^{\max}$, $p_2 \geq 0$, respectively. If the heavy users join (p_2, T_2, F_2), their optimal utility is $U_{\rm H}^{\rm opt}(p_2, T_2, F_2) = 0$, see Proposition 1.

1. If $V_{\rm H}(x_{\rm V,L}^{\max}) > V_{\rm L}(x_{\rm V,L}^{\max})$, they will prefer (p_1, T_1, F_1), see (32), since

$$U_{\rm H}^{\rm opt}(p_1, T_1, F_1) \geq U_{\rm H}(x_{\rm V,L}^{\max}, p_1, T_1, F_1) = V_{\rm H}(x_{\rm V,L}^{\max}) - F_1 = V_{\rm H}(x_{\rm V,L}^{\max}) - V_{\rm L}(x_{\rm V,L}^{\max}) > 0.$$

Hence, the firm will not extract the maximal revenue from the heavy users.

2. If $V_{\rm H}(x_{\rm V,L}^{\max}) < V_{\rm L}(x_{\rm V,L}^{\max})$, the heavy users will have a negative utility if they choose plan (p_1, T_1, F_1) and talk $x \leq x_{\rm V,L}^{\max}$ minutes, since $V_{\rm H}(x) \leq V_{\rm H}(x_{\rm V,L}^{\max}) < V_{\rm L}(x_{\rm V,L}^{\max}) = F_1$. The firm can make sure that they also have a negative utility if they choose plan (p_1, T_1, F_1) and talk $x_{\rm V,L}^{\max} < x \leq x_{\rm V,H}^{\max}$ minutes, by setting $T_1 = x_{\rm V,L}^{\max}$ and $p_1 \geq \max_{x_{\rm V,L}^{\max} < x \leq x_{\rm V,H}^{\max}} V_H'(x)$, since in that case

$$\begin{aligned} U_{\rm H}(x, p_1, T_1, F_1) &\leq V_{\rm H}(x) - F_1 - p_1(x - T_1) = V_{\rm H}(x_{\rm V,L}^{\max}) + (V_{\rm H}(x) - V_{\rm H}(x_{\rm V,L}^{\max})) - F_1 - p_1(x - x_{\rm V,L}^{\max}) \\ &< \underbrace{V_{\rm L}(x_{\rm V,L}^{\max}) - F_1}_{=0} + \underbrace{(V_{\rm H}(x) - V_{\rm H}(x_{\rm V,L}^{\max}))}_{< p_1(x - x_{\rm V,L}^{\max})} - p_1(x - x_{\rm V,L}^{\max}) < 0. \end{aligned}$$

Appendix D: Proof of Lemma 6

As in the proof of Proposition 4, if there is a plan that the light consumers sign up to, and if (33) holds, then the heavy consumers will prefer that plan to the one that extracts all their valuation, and that plan does not maximize the revenue from the heavy consumers. Conversely, if (33) does not hold, there exist x_0 such that $V_{\rm L}(x_0) \geq V_{\rm H}(x_0)$. Therefore, if we set $p_1 \geq \max_{x_0 \leq x \leq x_{\rm V,H}^{\max}} V_{\rm H}'(x)$, $T_1 = x_0$, $F_1 = V_{\rm L}(x_0)$, and $p_2 \geq 0$, $T_2 \geq x_{\rm V,H}^{\max}$, $F_2 = V_{\rm H}^{\max}$, the light and heavy consumers will sign up to plans (p_1, T_1, F_1) and (p_2, T_2, F_2), respectively.

