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Self-focusing on bounded domains

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Abstract

The critical nonlinear Schrödinger equation (NLS) on bounded domains models the propagation of cw laser beams in hollow-core fibers. Unlike the NLS on unbounded domains which models propagation in bulk media, the ground-state waveguide solutions are stable and the condition of critical power for singularity formation is generically sharp.

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1. Introduction

The critical nonlinear Schrödinger equation (NLS) on $\mathbb{R}^2$

$$i\psi_t(t,x,y) + \Delta\psi + |\psi|^2\psi = 0, \quad (x,y) \in \mathbb{R}^2, \ \ t \geq 0, \qquad \psi(0,x,y) = \psi_0(x,y), \quad (x,y) \in \mathbb{R}^2 \tag{1}$$

is the model equation for intense laser beam propagating in bulk media with Kerr nonlinearity. Here, ψ is the electric field amplitude, t is the distance in the direction of propagation, (x, y) are the transverse coordinates, $\Delta = \partial_{xx} + \partial_{yy}$ is the diffraction term and $|\psi|^2\psi$ is the focusing Kerr nonlinearity term. It is well known that solutions of Eq. (1) can self-focus and become singular in finite time. For recent reviews on singularity formation in the critical NLS, see [8,19,28].

Most research on self-focusing in the NLS has been carried out on unbounded domain (i.e., $\mathbb{R}^2$), corresponding to propagation in bulk media. Recently, however, there has been considerable interest in the propagation of intense laser beams in hollow-core fibers filled with a noble gas [6,13,21–24,29,30]. This physical setup offers various advantages, since the noble gas has a pure $\chi^{(3)}$ Kerr nonlinearity, whose magnitude can be controlled by varying the gas pressure. Because of the difference in the index of refraction between the fiber walls and the gas, when

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the fiber diameter is much greater than the beam wavelength, the walls reflect back almost all radiation [16,30]. Therefore, one can approximate the propagation of a laser beam inside a hollow fiber with the critical NLS on a smooth, bounded domain $\Omega \in \mathbb{R}^2$,

$$\mathrm{i}\psi_t(t,x,y) + \Delta\psi + |\psi|^2\psi = 0, \quad (x,y) \in \Omega,\ t \geq 0, \qquad \psi(0,x,y) = \psi_0(x,y), \quad (x,y) \in \Omega \tag{2}$$

with Dirichlet boundary conditions at the walls [6,29,30]

$$\psi(t,x,y) = 0, \quad (x,y) \in \partial\Omega. \tag{3}$$

In this study, we use a combination of rigorous, asymptotic and numerical results to address the following question: *What is the effect of reflecting boundaries (walls) on self-focusing, and in particular on waveguide solutions and on singularity formation*? To answer this question, we first note that the boundaries have a focusing effect, as they reflect back the diffracted part of the beam. As a result, the boundaries act together with the focusing nonlinearity and against the defocusing diffraction term. Indeed, we show that the threshold power for singularity formation is, in a sense, smaller than in free-space. More precisely, unlike in free-space, where the actual threshold power is somewhat higher than the theoretical lower-bound estimate N_c, on bounded domain the threshold power is equal, at least generically, to N_c. In addition, the boundaries stabilize the ground-state waveguides, since, by working together with the nonlinearity, they can 'support' waveguides whose power is below N_c. We note that stabilization of ground-state waveguides by an additional focusing mechanism was also observed in the case of NLS with an attractive linear potential [26]. Indeed, the properties of the ground-state waveguides in [26] and in this study are quite similar. The walls appear to have no effect, however, with regard to the behavior near the singularity, since singularity formation is a *local* phenomena. Thus, the power concentration property of NLS still holds, and one can expect the blowup rate to be the same as on unbounded domains.

We recall that the NLS in $\mathbb{R}^D$ with a general nonlinearity

$$\mathrm{i}\psi_t(t,x_1,\dots,x_D) + \left[\frac{\partial^2}{\partial x_1 x_1} + \cdots + \frac{\partial^2}{\partial x_D x_D}\right]\psi + |\psi|^{2\sigma}\psi = 0$$

is called *subcritical, critical* or *supercritical*, when σD is less than, equal to, or greater than 2, respectively. Although finite-time blowup can occur for both critical and supercritical NLS, there is a marked difference between these two cases, as near the singularity nonlinearity dominates over diffraction in the supercritical case, while they are of the same magnitude in the critical case. In this paper we consider the critical case $\sigma = 1$ and $D = 2$, which corresponds to the hollow fiber application. Our results, however, hold for critical NLS in all dimensions (e.g., $D = 1$ and $\sigma = 2$).

In order to simplify the presentation, we mainly consider radially symmetric solutions on the unit disc $B_1 := \{0 \leq r \leq 1\}$, where $r = \sqrt{x^2 + y^2}$. In this case, Eqs. (2) and (3) become

$$\begin{aligned} &\mathrm{i}\psi_t(t,r) + \Delta\psi + |\psi|^2\psi = 0, \quad 0 \leq r \leq 1,\ \ 0 \leq t, \\ &\psi(0,r) = \psi_0(r), \quad 0 \leq r \leq 1, \quad \frac{\mathrm{d}}{\mathrm{d}r}\psi(t,0) = 0, \quad \psi(t,1) = 0, \quad 0 \leq t, \end{aligned} \tag{4}$$

where $\Delta = \partial_{rr} + (1/r)\partial_r$. The assumption of radial symmetry is consistent with the hollow fiber application. However, almost all results in this paper are valid for a general bounded domain Ω so long as it is smooth and convex. In fact, the only case where the extension from the unit disc to a general domain is not obvious involve the variational characterization of the ground-state waveguides in Section 4.

Finally, in order to clarify the presentation, *all proofs are given in* Appendix A.

2. Invariance

In this paper, unless mentioned otherwise, all L^p norms are taken over a bounded domain $\Omega \in \mathbb{R}^2$, i.e.,

$$\|f(r)\|_p := \left(\int_\Omega |f(r)|^p \,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}.$$

We only consider functions in H_0^1, i.e., those satisfying (3) and

$$\|\psi(t,\cdot)\|_{H_0^1} < \infty, \quad \text{where } \|f\|_{H_0^1} := \sqrt{\|\nabla f\|_2^2 + \|f\|_2^2}.$$

The NLS on bounded domains, Eq. (2), has two important conserved quantities: The *power* [2]

$$\|\psi\|_2^2 \equiv \|\psi_0\|_2^2, \tag{5}$$

and the Hamiltonian

$$H(\psi) \equiv H(\psi_0), \qquad H(f) := \|\nabla f\|_2^2 - \tfrac{1}{2}\|f\|_4^4. \tag{6}$$

As in free-space, the two transformations

1. time-translation: $\psi(t,r) \to \psi(t-t_0, r)$,
2. phase change: $\psi \to \psi \mathrm{e}^{\mathrm{i}\theta}$ with θ real

leave the NLS on bounded domain invariant. However, unlike in free-space, space translation, Galilean transformation, the scaling transformation

$$\psi(t,x) \to \lambda\psi(\lambda^2 t, \lambda x), \tag{7}$$

and the lens transformation (pseudo-conformal transformation), do not leave Eq. (2) invariant. The absence of the last two symmetries, which play an important role in critical self-focusing theory, is related to some of the differences between self-focusing in free-space and in bounded domains.

3. Waveguide solutions

We begin in Section 3.1 with a short review of the theory of waveguide solutions of the NLS on $\mathbb{R}^2$. The corresponding theory for bounded domains is developed in Section 3.2.

3.1. Infinite domain

Eq. (1) has radially symmetric waveguide solutions of the form

$$\psi = \exp(\mathrm{i}\omega t) R_\omega(r),$$

where R_ω is the real solution of the nonlinear ODE

$$\Delta R_\omega - \omega R_\omega + R_\omega^3 = 0, \quad \frac{\mathrm{d}}{\mathrm{d}r} R_\omega(0) = 0, \quad R_\omega(\infty) = 0. \tag{8}$$

[2] In the nonlinear optics context, $\|\psi\|_2^2$ is the normalized beam power.

For $\omega > 0$, Eq. (8) has an infinite number of solutions, which can be arranged in order of increasing power (see [1,2])

$$\|R_\omega^{(0)}\|_2 < \|R_\omega^{(1)}\|_2 < \|R_\omega^{(2)}\|_2 < \cdots . \tag{9}$$

In addition, due to the scaling invariance of (8), we have that for $\omega > 0$

$$R_\omega^{(n)} = \sqrt{\omega} R^{(n)}(\sqrt{\omega} r), \quad R^{(n)} := R_{\omega=1}^{(n)}, \tag{10}$$

and, as a result,

$$\|R_\omega^{(n)}\|_2 = \|R^{(n)}\|_2. \tag{11}$$

Integration by parts of (8) shows that

$$\omega \|R_\omega^{(n)}\|_2^2 = \left\| \frac{\mathrm{d}R_\omega^{(n)}}{\mathrm{d}r} \right\|_2^2 = \frac{1}{2} \|R_\omega^{(n)}\|_4^4. \tag{12}$$

Therefore,

$$H(R_\omega^{(n)}) = 0. \tag{13}$$

From Eqs. (10) and (12) it follows that there are solutions in H^1 to Eq. (8) when $0 < \omega < \infty$, but not for $\omega \leq 0$. We also recall that by standard WKB,

$$R^{(n)}(r) \sim A_n \, \mathrm{e}^{-r} r^{-1/2} \quad \text{for } 1 \ll r, \tag{14}$$

where the constants A_n are given by the formula

$$A_n = \sqrt{\frac{\pi}{2}} \int_0^\infty (R^{(n)}(r))^3 I_0(r) r \, \mathrm{d}r,$$

and I_0 is the modified Bessel function.

A special role in NLS theory is played by the ground-state solution $R := R^{(0)}$, the so-called *Townes soliton*, which is the positive, monotonically decreasing solution of

$$\Delta R(r) - R + R^3 = 0, \quad \frac{\mathrm{d}}{\mathrm{d}r} R(0) = 0, \quad R(\infty) = 0. \tag{15}$$

For any $n \geq 0$, there exists a solution of (15) with exactly n zeros in the interval $0 < r < \infty$ [12]. It is conjectured that this solution is the nth-state solution $R^{(n)}$. However, rigorous proof exists only for the ground-state solution [15,31].

