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Gadi Fibich - Recent Research Highlights (2004-2007)

As in the past, my main area of research was on propagation of intense laser beams in Kerr media, which is modeled by the critical nonlinear Schrödinger equation (NLS). Previously, I mainly studied solutions whose input power (L^2 norm) was close to the critical power for collapse. Recently, however, my interests shifted towards solutions whose power is many times the critical power. This highly nonlinear regime turned out to be very different from the moderately-above-critical regime. Indeed, much to our surprise, we discovered that some of the results that were assumed to hold for "all" collapsing solutions of the NLS (blowup profile, blowup rate, scaling law for the self-focusing distance), are in fact not valid for very high-power solutions. In particular, the fact that the radially-symmetric, self-similar R (Townes) profile ceases to be an attractor for very high-power solutions, implies that a different analytical approach should be developed for this high-power regime.

I have also worked in several other fields (Auction Theory, Optical Imaging, Biochemistry, and Marketing). In each of these fields I collaborated with researchers from the "other" discipline, in order to work on problems which are not only mathematically challenging, but are also relevant to the "applied communities".

In the last three years I was fortunate to have fruitful collaboration with experimental groups in Nonlinear Optic (Gaeta's lab at Cornell University, Zigler's lab at Hebrew University, Dubietis' lab in Vilnius), in Biochemistry (Gutman's lab at Tel Aviv University), in Neurobiochemistry (Asheri's lab at Tel Aviv University), and in Biomedical Engineering (Gannot's lab at Tel Aviv University). Since I have always been interested in both the 'applied" and the "mathematics" components of applied mathematics, being able to see "my" mathematics "in action" in such a variety of applications was a real pleasure.

1 Self-focusing - theory

1.1 New singular solutions

It has been "known" since the 1980s that generically, "all" singular solutions of the d-dimensional critical NLS

$$i\psi_t(t,\mathbf{x}) + \Delta\psi + |\psi|^{4/d}\psi = 0, \qquad \mathbf{x} \in \mathbb{R}^d, \quad t \in \mathbb{R}^+,$$

1. Collapse with a universal self-similar blowup profile, which is, up to rescaling, the ground state of

$$R''(r) + \frac{d-1}{r} - R + R^{4/d+1} = 0, \quad R'(0) = 0, \quad R(\infty) = 0. \tag{1}$$

2. Collapse at a square root rate with a loglog correction (the *loglog law*).

These results have been the basis of practically all the asymptotic analysis of the critical NLS, with and without perturbations.[1] However, a fully rigorous proof of these results turned out to be a hard problem. Indeed, only recently (2003) Merle and Raphael proved these results (for which, in part, Merle won the prestigious 2005 Bôcher Prize).[2]

Although Merle and Raphael proved these results "only" for collapsing solutions whose power (L^2 norm) is slightly above the critical power for collapse, it was widely believed (based, in part, on more than twenty years of NLS simulations), that these results are true for "all' solutions. In 2005, however, we discovered a *new family of high-power singular solutions of the critical NLS that collapse with a self-similar* **ring** *profile* G, which is different from the R profile [9]. The blowup rate of these solutions turned out not to be given by the loglog law, but rather by a square root with no loglog correction. We showed that these new blowup solutions are stable under radially-symmetric perturbations, but unstable under azimuthal perturbations. In [10], we showed that and explained why high-power super-Gaussian initial conditions would collapse with the self-similar ring profile G, and not with the R profile.

The main importance of the discovery of these new blowup solutions is that they show that collapsing high-power NLS solutions can be very different from collapsing solutions whose power is moderately above critical. Hence, we are now in the process of assessing to what extent what "everyone believed" about "all" collapsing solutions of the NLS remains valid for very high-power solutions of the NLS.

1.1.1 Supercritical NLS

In 2006, Raphael discovered a new family of singular ring solutions of the radially-symmetric, 2D quintic NLS

$$i\psi_t(t,r) + \psi_{rr} + \frac{1}{r}\psi_r + |\psi|^4\psi = 0.$$

These solutions collapse on a sphere $r = r_c(T) > 0$, rather than at the point $r = 0$, and at the loglog law blowup rate.