Appendix E: Proof of Lemma 7

If plan (p_1, T_1, F_1) maximizes the revenue from the light consumers, then

$$F_1 = V_{\rm L}^{\max}, \qquad T_1 \geq x_{\rm V,L}^{\max}, \qquad p_1 \geq 0, \tag{53}$$

see (10). The heavy consumers will choose plan (p_2, T_2, F_2) provided that

$$U_{\rm H}^{\rm opt}(p_2, T_2, F_2) \geq U_{\rm H}^{\rm opt}(p_1, T_1, F_1), \tag{54}$$

see (32).[4] In addition, as in the proofs of Proposition 1 and 5, see (44), the firm revenue from a heavy consumer is bounded by the difference between his maximal valuation and his optimal utility if he joins (p_1, T_1, F_1), i.e., $\Pi_{\rm H}^{\rm opt}(p_2, T_2, F_2) \le V_{\rm H}^{\max} - U_{\rm H}^{\rm opt}(p_1, T_1, F_1)$. Since $p_1 \ge V'_{\rm H}(x_{\rm V,L}^{\max})$, if heavy consumers join (p_1, T_1, F_1), they do not increase their utility by exceeding the allowance T_1. Hence,

$$U_{\rm H}^{\rm opt}(p_1, T_1, F_1) = U_{\rm H}(x_{\rm V,L}^{\max}, p_1, T_1, F_1) = V_{\rm H}(x_{\rm V,L}^{\max}) - F_1 = V_{\rm H}(x_{\rm V,L}^{\max}) - V_{\rm L}^{\max}. \tag{55}$$

By the last two relations,

$$\Pi_{\rm H}^{\rm opt}(p_2, T_2, F_2) \le V_{\rm H}^{\max} - (V_{\rm H}(x_{\rm V,L}^{\max}) - V_{\rm L}^{\max}). \tag{56}$$

We now show that heavy consumers will choose plan (34b), and that the firm revenue from (34b) is equal to the right-hand-side of (56). Therefore, plan (34b) maximizes the revenue from the heavy consumers. If the heavy consumers join plan (34b), they talk $x_{\rm V,H}^{\max}$ minutes (i.e., as much as they want), and so their utility is $U_{\rm H}^{\rm opt}(p_2, T_2, F_2) = V_{\rm H}^{\max} - F_2 \overset{(34b)}{=} V_{\rm H}(x_{\rm V,L}^{\max}) - V_{\rm L}^{\max}$. Since $U_{\rm H}^{\rm opt}(p_2, T_2, F_2) = U_{\rm H}^{\rm opt}(p_1, T_1, F_1)$, see (55), they will sign up to plan (34b), see (54), and so the firm revenue from a heavy consumer is

$$\Pi_{\rm H}^{\rm opt}(p_2, T_2, F_2) = F_2, \tag{57}$$

which is the optimal revenue from a heavy consumer, see (56) and (34b).

In order to show that the revenue from a light consumer is F_1, we need to check that she does not prefer plan (34b), i.e., that $F_1 < F_2$. Now, by (34),

$$F_2 - F_1 = V_{\rm H}^{\max} - V_{\rm H}(x_{\rm V,L}^{\max}) = V_{\rm H}(x_{\rm V,H}^{\max}) - V_{\rm H}(x_{\rm V,L}^{\max}) > 0, \tag{58}$$

where the last inequality follows from (2).

We thus see the average firm revenue per consumer is

$$\Pi_{\text{ two plans}} = \gamma_{\rm H} F_2 + (1 - \gamma_{\rm H}) F_1 = \gamma_{\rm H}(V_{\rm H}^{\max} - V_{\rm H}(x_{\rm V,L}^{\max})) + (1 - \gamma_{\rm H}) V_{\rm L}^{\max}.$$

By Lemma 5, $\Pi_{\rm L-mainly} = V_{\rm L}^{\max} + \gamma_{\rm H} p_{\rm L}^{\rm opt}(x_{\rm U,H}^{\rm opt} - x_{\rm U,L}^{\max})$, where $x_{\rm U,L}^{\max} < x_{\rm U,H}^{\rm opt} < x_{\rm U,H}^{\max}$. Therefore, to show that $\Pi_{\text{ two plans}} > \Pi_{\rm L-mainly}$, it is enough to show that

$$V_{\rm H}^{\max} - V_{\rm H}(x_{\rm V,L}^{\max}) > p_{\rm L}^{\rm opt}(x_{\rm U,H}^{\rm opt} - x_{\rm V,L}^{\max}). \tag{59}$$