3.2. Bounded domain

We now derive the corresponding theory for waveguide solutions of the form

$$\psi = \exp(\mathrm{i}\omega t) Q_\omega(r)$$

of the NLS on a the unit disc, Eq. (4). The function Q_ω is the real solution of

$$\Delta Q_\omega(r) - \omega Q_\omega + Q_\omega^3 = 0, \quad \frac{\mathrm{d}}{\mathrm{d}r} Q_\omega(0) = 0, \quad Q_\omega(1) = 0. \tag{16}$$

For each ω we can define the functional

$$I_\omega(u) := H(u) + \omega\|u\|_2^2.$$

The nonlinear ODE (16) has an infinite number of solutions which can be arranged in order of increasing I_ω [1,2]:

$$I_\omega(Q_\omega^{(0)}) < I_\omega(Q_\omega^{(1)}) < I_\omega(Q_\omega^{(2)}) < \cdots .$$

The scaling properties (10) and (11) do not hold on bounded domains. This fact is used in Section 4 to provide a variational formulation for $Q_\omega^{(0)}$.

The bounded domain version of the identities (12) are given by (see Appendix A.1)

$$\omega\|Q_\omega^{(n)}\|_2^2 = \left\|\frac{\mathrm{d}Q_\omega^{(n)}}{\mathrm{d}r}\right\|_2^2 - \left[\frac{\mathrm{d}Q_\omega^{(n)}}{\mathrm{d}r}(1)\right]^2 = \frac{1}{2}\|Q_\omega^{(n)}\|_4^4 - \frac{1}{2}\left[\frac{\mathrm{d}Q_\omega^{(n)}}{\mathrm{d}r}(1)\right]^2. \tag{17}$$

Therefore,

$$H(Q_\omega^{(n)}) = \frac{1}{2}\left[\frac{\mathrm{d}Q_\omega^{(n)}}{\mathrm{d}r}(1)\right]^2 > 0. \tag{18}$$

The fact that the Hamiltonian of waveguides on bounded domains is positive, rather than zero as in the free-space case (13), has implications to the stability of waveguide solutions (Section 7). We also note that, as in free-space, it can be proved that the ground-state solutions $Q_\omega^{(0)}$ are strictly positive inside the unit disc and are monotonically decreasing ([2], and see also Section 4).

The following result relates changes in ω of the power and the Hamiltonian of $Q_\omega^{(n)}$:

Lemma 1.

$$\frac{\mathrm{d}}{\mathrm{d}\omega}H(Q_\omega^{(n)}) = -\omega\frac{\mathrm{d}}{\mathrm{d}\omega}\|Q_\omega^{(n)}\|_2^2.$$

We can identify two asymptotic regimes for Eq. (16):

Large amplitude regime $\|Q_\omega^{(n)}\|_\infty \gg 1$. In this case, if we substitute $Q_\omega^{(n)} = \sqrt{\omega}F_\omega^{(n)}(\sqrt{\omega r})$ with $\omega \gg 1$, then $F_\omega^{(n)}(r)$ is the solution of

$$\Delta F_\omega^{(n)} - F_\omega^{(n)} + (F_\omega^{(n)})^3 = 0, \quad \frac{\mathrm{d}}{\mathrm{d}r}F_\omega^{(n)}(0) = 0, \quad F_\omega^{(n)}(\sqrt{\omega}) = 0.$$

Therefore, we have, both in H_0^1 and pointwise,[3] that

$$\lim_{\omega\to\infty} F_\omega^{(n)} = R^{(n)},$$

where $R_\omega^{(n)}$ is the nth-state solution of (15). In particular, in L^2,

$$Q_\omega^{(n)} \approx R_\omega^{(n)} = \sqrt{\omega}R^{(n)}(\sqrt{\omega}r) \quad \text{for } \omega \gg 1. \tag{19}$$

The waveguides $Q_\omega^{(n)}$ are, therefore, localized around the origin, and

$$\lim_{\omega\to\infty}|Q_\omega^{(n)}|^2 = \|R^{(n)}\|_2^2\delta(r). \tag{20}$$

[3] These limits can be rigorously proved using Sturm–Liouville properties of ODEs to show that the zeros of F_ω remain bounded.

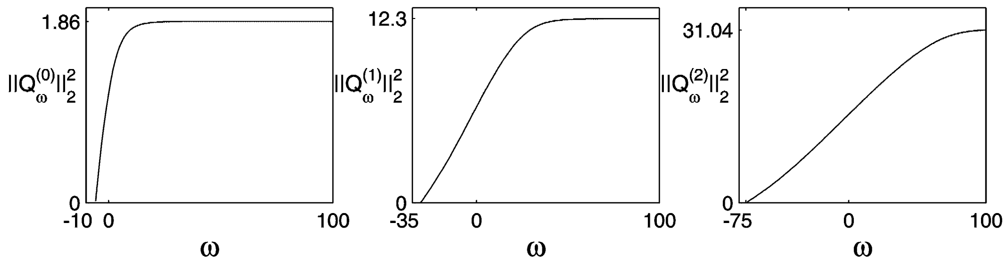

Fig. 1. $\|Q_\omega^{(n)})\|_2^2$ as a function of ω for the first three states.

From (9) and (19) and the conjecture that $R_\omega^{(n)}$ has n zeros in $0<r<\infty$, we have that for $\omega \gg 1$:

1. $\|Q_\omega^{(0)}\|_2 < \|Q_\omega^{(1)}\|_2 < \|Q_\omega^{(2)}\|_2 < \cdots$,
2. $Q_\omega^{(n)}$ has n zeros in $0<r<1$.

Numerical calculations suggest that properties (1) and (2) hold for all ω (see Figs. 1 and 3).

From (14) and (19) we have that

$$\frac{\mathrm{d}Q_\omega^{(n)}}{\mathrm{d}r}(1) \approx \omega \frac{\mathrm{d}R_\omega^{(n)}}{\mathrm{d}r}(\sqrt{\omega}) \to 0 \quad \text{as } \omega \to \infty.$$

Therefore, using (18), we have that

$$\lim_{\omega\to\infty} H(Q_\omega^{(n)}) = 0, \tag{21}$$

and also that

$$\frac{\mathrm{d}}{\mathrm{d}\omega} H(Q_\omega^{(n)}) < 0 \quad \text{for } \omega \text{ sufficiently large.} \tag{22}$$

Therefore, by Lemma 1, we have that

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \|Q_\omega^{(n)}\|_2^2 > 0 \quad \text{for } \omega \text{ sufficiently large.} \tag{23}$$

Small amplitude regime $\|Q_\omega^{(n)}\|_\infty \ll 1$. In this case, if we substitute $Q_\omega^{(n)} = \sqrt{\epsilon} G_\omega^{(n)}(r)$ with $0<\epsilon \ll 1$, then $G_\omega^{(n)}(r)$ is the solution of

$$\Delta G_\omega^{(n)} - \omega G_\omega^{(n)} + \epsilon (G_\omega^{(n)})^3 = 0, \quad \frac{\mathrm{d}}{\mathrm{d}r} G_\omega^{(n)}(0) = 0, \quad G_\omega^{(n)}(1) = 0.$$

Therefore,

$$G_\omega^{(n)} \approx c J_0(\sqrt{-\omega_n} r), \quad 0<\epsilon \ll 1,$$

where J_0 is the Bessel function of zero-order of the first kind, c a constant, $\omega_n = -k_n^2$ and k_n the nth positive root of $J_0(r)$. Thus, $\omega_0 \approx -5.8$, $\omega_1 \approx -30.5$, $\omega_2 \approx -74.8$, etc.

The following lemma, which is a standard bifurcation result, provides further information on the solution magnitude.

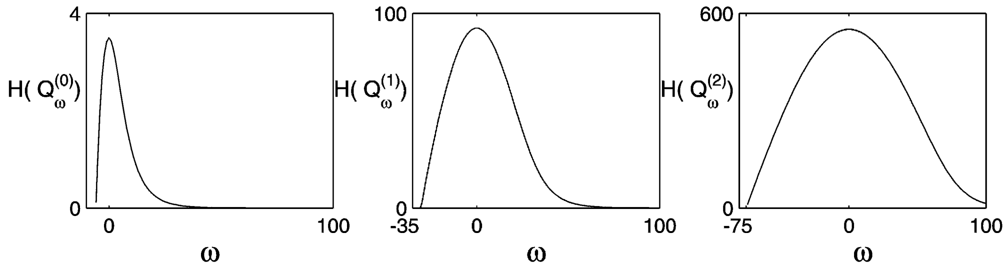

Fig. 2. $H(Q_\omega^{(n)})$ as a function of ω for the first three states.

Lemma 2. *When* $0 < \omega - \omega_n \ll 1$, *then*

$$Q_\omega^{(n)} \sim B_n\sqrt{\omega - \omega_n}\,J_0(\sqrt{-\omega_n}r), \qquad B_n = \left(\frac{\int_0^1 J_0^2(\sqrt{-\omega_n}r)r\,\mathrm{d}r}{\int_0^1 J_0^4(\sqrt{-\omega_n}r)r\,\mathrm{d}r}\right)^{1/2}.$$

Note that Lemma 2 implies that in the domain $0 < \omega - \omega_n \ll 1$,

$$\|Q_\omega^{(n)}\|_2^2 = \mathrm{O}(\omega - \omega_n), \qquad H(Q_\omega^{(n)}) = \mathrm{O}(\omega - \omega_n), \tag{24}$$

and that $Q_\omega^{(n)}(r)$ has n zeros in $0 < r < 1$. In particular, Lemma 1 and (24) imply the following corollary.

Corollary 1. *There exists* $0 < \Omega_n$ *such that*

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\|Q_\omega^{(n)}\|_2^2 > 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\omega}H(Q_\omega^{(n)}) > 0 \tag{25}$$

for $\omega_n < \omega < \omega_n + \Omega_n$.