In [23], we showed that our critical G-profile ring solutions and Raphael's supercritical ring solutions are in fact "related", as they are two special cases of *a new, two-parameter family of singular ring solutions* of the NLS

$$i\psi_t(t,r) + \psi_{rr} + \frac{d-1}{r}\psi_r + |\psi|^{2\sigma}\psi = 0, \qquad 2/d \le \sigma \le 2.$$

Here, the key step was to discover the asymptotic profile of these new solutions. This was quite a challenge, because this profile is different from the profile of all previously known singular NLS solutions. The new blowup rings solutions are very different from the 'old' peak-type singular solutions. For example, they can blowup at any rate between 1/2 and 1, whereas the blowup rate of the 'old' peak-type supercritical singular solutions is 1/2.

[1] For example, we previously used these results to derive *modulation theory*, which is a singular perturbation method for analyzing the effect of small perturbations in the critical NLS.

[2] The proof of Merle and Raphael relies on a spectral conjecture which was proved analytically only for $d = 1$. For dimension larger than one, we recently provided a numerically-aided proof in [16].

1.2 Multiple filamentation and the 1/P scaling law

A phenomenon that is unique to the high-power regime is *multiple filamentation*, whereby a single laser beam breaks up into several, well-separated filaments. The standard model for *multiple filamentation* of laser beams, due to Bespalov and Talanov (1996), has been that it is due to instabilities that are excited by the noise in the initial condition. Although the analysis of Bespalov and Talanov was performed for infinite-power plane waves, it has been generally believed that it applies to all solutions whose power is $\gg$ critical power for collapse. A few years ago, however, we questioned the validity of the Bespalov-Talanov analysis to finite-power solutions, by showing numerically that noisy Gaussian beams with 20 times the critical power do not break into multiple filaments, but rather collapse at a single point. In [6], we "reconciled" the results of these two studies, by showing the existence of a *second power threshold*, of the order of 100 times the critical power, such that the Bespalov-Talanov analysis is only valid above this threshold.

An important consequence of the discovery of the second power threshold is as follows. Ever since the pioneering work of Kelley in 1965, it has been known that the time/distance to the singularity scales as $1/P^{1/2}$, where $P = \int |\psi_0|^2 \, dxdy$ is the input beam power. In [6], we showed that the $1/P^{1/2}$ scaling is valid when P is moderately above the critical power for collapse. When, however, P is above the second power threshold, noise effects dominate, the solution breaks into multiple filaments, hence the self-focusing time/distance scales as $1/P$.

1.3 NLS with a nonlinear microstructure

In the last decade or so there had been a lot of interest, both theoretically and experimentally, in propagation of laser beams in a medium in which the linear index of refraction has a periodic microstructure (linear microstructure). Recently, some of the attention has shifted to the effect of spatial variations in the nonlinear index of refraction (a nonlinear microstructure). The NLS with a nonlinear microstructure also models the dynamics of Bose-Einstein Condensates (BEC) in a medium with a spatially dependent scattering length.

In [12], we considered nonlinear bound states $\psi = e^{i\nu t}\phi(x;\nu)$ of the NLS in the presence of a periodic nonlinear microstructure $m(Nx)$, i.e.,

$$i\psi_t(t,x) + \psi_{xx} + (1 + m(Nx))|\psi|^{p-1}\psi = 0.$$

In the nonlinear optics context, $N = r_{beam}/r_{ms}$ denotes the ratio of beam width to microstructure characteristic scale. We studied the profiles and stability of the nonlinear bound states using a multiple scale (homogenization) expansion for $N \gg 1$ (wide beams), a perturbation analysis for $N \ll 1$ (narrow beams) and numerical simulations for $N = O(1)$; for both the subcritical ($p < 5$) and critical ($p = 5$) cases.

The novelties of that study were as follows. It pointed out the importance of the parameter N, and carried out a systematic study for the three distinct regimes $N \gg 1$, $N \ll 1$ and $N = O(1)$. It showed that in the critical case, a nonlinear microstructure can only stabilize beams provided that 1) the beam is narrow, 2) the beam is centered at a local maximum of the microstructure, and 3) the microstructure also satisfies a certain local condition. Even in this case, the stability region is so narrow that such beams are "mathematically stable" but "physically unstable" (i.e. stable under infinitesimal perturbations, but unstable under perturbations in actual physical setups).