Now, $V'_{\rm H}(x_{\rm U,H}^{\rm opt}) \ge p_{\rm L}^{\rm opt}$, since otherwise the heavy consumer will not use the $x_{\rm U,H}^{\rm opt}$ minute. Therefore, since $V''_{\rm H} < 0$,

$$V'_{\rm H}(x) > p_{\rm L}^{\rm opt}, \qquad 0 \le x < x_{\rm U,H}^{\rm opt}. \tag{60}$$

Hence, $p_{\rm L}^{\rm opt}(x_{\rm U,H}^{\rm opt} - x_{\rm V,L}^{\max}) = \int_{x_{\rm V,L}^{\max}}^{x_{\rm U,H}^{\rm opt}} p_{\rm L}^{\rm opt}\, dx \overset{(60)}{<} \int_{x_{\rm V,L}^{\max}}^{x_{\rm U,H}^{\rm opt}} V'_{\rm H}(x)\, dx = V(x_{\rm U,H}^{\rm opt}) - V(x_{\rm V,L}^{\max})$. Since $x_{\rm U,H}^{\rm opt} < x_{\rm U,H}^{\max}$, then $V(x_{\rm U,H}^{\rm opt}) < V_{\rm H}^{\max}$. Therefore, we proved (59).

Appendix F: Proof of Lemma 8

For $0 < \Delta T \ll 1$ we define the plans

$$T_1^- = x_{\mathrm{V,L}}^{\max} - \Delta T, \quad F_1^- = V_{\mathrm{L}}(T_1^-), \quad p_1^- = \infty, \quad T_2^- = x_{\mathrm{V,H}}^{\max}, \quad F_2^- = V_{\mathrm{H}}^{\max} - (V_{\mathrm{H}}(T_1^-) - F_1^-), \quad p_2^- \geq 0.$$

We now show that there exist ΔT sufficiently small such that

$$\Pi_{\text{ two plans}}(p_1, T_1, F_1, p_2, T_2, F_2) < \Pi_{\text{ two plans}}(p_1^-, T_1^-, F_1^-, p_2^-, T_2^-, F_2^-).$$

Suppose the firm offers (p_i^-, T_i^-, F_i^-), $i = 1, 2$. Light consumers join (p_1^-, T_1^-, F_1^-), because $V_{\mathrm{L}}^{\max} < F_2^-$, see the proof of Lemma 7. Compared to (p_1, T_1, F_1) the firm revenue from light consumers decreases by $n_{\mathrm{L}}(F_1 - F_1^-) = n_{\mathrm{L}}(V_{\mathrm{L}}(x_{\mathrm{V,L}}^{\max}) - V_{\mathrm{L}}(x_{\mathrm{V,L}}^{\max} - \Delta T))$. Since $V_{\mathrm{L}}'(x_{\mathrm{V,L}}^{\max}) = 0$, see (2),

$$n_{\mathrm{L}}(F_1 - F_1^-) \approx n_{\mathrm{L}} V_{\mathrm{L}}''(x_{\mathrm{V,L}}^{\max})(\Delta T)^2, \qquad \Delta T \ll 1.$$

Similarly to Lemma 7, heavy consumers will join (p_2^-, T_2^-, F_2^-) and pay $F_2^- = V_{\mathrm{H}}^{\max} - V_{\mathrm{H}}(T_1^-) + V_{\mathrm{L}}(T_1^-)$. Hence, the firm revenue from the heavy consumers increases by

$$n_{\mathrm{H}}(F_2^- - F_2) = n_{\mathrm{H}}\big(V_{\mathrm{H}}(x_{\mathrm{V,L}}^{\max}) - V_{\mathrm{H}}(x_{\mathrm{V,L}}^{\max} - \Delta T)\big) - n_{\mathrm{H}}\big(V_{\mathrm{L}}(x_{\mathrm{V,L}}^{\max}) - V_{\mathrm{L}}(x_{\mathrm{V,L}}^{\max} - \Delta T)\big).$$