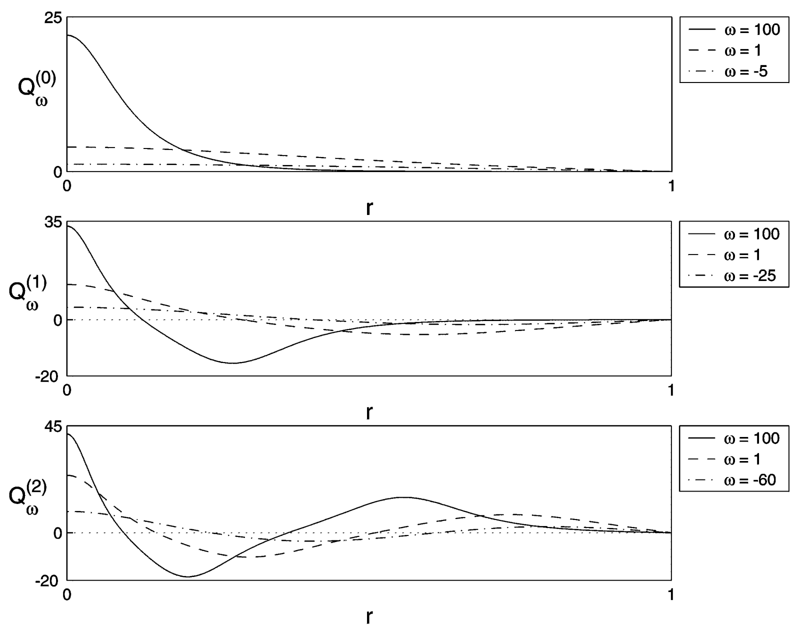

Fig. 3. Spatial profile of Q_ω for various ω for the first three states.

The above results, together with numerical results shown in Figs. 1–3, suggest the following picture. The nth-state solution of (16) exists for $\omega_n < \omega < \infty$,[4] and has n zeros inside the domain $r \in (0, 1)$. The nonlinearity becomes more pronounced as ω increases and it becomes negligible as $\omega \to \omega_n+$. As ω increase from ω_n to ∞, $\|Q_\omega^{(n)}\|_2^2$ is monotonically increasing from zero (see (24)) to $\|R^{(n)}\|_2^2$ (see (20)). The monotonicity of $\|Q_\omega^{(n)}\|_2^2$ is observed numerically (Fig. 1) and is proved in the two limiting cases $\omega \to \infty$ (see (23)) and $\omega \to \omega_n$ (Corollary 1). In addition, as ω increases from ω_n to zero, $H(Q_\omega^{(n)})$ is monotonically increasing from zero (see (24)) to its maximal value and that as ω increases from zero to infinity, $H(Q_\omega^{(n)})$ is monotonically decreasing to zero (see (21)). The monotonicity of $H(Q_\omega^{(n)})$ for $\omega_n < \omega < 0$ and for $0 < \omega < \infty$ follows from the monotonicity of $\|Q_\omega^{(n)}\|_2^2$ and Lemma 1.

4. Variational characterization of Q_ω

In the free-space case, the scaling invariance of Eq. (8) implies that $H(R_\omega^{(n)})$ and $\|R_\omega^{(n)}\|_2$ are independent of ω (see (11) and (13)). These properties are not true for Eq. (16) on bounded domains, and can be used to give the following variational characterization of the ground-state solutions Q_ω.

Theorem 1. *For all $\omega \in (-\omega_0, +\infty)$, $Q_\omega(r)$ is the unique real minimizer over all $U(x, y) \in H_0^1(B_1)$ of*

$$\inf_{\|U\|_2^2=\|Q_\omega\|_2^2} H(U).$$

In addition, for all $0 < M < N_c$, there exists a unique $\omega_M \in (-\omega_0, +\infty)$ such that $Q_{\omega_M}(r)$ is the unique real minimizer of

$$\inf_{\|U\|_2^2=M} H(U).$$

Thus, the variational structure of Q_ω is based on the absence of scaling invariance on bounded domains.

Corollary 2. *If the infimum in Theorem 1 is taken over functions U which are not necessarily real, then the minimizers are given by the one-parameter family $\{e^{i\theta} Q_\omega(r), \theta$ real$\}$.*

In the remainder of this section we give the proof of Theorem 1. Let us first recall the following result from nonlinear ODE theory.

Lemma 3 (Kwong [15]). *For any given $\omega > 0$, there is a unique real positive solution to the nonlinear ODE* (16).

Kwong's result holds only for $\omega > 0$. We are not aware of a similar uniqueness result for $\omega_0 < \omega \le 0$, which is, in fact, a simpler case, since $f(u) = -\omega u + u^3$ is monotonic in u. Since this result is widely expected to be true, in the remainder of this paper we assume that Lemma 3 holds for $\omega_0 < \omega < \infty$. Note that the question here is about uniqueness, as existence follows from classical arguments (see below).

We also recall the Gidas et al. result [9,10] that all positive solutions of the elliptic PDE in the unit disc

$$\Delta Q(x, y) + Q^3 = \omega Q \quad \text{for } (x, y) \in B_1, \qquad Q(x, y) = 0 \quad \text{for } (x, y) \in \partial B_1, \tag{26}$$

[4] Unlike in free-space, where solutions of all states only exist for $0 < \omega < \infty$.

are radially symmetric and a decreasing function of the variable r. Therefore, by Lemma 3, there is a unique positive solution to the PDE (26) for $\omega > 0$, which is the positive solution of the ODE (16). We assume that this also holds for $\omega_0 < \omega \le 0$. We emphasize that here we use the fact that Ω is a disc. The uniqueness result is believed to remain true for positive ground-state solutions in general bounded regular convex domains, but, at present, there is no proof for that.

As before, we denote by Q_ω the unique positive solution of (16). From numerical evidence (Fig. 1) we claim the following result.

Lemma 4. *$\|Q_\omega\|_2$ is a strictly monotonically increasing function of ω.*

We have proved Lemma 4 for $0 < \omega - \omega_0 \ll 1$ (Corollary 1) and for $\omega \gg 1$ (23). From now on we assume that Lemma 4 holds for all ω. Thus, the results in the remainder of this section, as well as in Section 7, are completely proved only for $0 < \omega - \omega_0 \ll 1$ and for $\omega \gg 1$.

We now introduce the following variational problem, which will later turn out to be the variational characterization of Q_ω.

Lemma 5. *For any $M \in (0, N_c)$, there is a unique $U_M(x, y) \in H_0^1(B_1)$, such that*

$$I(M) := \inf_{\int |U|^2 = M} H(U) = H(U_M).$$

The infimum in Lemma 5 is taken over functions in H_0^1 which are not necessarily radial. However, using Steiner symmetrization [25], it follows that $U_M(r)$ is positive, radially symmetric and monotonically decreasing. Application of the Euler–Lagrange equation with a side-condition shows that there is an $\omega_M = \omega(M)$ such that U_M is the positive solution of

$$\Delta U_M + U_M^3 = \omega_M U_M. \tag{27}$$

Therefore, by Lemma 3 we have that

$$U_M = Q_{\omega_M} \quad \text{and} \quad I(M) = H(Q_{\omega_M}). \tag{28}$$

We now claim the following properties of the function $\omega(M)$.

Lemma 6. *The function $\omega(M)$ is continuous and strictly monotonic. In addition,*

$$\lim_{M \to N_c -} \omega(M) = \infty, \tag{29}$$

and

$$\lim_{M \to 0+} \omega(M) = \omega_0, \tag{30}$$

where $\omega_0 = -\lambda_0$ and $\lambda_0 > 0$ is the smallest eigenvalue of

$$-\Delta U = \lambda U, \qquad \frac{\mathrm{d}U}{\mathrm{d}r}(0) = 0, \quad U(1) = 0.$$

From Lemma 6, it follows that as M goes from zero to N_c, $\omega(M)$ is monotonically increasing from ω_0 to infinity. In particular, for all $\omega \in (-\omega_0, +\infty)$, there is a unique $M = M(\omega) \in (0, N_c)$, such that $\omega = \omega(M)$ and $M(\omega)$ is an increasing function of ω (see Fig. 1).

Summarizing the results of Lemmas 3–6 lead to Theorem 1.

5. Global existence

Local existence in time in H^1 for the Cauchy problem (2) on bounded domains has been proved by Bourgain in the case of periodic boundary conditions using function expansion in Fourier series [3]. An adaptation of this technique can be used to prove local existence in the case of Dirichlet boundary conditions. We also recall that Ginibre and Velo proved local existence in H^1 for the NLS in free-space (1), and that if blowup occurs $\|\psi\|_{H^1}$ goes to infinity at the blowup time [11]. These results were later extended to the NLS (2) on bounded domains by Brezis and Gallouet [4]. Since by (5), $\|\psi\|_2^2$ is bounded, global existence is equivalent to $\|\nabla\psi\|_2^2$ being bounded. In order to bound $\|\nabla\psi\|_2^2$, we rewrite Eq. (6) as

$$\|\nabla\psi\|_2^2 = H(0) + \tfrac{1}{2}\|\psi\|_4^4.$$

We also recall the Gagliardo–Nirenberg inequality

$$\|u\|^4_{L^4(\Omega)} \le C_{1,2}(\Omega)\|u\|^2_{L^2(\Omega)}\|\nabla u\|^2_{L^2(\Omega)}. \tag{31}$$

Combining the last two equations gives

$$\|\nabla\psi\|_2^2 \le H(0) + \frac{C_{1,2}(\Omega)}{2}\|\psi\|_2^2\|\nabla\psi\|_2^2.$$

Therefore, if $C_{1,2}(\Omega)\|\psi_0\|_2^2 < 2$ or, equivalently, if

$$\|\psi_0\|_2^2 < N_c(\Omega), \quad \text{where } N_c(\Omega) := \frac{2}{C_{1,2}(\Omega)},$$

then $\|\nabla\psi\|_2^2$ remains bounded and the solution exists globally.

The optimal constant $C_{1,2}(\Omega)$ in the Sobolev inequality (31) satisfies

$$\frac{1}{C_{1,2}(\Omega)} = \inf_{u\not\equiv 0 \in H_0^1(\Omega)} J_\Omega(u), \quad \text{where } J_\Omega(u) = \frac{\int_\Omega|\nabla u|^2\int_\Omega|u|^2}{\int_\Omega|u|^4}.$$

We recall that the calculation of $C_{1,2}(\mathbb{R}^2)$ was done in [31] by minimization of the functional

$$J_{\mathbb{R}^2}(u) = \frac{\int|\nabla u|^2\int|u|^2}{\int|u|^4},$$

over all functions $u \in H^1(\mathbb{R}^2)$ (see for existence [31] for uniqueness [15]). The minimum of $J_{\mathbb{R}^2}$ is achieved by the ground-state Townes soliton $R(r)$, and is unique up to scaling, phase and translation parameters. In addition, $J_{\mathbb{R}^2}(R) = \frac{1}{2}\|R\|_2^2$. Therefore, $C_{1,2}(\mathbb{R}^2) = 2/\|R\|_2^2$ and

$$N_c := N_c(\mathbb{R}^2) = \|R\|_2^2 \approx 2\pi\cdot 1.86.$$

The optimal constant in Sobolev inequalities, such as (31), depends, in general, on the domain Ω. In the critical case, however, Fibich [5] showed that the following results hold.