Let us elaborate a little more on the last result. It is well-known that a necessary condition for stability is that the *sign* of the slope of the curve $\nu \longmapsto \int |\phi(x;\nu)|^2\,dx$ would be positive. However, to the best of our knowledge, this is the first study that pointed out that

1. *The magnitude of the slope determines the size of the stability region.*

2. *Hence, when the slope is positive but small, the bound state is "mathematically stable" but "physically unstable".*

Clearly, the importance of these observations goes beyond the specific problem of propagation in a nonlinear microstructure.

In [17], we showed that our results can be extended to the 2D NLS with anisotropic nonlinear microstructure

$$i\psi_t(t,x,y) + \Delta_{x,y}\psi + (1 + m(Nx))|\psi|^2\psi = 0,$$

which models the propagation of anomalous ultrashort laser pulses in a medium for which the Kerr nonlinearity varies periodically along the transverse spatial coordinate x, and also the evolution of 2D Bose-Einstein condensates in which the scattering length varies periodically in one spatial direction.

1.4 Optical bullets

Ever since proposed by Silberberg in 1990, one of the holy grails of nonlinear optics has been to achieve an "optical bullet", i.e., a laser pulse, localized in both space and time, that can propagate over long distances unchanged.

Previously, we showed that the critical exponent of NLS equations with a negative anisotropic fourth-order dispersion

$$i\psi_t(t,x_1,\ldots,x_d) + \Delta\psi + |\psi|^{2\sigma}\psi - \sum_{i=1}^{k} \psi_{x_ix_ix_ix_i} = 0, \qquad 0 < k < d,$$

is given by $\sigma^* = 2/(d - k/2)$. In [1], we used this theoretical result to show that small negative fourth-order dispersion can arrest collapse and stabilize the solitary waves that propagate in a Kerr medium with a slab geometry in the anomalous regime, i.e., for solutions of[3]

$$i\psi_z(z,x,t) + \psi_{xx} - \beta_2\psi_{tt} + \beta_4\psi_{tttt} + |\psi|^2\psi = 0, \qquad \beta_2 < 0, \quad \beta_4 < 0.$$

To the best of our knowledge, this has been the first theoretical model of optical bullets in a pure[4] Kerr medium. These bullets turned out to be robust, and even to undergo elastic collisions.

We also showed that small negative fourth-order dispersion cannot arrest collapse for pulses propagating in a bulk medium in the anomalous regime, i.e., when the equation is given by

$$i\psi_z(z,x,y,t) + \psi_{xx} + \psi_{yy} - \beta_2\psi_{tt} + \beta_4\psi_{tttt} + |\psi|^2\psi = 0.$$

[3] Here we use physical notations, where z plays the role of "time" and t plays the role of an additional "spatial" variable.

[4] i.e., a Kerr medium without additional collapse-arresting mechanisms such as nonlinear saturation.

2 Self-focusing - numerical methods

The NLS, which is routinely used to model propagation of intense beams in a Kerr medium, is derived by neglecting the backscattered field and under the paraxial approximation. A more comprehensive physical model, which does not make these approximations, is the Nonlinear Helmholtz equation. Previously, we developed the first numerical algorithm capable of solving the two-dimensional nonlinear Helmholtz equation

$$E_{zz}(z,x) + E_{xx} + k_0^2(1 + |E|^{2\sigma})E = 0$$

as a true nonlinear boundary problem. A key issue here was the derivation of a two-way absorbing boundary condition that was capable of fully transmitting the incoming beam into the medium while simultaneously allowing the backscattered waves to radiate outside. *This algorithm was the first to allow for a quantitative calculation of nonlinear backscattering.*

In that algorithm, we applied Dirichlet boundary conditions in the transverse direction. This enabled us to carry out a separation-of-variables expansion in an orthonormal basis. The Dirichlet boundary condition was far from ideal, however, since it reflected back all waves hitting the transverse boundary. Therefore, in [7] we introduced instead a Sommerfeld-type boundary condition in the transverse direction. This change was far from trivial, both theoretically- and numerically-wise, as it lead to an *expansion in a nonorthogonal basis.* Nevertheless, our analysis and simulations showed the advantage of this approach.

In [11], we extended the algorithm to a three-dimensional cylindrically symmetric setting, in order to be able to model beam propagation in a bulk medium. More importantly, we introduced a major improvement to the algorithm, by solving the system of one-dimensional Helmholtz equations (that follow from the separation of variables) using an integral, Green's function formulation, rather than the differential formulation that we previously used.