Since $V_{\mathrm{L}}'(x_{\mathrm{V,L}}^{\max}) = 0$ and $V_{\mathrm{H}}'(x_{\mathrm{V,L}}^{\max}) > 0$, see (33),

$$n_{\mathrm{H}}(F_2^- - F_2) \approx n_{\mathrm{H}} V_{\mathrm{H}}^-(x_{\mathrm{V,L}}^{\max})\Delta T.$$

Therefore, if the firm changes to plans (p_i^-, T_i^-, F_i^-), $i = 1, 2$, it gains $O(\Delta T^2)$ less from the light consumers but $O(\Delta T)$ more from the heavy ones.

Appendix G: Proof of Lemma 11

By Proposition 5, the optimal firm revenue is $\bar{\Pi} = \bar{V}^{\max}$. In particular, the optimal firm revenue from deterministic consumers is $\Pi = V^{\max}$. The result follows from the inequality $\bar{V}^{\max} = \bar{V}(x_{\bar{\mathrm{V}}}^{\max}) \overset{(36)}{=} \int_0^{M(x)} V(y) f_{x_{\bar{\mathrm{V}}}^{\max}}(y)\, dy < V(x_{\bar{\mathrm{V}}}^{\max}) \int_0^{M(x)} f_{x_{\bar{\mathrm{V}}}^{\max}}(y)\, dy = V^{\max}$, where the sharp inequality follows from (2).

Appendix H: Proof of Proposition 6

By Proposition 5, the optimal firm revenue is $\bar{\Pi}(w) = \bar{V}_w^{\max}$, where $V_{\mathrm{w}}^{\max}$ is the maximal expected valuation when the stochastic demands are X_x^w. Therefore, we need to show that

$$\bar{V}_w^{\max} < \bar{V}_{w'}^{\max}, \qquad 0 \leq w' < w. \tag{61}$$

In what follows, we will show that for every x and $0 \leq w' < w$, there exists $y = y(x, w, w')$ such that

$$V(x + wz) < V(y + w'z). \tag{62}$$

From this, it follows that $\bar{V}_w(x) = \int V(x + wz)\, f_Z(z)\, dz < \int V(y + w'z)\, f_Z(z)\, dz = \bar{V}_{w'}(y)$, where $f_Z(z)$ is the density distribution of Z. In particular, substituting $x = x^{\max}_{\bar{\mathrm{V}},\mathrm{w}}$ yields $\bar{V}^{\max}_w = \bar{V}_w(x^{\max}_{\bar{\mathrm{V}},\mathrm{w}}) < \bar{V}_{w'}(y) \le \bar{V}^{\max}_{w'}$, which is (61).

To prove (62), let $y = x^{\max}_{\mathrm{V}} + \frac{w'}{w}(x - x^{\max}_{\mathrm{V}})$. Then

$$x^{\max}_{\mathrm{V}} - (y + w'z) = \frac{w'}{w}\left(x^{\max}_{\mathrm{V}} - (x + wz)\right). \tag{63}$$

Since $w' < w$, then $|x^{\max}_{\mathrm{V}} - (y + w'z)| = \frac{w'}{w}|x^{\max}_{\mathrm{V}} - (x + wz)| < |x^{\max}_{\mathrm{V}} - (x + wz)|$, i.e., $y + w'z$ is closer to $x^{\max}_{\mathrm{V}}$ than $x + wz$. In addition, since $\frac{w'}{w} > 0$, $x + wz$ and $y + w'z$ are on the same side of $x^{\max}_{\mathrm{V}}$, see (63). Therefore, since $V(x)$ is concave and since the global maximum of $V(x)$ is attained at $x^{\max}_{\mathrm{V}}$, we have (62).