Lemma 7. *In critical case* $\sigma D = 2$,

1. $C_{1,2}(\Omega)$ *is independent of* Ω.
2. $C_{1,2}(\Omega) = C_{1,2}(\mathbb{R}^2)$.
3. $\inf_{u\in H_0^1(\Omega)} J_\Omega(u) = \inf_{u\in H^1(\mathbb{R}^2)} J_{\mathbb{R}^2}(u)$.

The result of Lemma 7 holds only in the critical case $\sigma D = 2$. Indeed, the proof utilizes the fact that in the critical case the rescaling (A.11) does not change the relative sizes of diffraction and nonlinearity.

An immediate consequence of Lemma 7, which shows an important difference between bounded domains and $\mathbb{R}^2$, is the following corollary.

Corollary 3. *The infimum of $J_\Omega(u)$ over all functions $u \in H_0^1(\Omega)$ is not achieved.*

From Lemma 7 it follows that the necessary condition for singularity formation in $\mathbb{R}^2$ is also true for bounded domains.

Theorem 2 ([5]). *The condition $\|\psi_0\|^2_{L^2(\Omega)} \geq N_c$ is necessary for singularity formation in* (2)–(3), *where $N_c = N_c(\mathbb{R}^2)$.*

The question whether the condition $\|\psi_0\|_2^2 \geq N_c$ is also sufficient for singularity formation is addressed in Section 6.2.

6. Singularity formation

As in free-space, the main analytical tool for proving blowup of NLS solutions on bounded domains is the variance identity.

6.1. Variance identity and Hamiltonian condition for blowup

Let us define the variance of $\psi \in H_0^1(B_1)$ by

$$V(t) = \int r^2 |\psi|^2 r \, \mathrm{d}r.$$

Then, differentiating V with respect to time, using (4) and integrating by parts gives

$$V_t = -2\mathrm{i} \int r^2 \psi^* \psi_r + \text{c.c.},$$

where c.c. is complex conjugate. Differentiating a second time and using (6) gives the *variance identity on bounded domains*:

$$V_{tt} = 8H(0) - 4|\psi_r(t, r = 1)|^2. \tag{32}$$

Therefore, we have the following result [14].

Lemma 8. *If $H(0) < 0$ then ψ becomes singular at a finite time $T_c < \infty$.*

Remarks:

- The variance identity on bounded domains differs from the free-space one by the boundary term. This term is negative, because reflecting boundaries enhance blowup.
- The difference between bounded and unbounded domains is also evident when $H(0) > 0$ and ψ exists globally. In this case, from the variance identity it follows that in free-space case $\lim_{t\to\infty} V(t) = \infty$. On bounded domains, however, V always remains bounded. For example, when $\Omega = B_1$, we have that $V(t) \leq \|\psi_0\|_2^2$.

- A general rule of thumb in nonlinear wave equations is that waveguides are stable if and only if there is no singularity formation in the equation. We see, thus, that NLS on bounded domains is an exception to this 'rule'.
- If we substitute the waveguide solution $\psi = \exp(\mathrm{i}\omega t)Q_\omega(r)$ into the variance identity, we recover Eq. (18).

6.2. Critical power condition for blowup

We have seen that, as in free-space, a necessary condition for singularity formation is $\|\psi_0\|_2^2 \geq N_c$, while the condition $H(0) < 0$ is a sufficient one. The following lemma shows that, as in free-space, the condition $\|\psi_0\|_2^2 \geq N_c$ is sharp in the following (weak) sense.

Lemma 9. *For all $\epsilon > 0$, there exists a blowup solution with initial condition ψ_0, such that $\|\psi_0\|_2^2 \leq (1+\epsilon)N_c$.*

It is well-known that in free-space the condition of a negative Hamiltonian is not really necessary for singularity formation, as solutions with positive Hamiltonian can blowup [6,7]. On the other hand, any initial condition which is not $R(r)$ and does blowup has power strictly above N_c [17,18]. Therefore, the actual critical power in free-space of the one-parameter family of initial conditions $\psi_0 = cf(x,y)$ is above N_c, but below the upper bound $2J_{\mathbb{R}^2}(f)$ which correspond to the condition $H(0) = 0$ [6,7]. As first pointed out by Fibich and Gaeta in [6], numerical simulations suggest that in a bounded domain the condition $\|\psi_0\|_2^2 < N_c$ is in fact sharp, at least generically. In other words, all initial profiles with power above N_c ultimately lead to blowup. In this study, we have carried additional numerical simulations which support this observation.

In order to motivate this observation, we note that in free-space

1. Unless the initial condition is $R(r)$, a finite amount of power always radiates away from the singularity due to diffraction.
2. The amount of power going into the singularity is $\geq N_c$ [7], and numerical experiments show that the amount of power going into the singularity is, in fact, equal to N_c.

The second fact also holds for bounded domains. However, with regard to the first fact, in the case of bounded domains all power which radiates away from the singularity is reflected back by the boundary. As a result, as the solution tries to reorganize itself in the form of the Q_ω function, no power is lost, or radiated, to the background. As more and more power is trapped in this profile, ω in increasing, which implies that the solution becomes narrower and narrower.

Of course, the condition $\|\psi_0\|_2^2 < N_c$ cannot be sharp for all initial profiles, since the initial condition $\psi_0 = Q_\omega^{(2)}$ for ω sufficiently large, has power above N_c yet the corresponding waveguide solution does exist globally in time. Our numerical simulations suggest, however, that these solutions are unstable. One can conjecture, therefore, that on bounded domains all stable solutions which are global in time must have power strictly below N_c.

6.3. Concentration result

The local nature of blowup in the critical NLS is manifested in the concentration theorem, which shows that the amount of power going into the singularity is given by a constant which is $\geq N_c$. This result was proved in free-space by Merle and Tsutsumi in [20]. The same proof applies also in the bounded domain case, since it is based on (i) power conservation, (ii) Hamiltonian conservation, and (iii) the Nirenberg–Gagliardo inequality. In fact, one can use the free-space version of the Nirenberg–Gagliardo inequality, applied to $\tilde{u}(t)$, as defined in (A.10). We thus have the following result.

Theorem 3. *Let $\psi(t,r)$ be a solution of* (4) *which blows-up as $t \to T_c$. Then, for all $\rho > 0$,*

$$\lim_{t\to T_c} \|\psi(t)\|_{L^2(\{r\le\rho\})} \ge N_c.$$

Note that from Theorem 3 we can recover Theorem 2. Clearly, the result of Theorem 3 is of most interest when ρ is small. In some sense, this result says that at the blowup time a power of at least N_c goes into the singularity point (which is the origin in the case of a radially symmetric solutions). We expect the amount of power going into the singularity to be exactly N_c.

In light of the power concentration property, one can expect that the boundary has no effect near the singularity and thus that blowup on bounded domains is a local phenomena, much as it is in free-space. If true, this implies that the asymptotic profile near the singularity and the rate of blowup are the same as in free-space. The last statement is, however, only a conjecture at present.

7. Stability of ground-state waveguide solutions

We recall that the ground-state waveguides of the NLS on $\mathbb{R}^D$ are orbitally stable in the subcritical case, but are unstable in the critical and supercritical case. The instability in the critical case is related to the fact that the Hamiltonian of ground-state waveguides is equal to zero (13). As a result, small perturbations can make the Hamiltonian negative and thus result in blowup [31].

The situation on bounded domains is quite different, since these waveguides have strictly positive Hamiltonian (18). In other words, unlike the free-space case, where nonlinearity and diffraction are completely balanced only when the power is equal to N_c, the reflecting boundary 'enables' the nonlinearity to support weaker waveguides whose power is below N_c. Indeed, we now show that the ground-state waveguides in the critical NLS on bounded domains are orbitally stable.

Lemma 10. *The ground-state waveguides*

$$\psi = \exp(\mathrm{i}\omega t) Q_\omega^{(0)}(r) \tag{33}$$

are orbitally stable in H_0^1. That is, for all $\epsilon > 0$ there is δ such that if $\inf_{\theta\in\mathbb{R}} \|\psi_0 - \mathrm{e}^{\mathrm{i}\theta} Q_\omega^{(0)}\|_{H_0^1} \le \delta$, then

$$\inf_{\theta\in\mathbb{R}} \|\psi(t,\cdot) - \mathrm{e}^{\mathrm{i}\theta} Q_\omega^{(0)}\|_{H_0^1} \le \epsilon \quad \textit{for all} \ \ t \ge 0.$$

We recall that a generic condition for stability of ground-state waveguides is that [27,32]

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \|Q_\omega^{(0)}\|_2^2 > 0. \tag{34}$$

Thus, the result of Lemma 10 is consistent with Lemma 4.

Since the ground-state waveguides (33) are orbitally stable, a natural question is whether they are local or global attractors. The following lemma shows that they are not.

Lemma 11. *Let ψ_0 be an initial condition. Then*

$$\lim_{t\to\infty} \inf_{\theta\in\mathbb{R}} \|\psi(t,\cdot) - \mathrm{e}^{\mathrm{i}\theta} Q_{\bar\omega}^{(0)}\|_{H_0^1} = 0 \quad \textit{for some} \ \ \bar\omega, \tag{35}$$

if and only if $\psi_0 \equiv \mathrm{e}^{\mathrm{i}\theta_0} Q_{\bar\omega}^{(0)}$ for some θ_0.

The results of Lemmas 10 and 11 are intuitive, because fixed points in conservative systems are centers, rather than attractors.

7.1. Stability of higher-order waveguides

We have carried numerical simulations which showed that, indeed, the ground-state waveguides are orbitally stable. When we tested the stability of higher-state waveguides, our simulations clearly indicate that those whose power is above N_c are unstable and blowup in finite time. However, higher-state waveguides whose power is sufficiently below N_c appear to be numerically stable for quite a long time. In fact, for given numerical parameters (grid size, time-step size, etc.), higher-state waveguides seem to have a similar deviation from the exact solution as the ground-state waveguides.

8. Numerical results

8.1. Numerical methods

We solve the NLS on a unit disc, Eq. (4), by combining a Crank–Nicholson implicit method for the Laplacian with Adams–Bashford extrapolation for the nonlinearity. Thus, the predictor stage is

$$\left(1 - \mathrm{i}\frac{\mathrm{d}t}{2}\Delta\right)\psi^{\text{pred.}} = \left(1 + \mathrm{i}\frac{\mathrm{d}t}{2}\Delta\right)\psi(t,\cdot) + \mathrm{i}\frac{3}{2}|\psi|^2\psi(t,\cdot) - \mathrm{i}\frac{1}{2}|\psi|^2\psi(t-\mathrm{d}t,\cdot),$$

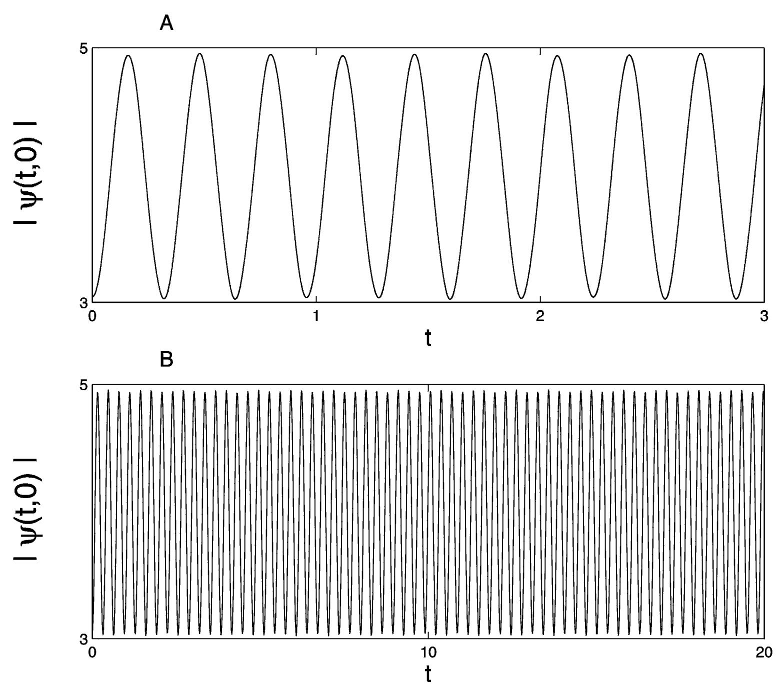

Fig. 4. On-axis amplitude of the solution of Eq. (4) with the initial condition (36).

and the corrector stage is

$$\left(1-\mathrm{i}\frac{\mathrm{d}t}{2}\Delta\right)\psi(t+\mathrm{d}t,\cdot)=\left(1+\mathrm{i}\frac{\mathrm{d}t}{2}\Delta\right)\psi(t,\cdot)+\mathrm{i}\frac{1}{2}|\psi|^2\psi(t,\cdot)+\mathrm{i}\frac{1}{2}|\psi^{\mathrm{pred.}}|^2\psi^{\mathrm{pred.}}.$$

We use forth-order schemes for the spatial derivatives:

$$\psi_r(\cdot,r)\approx\frac{\psi(\cdot,r+2\,\mathrm{d}r)-8\psi(\cdot,r+\mathrm{d}r)+8\psi(\cdot,r-\mathrm{d}r)-8\psi(\cdot,r-2\,\mathrm{d}r)}{12\,\mathrm{d}r},$$
$$\psi_{rr}(\cdot,r)\approx\frac{-\psi(\cdot,r+2\,\mathrm{d}r)+16\psi(\cdot,r+\mathrm{d}r)-30\psi(\cdot,r)+16\psi(\cdot,r-\mathrm{d}r)-\psi(\cdot,r-2\,\mathrm{d}r)}{12\,\mathrm{d}r^2}.$$

Boundary conditions are implemented as follows: Near $r=0$, we use radial symmetry to add the two fictitious points $\psi(-\mathrm{d}r)=\psi(\mathrm{d}r)$ and $\psi(-2\,\mathrm{d}r)=\psi(2\,\mathrm{d}r)$. At $r=1$ we use the Dirichlet condition $\psi(r=1)=0$. At

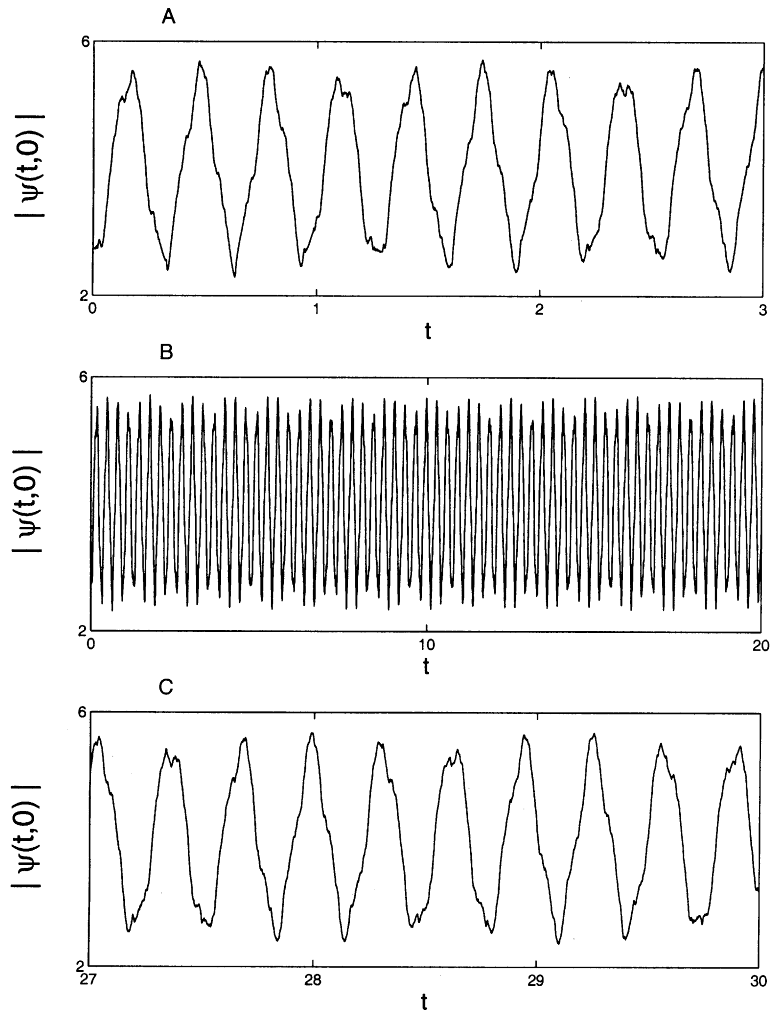

Fig. 5. Same as Fig. 4 with the initial condition (37).

$r = (1 - \mathrm{d}r)$ we use the one-sided forth-order schemes

$$\psi_r(\cdot, 1-\mathrm{d}r) \approx \frac{-\psi(\cdot, 1-4\,\mathrm{d}r) + 6\psi(\cdot, 1-3\,\mathrm{d}r) - 18\psi(\cdot, 1-2\,\mathrm{d}r) + 10\psi(\cdot, 1-\mathrm{d}r) + 3\psi(\cdot, 1)}{12\,\mathrm{d}r},$$
$$\psi_{rr}(\cdot, 1-\mathrm{d}r) \approx \frac{-\psi(\cdot, 1-4\,\mathrm{d}r) + 4\psi(\cdot, 1-3\,\mathrm{d}r) + 6\psi(\cdot, 1-2\,\mathrm{d}r) - 20\psi(\cdot, 1-\mathrm{d}r) + 11\psi(\cdot, 1)}{12\,\mathrm{d}r^2}.$$

The Laplacian is evaluated at the origin using

$$\Delta\psi(\cdot, 0) = 2\psi_{rr}(\cdot, 0).$$

Since the left-hand side matrix $I - (\frac{1}{2}\mathrm{d}t)\Delta$ of the predictor and the corrector stages is the same and does not change during the iterations, it is LU-decomposed only once. As a consistency check, we monitor the conservation of $\|\psi\|_2^2$ and $H(\psi)$.

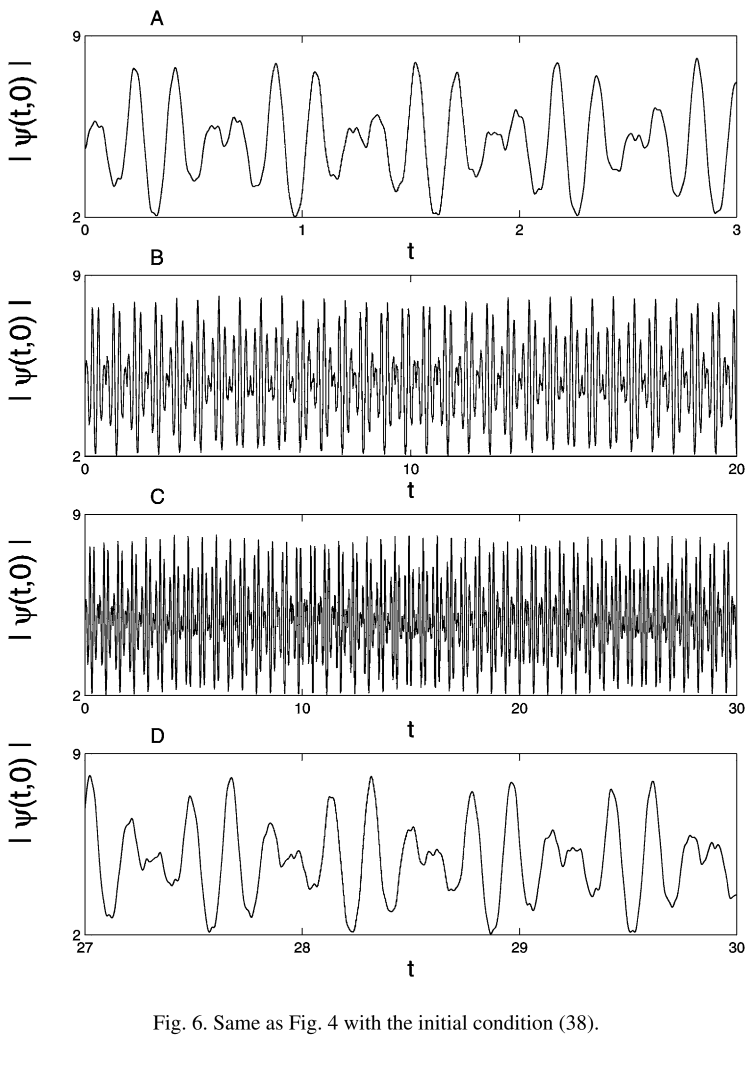

Fig. 6. Same as Fig. 4 with the initial condition (38).

8.2. Numerical results

The theory presented so far tells us very little on the dynamics of solutions whose initial power is below critical, except that they exist for all time. In the following, we present results of numerical simulations which reveal some features of the dynamics. In our simulations we use the initial conditions

$$\psi_0^{(1)}(r) = c_1 J_0(k_0 r), \quad k_0 \approx 2.405, \tag{36}$$

$$\psi_0^{(2)}(r) = c_2(1 - r^2), \tag{37}$$

$$\psi_0^{(3)}(r) = c_3 J_0(k_1 r), \quad k_1 \approx 5.520. \tag{38}$$

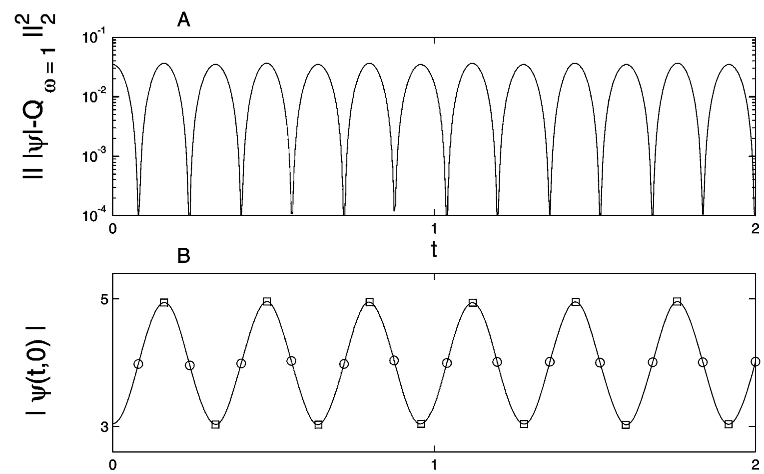

Fig. 7. Solution of Eq. (4) with the initial condition (36). (A) Distance from waveguide. (B) On-axis amplitude. Squares and circles mark the locations where distance from waveguide is maximal and minimal, respectively.

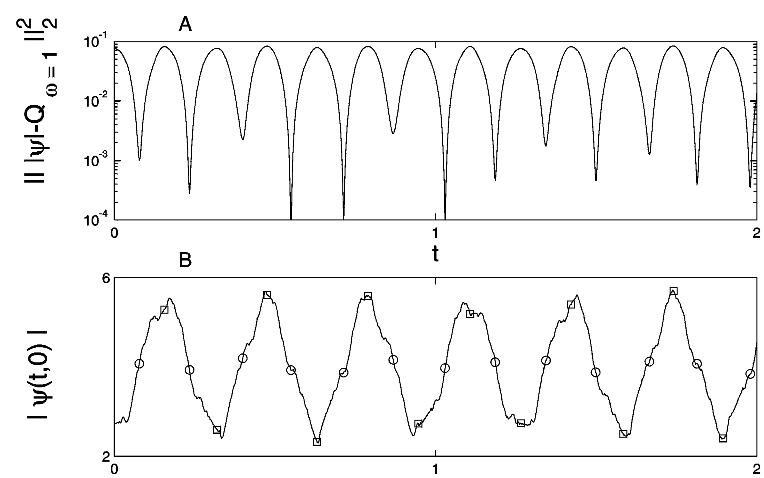

Fig. 8. Same as Fig. 7 with the initial condition (37).

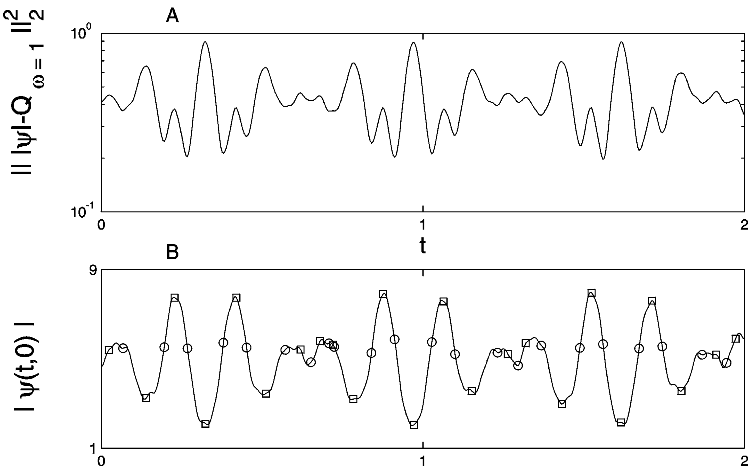

Fig. 9. Same as Fig. 7 with the initial condition (38).

In Figs. 4–9, the value of the coefficients are $c_1 = 3.0444$, $c_2 = 2.7375$ and $c_3 = 4.645$. These values are chosen so that all initial conditions have the same power as $Q^{(0)}_{\bar\omega=1}$, i.e.,

$$\|\psi_0^{(1)}\|_2^2 = \|\psi_0^{(2)}\|_2^2 = \|\psi_0^{(3)}\|_2^2 = \|Q^{(0)}_{\bar\omega=1}\|_2^2 \approx 0.67N_\mathrm{c}.$$

Both $\psi_0^{(1)}(r)$ and $\psi_0^{(2)}(r)$ are monotonically decreasing, while $\psi_0^{(3)}(r)$ is not. The initial conditions are ordered according to their H^1 distance from $Q^{(0)}_{\bar\omega=1}$:

$$\|\psi_0^{(1)} - Q^{(0)}_{\bar\omega=1}\|_{H_0^1} \approx 1.10, \qquad \|\psi_0^{(2)} - Q^{(0)}_{\bar\omega=1}\|_{H_0^1} \approx 2.38, \qquad \|\psi_0^{(3)} - Q^{(0)}_{\bar\omega=1}\|_{H_0^1} \approx 34.1.$$

The generic behavior of solutions with power below critical is *focusing–defocusing oscillations* (Figs. 4–6). Strictly speaking, these focusing–defocusing oscillations are not periodic. However, in the case of $\psi_0^{(1)}$, which is the closest to $Q^{(0)}_{\bar\omega=1}$, the oscillations are smooth and almost periodic. The leading-order dynamics is similar in the case of $\psi_0^{(2)}$, but the oscillations have little bumps, or wiggles. In the case of $\psi_0^{(3)}$, the oscillation pattern is much more complex.

The envelope of the oscillations undergoes changes which occur on time-scales that are longer than that of the primary oscillations. Note, in particular, Fig. 6B and C, where several long time-scales can be observed. In addition, the small wiggles in Figs. 5 and 6 occur on much shorter time-scales. We checked that the wiggles are a feature and not a numerical artifact, by varying the values of $\mathrm{d}r$ and $\mathrm{d}t$. In addition, we note that wiggles can be observed in Fig. 2 in [6], where a different numerical scheme was used. The wiggles do not disappear with time (Figs. 4C and 6D), since the equation is conservative and time-reversible, rather than dissipative. For the same reason, the solutions in all cases do not approach a limit cycle as $t \to \infty$.

In Figs. 7–9, we plot the L^2-distance between $|\psi|$ and the ground-state waveguide with the same power, i.e., $\||\psi| - Q^{(0)}_{\bar\omega=1}\|_2^2$. The distance is maximal around the solution peaks and valleys, and it is minimal half-way in-between. As a result, the frequency of the oscillations of the distance is twice that of the solution. To better see this, we add squares and circles in Figs. 7–9 at the locations where the distance is maximal and minimal, respectively.

We can explain the dynamics of the distance by employing the standard nonlinear optics ansatz

$$\psi(t,\cdot) \sim S(r,t)\exp\left(\mathrm{i}\tau(t) + \mathrm{i}\frac{\alpha(t)r^2}{2}\right),$$

where $S(t, r)$, $\tau(t)$ and $\alpha(t)$ are real. From Hamiltonian conservation we have that

$$H(S) + \alpha^2 V(S) \equiv (\psi_0). \tag{39}$$

Since $\|S\|_2^2 = \|Q_{\bar{\omega}}^{(0)}\|_2^2$, the potential energy $H(S)$ is minimal when $S = Q_{\bar{\omega}}^{(0)}$ (Theorem 1). Therefore, the distance between S and $Q_{\omega=1}^{(0)}$ is minimal at the bottom of the potential well, which is roughly half-way between the focusing peaks and valleys. On the other hand, at the focusing peaks and valleys $\alpha = 0$. Hence, at these points the kinetic energy vanishes, the potential energy is maximal and the distance between S and $Q_{\omega=1}^{(0)}$ is the largest.

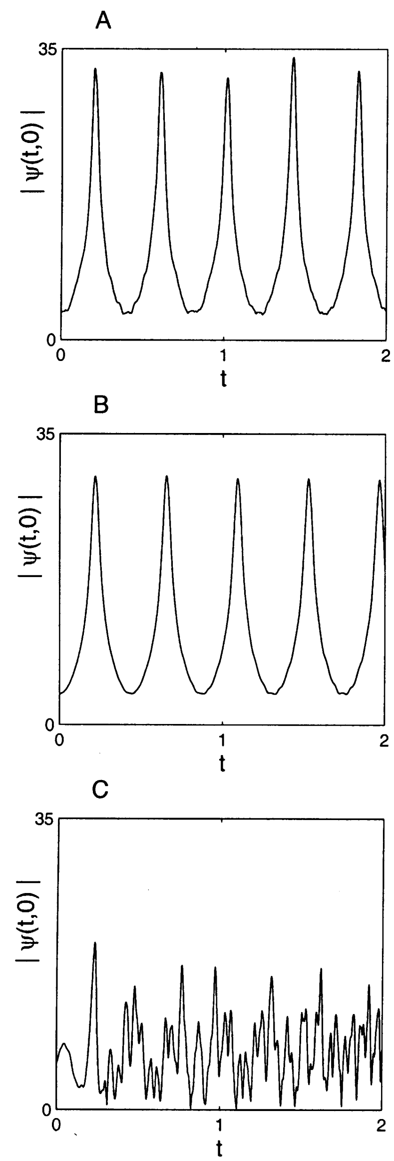

Fig. 10. On-axis amplitude for the initial conditions. (A) $\psi_0^{(1)}$; (B) $\psi_0^{(2)}$ and (C) $\psi_0^{(3)}$, when $\|\psi_0^{(i)}\|_0^2 = 0.99N_c$.

In Fig. 10, we plot the dynamics of the solutions when the values of the coefficients c_is in (36)–(38) are raised so that the power of the initial conditions is 1% *below* critical, i.e.

$$\|\psi_0^{(1)}\|_2^2 = \|\psi_0^{(2)}\|_2^2 = \|\psi_0^{(3)}\|_2^2 = 0.99N_c.$$

In the case of the initial conditions (36) and (37), the dynamics remains roughly the same. In fact, at this power, the two solutions are more similar than at the lower power. In contrast, the dynamics in the case of the initial condition (38) is quite different from the lower power case, looking more chaotic than periodic. Note that the H^1 distance

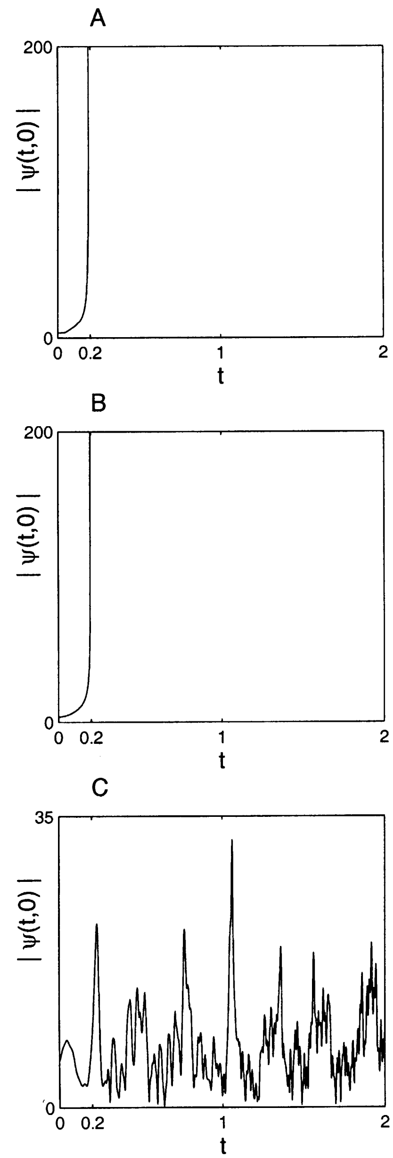

Fig. 11. Same as Fig. 10 with $\|\psi_0^{(i)}\|_0^2 = 1.01N_c$.

from the ground-state with the same power, $Q^{(0)}_{\tilde{\omega}=17}$, are comparable for all the three initial conditions:

$$\|\psi_0^{(1)} - Q^{(0)}_{\tilde{\omega}=17}\|_{H_0^1} \approx 25.2, \qquad \|\psi_0^{(2)} - Q^{(0)}_{\tilde{\omega}=17}\|_{H_0^1} \approx 29.3, \qquad \|\psi_0^{(3)} - Q^{(0)}_{\tilde{\omega}=17}\|_{H_0^1} \approx 33.1.$$

In Fig. 11, we plot the dynamics of the solutions when the values of the c_is in (36)–(38) are set so that the power of the initial conditions is 1% *above* critical, i.e.

$$\|\psi_0^{(1)}\|_2^2 = \|\psi_0^{(2)}\|_2^2 = \|\psi_0^{(3)}\|_2^2 = 1.01N_{\mathrm{c}}.$$

In the case of the initial conditions (36) and (37), the solutions blowup in a finite time, in accordance with the observation in [6] that the condition of critical power is generically sharp on bounded domains. For comparison, we note that on unbounded domains, Gaussians with the same power do not blowup, as the critical power for Gaussians is 1.8% above N_{c} [6]. The dynamics in the case of the initial condition (38) looks chaotic, but does not lead to blowup until $t = 2$ (Fig. 11C). In fact, in our simulations we do not see any evidence for blowup until $t = 50$. We caution, though, that as t increases, the numerical solution becomes less and less reliable, which may be the result of the ill-posedness of the equation with this initial condition. Therefore, we cannot rule out the possibility that this initial condition does lead to a finite-time blowup.

Acknowledgements

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Appendix A. Proofs

A.1. Eq. (17)

If we multiply (16) by $rQ^{(n)}_\omega$ and integrate, we get that

$$-\left\|\frac{\mathrm{d}Q^{(n)}_\omega}{\mathrm{d}r}\right\|_2^2 - \omega\|Q^{(n)}_\omega\|_2^2 + \|Q^{(n)}_\omega\|_4^4 = 0.$$

Similarly, if we multiply (16) by $r^2(\mathrm{d}Q^{(n)}_\omega/\mathrm{d}r)$ and integrate, we get that

$$\omega\|Q^{(n)}_\omega\|_2^2 = \frac{1}{2}\|Q^{(n)}_\omega\|_4^4 - \frac{1}{2}\left[\frac{\mathrm{d}Q^{(n)}_\omega}{\mathrm{d}r}(1)\right]^2.$$

Combining these two relations gives the desired identities.

A.2. Lemma 1

From (16), we have that

$$\frac{\mathrm{d}}{\mathrm{d}\omega}H(Q^{(n)}_\omega) = -2\int\left(\left[\Delta Q^{(n)}_\omega + \left(Q^{(n)}_\omega\right)^3\right]\frac{\mathrm{d}Q^{(n)}_\omega}{\mathrm{d}\omega}\right) r\,\mathrm{d}r = -\omega\frac{\mathrm{d}}{\mathrm{d}\omega}\|Q^{(n)}_\omega\|_2^2.$$

It should be noted that in the above calculation we implicitly assumed that $Q_\omega^{(n)}$ is differentiable with respect to ω. By standard theory, one can deduce differentiability if one proves that $Q_\omega^{(n)}$ is unique. A proof of uniqueness result is, however, only available for ground-state solutions (Lemma 3).

A.3. *Lemma 2*

We expand $Q_\omega^{(n)}$ in an asymptotic series

$$Q_\omega^{(n)} \sim Q_{\omega,0}^{(n)} + \delta Q_{\omega,1}^{(n)} + \cdots, \quad 0 < \delta \ll 1,$$

and substitute into (16). The leading-order equation is

$$\Delta Q_{\omega,0}^{(n)} - \omega_n Q_{\omega,0}^{(n)} = 0,$$

and its solution is $Q_{\omega,0}^{(n)} = c(\delta) J_0(\sqrt{-\omega_n} r)$. The equation for next-order terms is

$$\delta\left[\Delta Q_{\omega,1}^{(n)} - \omega_n Q_{\omega,1}^{(n)}\right] = (\omega - \omega_n) Q_{\omega,0}^{(n)} - (Q_{\omega,0}^{(n)})^3.$$

Since the eigenvalues of the radial Laplacian in the unit disc among radial functions are simple, the solvability condition for this equation is that $Q_{\omega,0}^{(n)}$ is orthogonal to the right-hand side. Since $Q_{\omega,0}^{(n)}$ is not orthogonal to either of these terms separately, both should be kept, implying that $c^2(\delta) = (\omega - \omega_n) B_n^2$.

A.4. *Corollary 2*

From Theorem 1 we have that $|U| = Q_\omega$. Therefore, $U = \mathrm{e}^{\mathrm{i}\theta(r)} Q_\omega$ and $H(U) = H(Q) + \|Q\theta'(r)\|_2^2$. Since U is a minimizer, we have that $H(U) = H(Q)$, implying that $\theta(r) \equiv$ constant.

A.5. *Lemma 5*

Existence. For all $U \in H_0^1$ with $\|U\|_2^2 = M < N_\mathrm{c}$, by the Gagliardo–Nirenberg inequality (31),

$$H(U) = \int |\nabla U|^2 - \frac{1}{2}\int |U|^4 \geq \int |\nabla U|^2 - \frac{M}{N_\mathrm{c}}\int |\nabla U|^2 = \frac{N_\mathrm{c} - M}{N_\mathrm{c}}\int |\nabla U|^2 \geq 0. \tag{A.1}$$

Since $H(U) \geq 0$, there exists a sequence U_n such that

$$\|U_n\|_2^2 = M \quad \text{and} \quad H(U_n) \to I(M).$$

By (A.1), we have that $\int |\nabla U_n|^2$ is bounded. Since, in addition, $\|U_n\|_2^2 = M$, there is a subsequence of U_n which converges weakly in H_0^1 to U_∞. Since on bounded domain H_0^1 is compactly embedded in L^2 and L^4, the subsequence also converges strongly in L^2 and in L^4 to U_∞. Therefore,

$$\int |U_\infty|^2 = M \quad \text{and} \quad \int |\nabla U_\infty|^2 \leq \liminf_n \int |\nabla U_n|^2,$$

from which it follows that

$$H(U_\infty) \leq \liminf_n H(U_n) = I(M).$$

Therefore, $U_\infty = U_M$ and $H(U_\infty) = I(M)$.

Uniqueness. We claim that for a given M, U_M is unique. By contradiction, let us consider two different minimizers $U_M \neq \bar{U}_M$, and the corresponding ω_M and $\bar{\omega}_M$. The uniqueness result of Kwong (Lemma 3) implies that $\omega_M \neq \bar{\omega}_M$. Therefore, from the strict monotonicity of properties $\|U_\omega\|_2$ (Lemma 4) we have that $\|U_M\|_2 \neq \|\bar{U}_M\|_2$, which is a contradiction.

A.6. Lemma 6

Let $M_0 < N_c$, we first show that $\omega(M)$ is continuous at M_0. Let $\{M_n\}$ be a sequence such that $\lim_{n\to\infty} M_n = M_0$. By definition, the sequence $Q_{M_n} := Q_{\omega(M_n)}$ satisfies $\|Q_{M_n}\|_2^2 = M_n \to M_0$. We claim that

$$H(Q_{M_n}) \to H(Q_{M_0}). \tag{A.2}$$

To see that, let ϵ_n be defined so that

$$\|(1+\epsilon_n)Q_{M_0}\|_2^2 = M_n. \tag{A.3}$$

Therefore, $\epsilon_n \to 0$ and $\lim_{n\to\infty} H((1+\epsilon_n)Q_{M_0}) = H(Q_{M_0})$. In addition, by Lemma 5, (28) and (A.3), we have $H(Q_{M_n}) \le H((1+\epsilon_n)Q_{M_0})$. Combining the last two relations yields

$$\limsup_n H(Q_{M_n}) \le H(Q_{M_0}). \tag{A.4}$$

Similarly, let $\tilde{\epsilon}_n$ be defined so that

$$\|(1+\tilde{\epsilon}_n)Q_{M_n}\|_2^2 = M_0. \tag{A.5}$$

Therefore, $\tilde{\epsilon}_n \to 0$ and $\lim_{n\to\infty}[H((1+\tilde{\epsilon}_n)Q_{M_n}) - H(Q_{M_n})] = 0$. In addition, by Lemma 5, (28) and (A.5), we have $H(Q_{M_0}) \le H((1+\tilde{\epsilon}_n)Q_{M_n})$. Combining the last two relations yields

$$H(Q_{M_0}) \le \liminf_n H(Q_{M_n}). \tag{A.6}$$

Therefore, (A.2) follows from (A.4) and (A.6).

From (A.2), it follows that $H(Q_{M_n})$ is bounded. Since $M_n < N_c$, by the Gagliardo–Nirenberg inequality, Q_{M_n} is bounded in H_0^1. Therefore, there is a subsequence of Q_{M_n} which converges weakly in H_0^1, hence strongly in L^2 and L^4 to U_∞. In addition, $\|U_\infty\|_2^2 = M_0$ and $H(U_\infty) \le H(Q_{M_0})$. Therefore, by Lemma 5 and (28), $U_\infty = Q_{M_0}$. Since Q_{M_n} converges strongly in L^4 to Q_{M_0}, (A.2) implies that $\|\nabla Q_{M_n}\|_2 \to \|\nabla Q_{M_0}\|_2$. Therefore, $\|Q_{M_n}\|_{H_0^1} \to \|Q_{M_0}\|_{H_0^1}$ and Q_{M_n} converges strongly in H_0^1 to Q_{M_0}.

Now, on one hand, since Q_{M_n} converges strongly to Q_{M_0} in H_0^1 and in L^4,

$$\int Q_{M_0} Q_{M_n} \to \int Q_{M_0}^2,$$

and

$$\int (\Delta Q_{M_n} + |Q_{M_n}|^2 Q_{M_n}) Q_{M_0} \to \int (\Delta Q_{M_0} + |Q_{M_0}|^2 Q_{M_0}) Q_{M_0} = \omega_{M_0} \int Q_{M_0}^2.$$

On the other hand, since Q_{M_n} satisfies Eq. (27),

$$\int (\Delta Q_{M_n} + |Q_{M_n}|^2 Q_{M_n}) Q_{M_0} = \omega_{M_n} \int Q_{M_0} Q_{M_n}.$$

Therefore, $\omega_{M_n} \to \omega_{M_0}$.

From the uniqueness result of Kwong (Lemma 3) we have that the function $\omega(M)$ is injective. Since it is also continuous, it follows that $\omega(M)$ is strictly monotonic.

Since $\omega(M)$ is monotonic, Eq. (29) can be proved by showing that $M \to N_c$ as $\omega \to \infty$. To see that, let $U = \omega V(\omega r)$. Then, V is the solution of

$$\Delta V - V + V^3 = 0, \quad V_r(0) = 0, \quad V(\omega) = 0 \quad \text{and} \quad \forall r \in [0, \omega), \quad V(r) > 0.$$

Therefore, using the positivity, we have as $\omega \to \infty$, $V \to R(r)$ (see [2]), the Townes soliton whose power is given by N_c.

In order to prove Eq. (30), let us note that from Gagliardo–Nirenberg inequality (31),

$$(1 - C_{1,2}M)\frac{\int \nabla u^2}{\int u^2} \le \frac{H(u)}{\int u^2} \le \frac{\int \nabla u^2}{\int u^2}. \tag{A.7}$$

In addition, by Lemma 5 and (27), we have that

$$\inf_{\int u^2 = M} \frac{H(u)}{\int u^2} = \frac{H(U_M)}{M} = -\omega_M. \tag{A.8}$$

We recall that

$$\inf_{\int u^2 = M} \frac{\int \nabla u^2}{\int u^2} = \lambda_0. \tag{A.9}$$

Therefore, taking the infimum on all sides of (A.7), and using (A.8) and (A.9), gives

$$(1 - C_{1,2}M)\lambda_0 \le -\omega_M \le \lambda_0,$$

from which (30) follows.

A.7. Lemma 7

For completeness, we repeat here the proof given in [5]. Clearly, it is sufficient to prove (3). For all $u \in H_0^1(\Omega)$ let us define the extended function $\tilde{u}$ as follows:

$$\tilde{u} = \begin{cases} u, & (x, y) \in \Omega, \\ 0, & (x, y) \notin \Omega. \end{cases} \tag{A.10}$$

Since $\tilde{u} \in H^1(\mathbb{R}^2)$ and $J_\Omega(u) = J_{\mathbb{R}^2}(\tilde{u})$, it follows that

$$\inf_{u \in H_0^1(\Omega)} J_\Omega(u) \ge \inf_{u \in H^1(\mathbb{R}^2)} J_{\mathbb{R}^2}(u).$$

On the other hand, let us define

$$u_\epsilon(r) = \begin{cases} \frac{1}{\epsilon} R\left(\frac{r}{\epsilon}\right), & r \le M - 1, \\ g_\epsilon(r), & M - 1 \le r \le M, \\ 0, & |x| \ge M, \end{cases} \tag{A.11}$$

where M is a positive number such that $\{|x| \le M\} \subset \Omega$, R is the Townes soliton (15), and $g_\epsilon(r)$ is a smooth monotonically decreasing function such that $g_\epsilon(M) = 0$ and $g_\epsilon(M-1) = (1/\epsilon)R((M-1)/\epsilon)$. Since $u_\epsilon \in H_0^1(\Omega)$ and $R(r)$ decays exponentially (14), we have that

$$\inf_{u\in H_0^1(\Omega)} J_\Omega(u) \le \lim_{\epsilon\to 0} J_\Omega(u_\epsilon) = J_{\mathbb{R}^2}(\epsilon^{-1}R(r/\epsilon)) = J_{\mathbb{R}^2}(R) = \inf_{u\in H_0^1(\mathbb{R}^2)} J_{\mathbb{R}^2}(u).$$

A.8. Corollary 3

Assume that the minimum of J_Ω is achieved by a function u, and let $\tilde{u}$ be defined by (A.10). In light of Lemma 7, $\tilde{u}$ is a minimizer of $J_{\mathbb{R}^2}(u)$ and is thus equal to the R function, up to scaling and phase shift. In particular, it follows that $\tilde{u}$ does not vanish on the boundary of the domain, which is a contradiction.

A.9. Lemma 8

If $H(0) < 0$, then from (32) we have that there exists a $0 < T^* < \infty$ such that $V(T^*) = 0$. However, from the uncertainty principle we have that

$$\|\psi\|_2^2 \le V(\psi)\|\nabla\psi\|_2^2.$$

Therefore, we see that $T_c \le T^*$.

A.10. Lemma 9

Let us define $\psi_0 = (1+\epsilon)A(R(Ar) - R(A))$. Then, it is clear that $\psi_0 \in H_0^1(\Omega)$, and that $\|\psi_0\|_2^2 \le (1+\epsilon)N_c$. In addition, $\lim_{A\to\infty} H(\psi_0) = -\infty$. Therefore, $H(\psi_0) < 0$ for sufficiently large A, implying that the solution blows-up in finite time.

A.11. Lemma 10

By negation. If not, then if we take a sequence $\psi_n(t, r)$ such that $\|\psi_n(0,\cdot) - Q_\omega^{(0)}\|_{H_0^1} \to 0$, there exists a sequence $t_n \to \infty$ such that for all n

$$\inf_{\theta\in\mathbb{R}} \|\psi_n(t_n,\cdot) - \mathrm{e}^{\mathrm{i}\theta} Q_\omega^{(0)}\|_{H_0^1} \ge \epsilon_0 > 0.$$

Let $V_n(r) := \psi_n(t_n, r)$. Then

$$\|V_n\|_2 = \|\psi_n(0,r)\|_2 \to \|Q_\omega^{(0)}\|_2, \qquad H(V_n) = H(\psi_n(0,r)) \to H(Q_\omega^{(0)}).$$

Therefore, $\|V_n\|_{H_0^1} \le C$, and there is a subsequence V_n which converges weakly in H_0^1, hence strongly in L^2 and in L^4 to $\tilde{U}$. Therefore,

$$\|\tilde{U}\|_2 = \|Q_\omega^{(0)}\|_2, \qquad H(\tilde{U}) \le \liminf_n H(V_n) = H(Q_\omega^{(0)}).$$

In light of Theorem 1 and Corollary 2, the above relations show that $\tilde{U} = \mathrm{e}^{\mathrm{i}\theta_0} Q_\omega^{(0)}$ for some θ_0 and $H(\tilde{U}) = H(Q_\omega^{(0)})$. Therefore, $H(V_n) \to H(\tilde{U})$, $\|V_n\|_{H_0^1} \to \|\tilde{U}\|_{H_0^1}$ and V_n converges strongly in H_0^1 to $\tilde{U}$. We thus have that

$$\inf_{\theta\in\mathbb{R}} \|\psi_n(t_n,\cdot) - \mathrm{e}^{\mathrm{i}\theta} Q_\omega^{(0)}\|_{H_0^1} \le \|V_n - \tilde{U}\|_{H_0^1} \overset{n\to\infty}{\to} 0.$$

Contradiction.

A.12. Lemma 1

If (35) holds, then

$$\lim_{t\to\infty}\|\psi\|_2^2=\|Q_{\tilde{\omega}}^{(0)}\|_2^2 \quad\text{and}\quad \lim_{t\to\infty}H(\psi)=H(Q_{\tilde{\omega}}^{(0)}).$$

From power and Hamiltonian conservation we have that $\|\psi_0\|_2^2=\|Q_{\tilde{\omega}}^{(0)}\|_2^2$ and $H(\psi_0)=H(Q_{\tilde{\omega}}^{(0)})$. Hence, by Corollary 2, $\psi_0=\mathrm{e}^{\mathrm{i}\theta_0}Q_{\tilde{\omega}}^{(0)}$.

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