In spite of all the improvements in the algorithm, we have been unable to solve the NLH for "initial conditions" that would lead to singularity formation in the NLS model, thus leaving the question of global existence in the NLH open. Indeed, the algorithm that we used is based on iterations in which we freeze the nonlinearity, and these iterations diverged as we approached the (NLS) critical power from below. It was unclear, however, whether this divergence is due to nonexistence of solutions to the NLH, or due to the numerical methodology used. In order to resolve this question, in [24] we study the one-dimensional nonlinear Helmholtz equation

$$E_{zz}(z) + k_0^2(1 + |E|^{2\sigma})E = 0.$$

Although this equation does not have blowup solutions, as the initial condition increases , its solutions become multivalued. Our simulations revealed that the iterations based on freezing the nonlinearity diverged far below the threshold for non-uniqueness. To overcome this limitation, replaced the iterations based on freezing the nonlinearity with Newton's iterations. Because the Kerr nonlinearity contains absolute values of the field, the NLH has to be recast as a system of two real equations in order to apply Newtons method. Our numerical simulations show that Newtons method converges rapidly and, in contradistinction with the iterations based on freezing the nonlinearity, enables computations for very high levels of nonlinearity. In addition, in [24] we introduced a novel compact finite-volume fourth order discretization for the NLH with material discontinuities.

3 Self focusing - experiments

In the last few years I have been involved in collaborations with several experimental groups on research projects in Nonlinear Optics in which the close collaboration between theory and experiments was vital.

3.1 Control of multiple filamentation

As noted in Section 1.2, the standard model for multiple filamentation has been that it is initiated by input beam noise. This implies, in particular, that the number and location of the filaments can change from shot to shot, a highly undesirable feature in applications. Previously, we were the first to suggest that deterministic mechanisms, such as polarization (vectorial) effects or input beam ellipticity, can lead to a deterministic multiple filamentation. In [2], we provided the first experimental demonstration of a deterministic multiple filamentation (in water), and showed that it was induced by input beam ellipticity, but not by polarization (vectorial) effects.

In the experiments in [2], the input beams where "cleaned", in order to focus on the effect of input beam ellipticity. In [4], however, we adopted the "opposite" approach: We introduced a large "ellipticity" (with a titled lens setup) without making any effort to clean the input pulses. "Nevertheless", we were able to control deterministically the multiple filamentation pattern of very noisy input beams (that propagate in air) with the tilted lens. This study showed, in particular, that input beam "ellipticity" can be stronger than noise in determining the multiple filamentation pattern, even for noisy input pulses that change from shot to shot. This finding has been very important for atmospheric propagation, where input beams are typically very noisy and unstable from shot to shot, but random multiple filamentation is undesirable.

3.2 $1/P$ scaling law

As noted in Section 1.2, in [6] we showed theoretically that the self-focusing distance scales as $1/P^{1/2}$ at moderate powers, but as $1/P$ for powers above the *second power threshold*. In that study, we also observed the two scaling laws experimentally for laser pulses propagating in air.

3.3 Observations of collapsing ring and vortex beams

As noted in Section 1.1, in [9] we predicted the existence of collapsing ring solutions of the NLS. In [10] we observed experimentally that (and provided a theoretical explanation of why) high-power Gaussian input beams would collapse with the R profile, but equal-power Super-Gaussian input beams would collapse with a ring profile. We also observed experimentally the azimuthal instability of the collapsing rings predicted theoretically.

In [14] we observed experimentally a yet another new family of singular NLS solutions, namely, collapsing vortex solutions. A linear stability analysis showed that these vortices are azimuthally unstable, hence they break up into filaments. The number of filaments predicted by the analysis was confirmed both in simulations and experimentally.

3.4 Control of the collapse distance in atmospheric propagation

One of the key challenges in atmospheric propagation is to be able to delay and control the collapse distance. Until recently, the only known method to do that has been by negatively chirping the input pulse. In [15], we used the lens transformation property of the critical NLS to suggest that one can also delay the collapse distance with a defocusing lens. Moreover, we suggested a simple double-lens setup that allows for a continuous control of the collapse distance. In our atmospheric propagation experiments, we observed a very good agreement between the predicted and measured collapse distance. One advantage of the double lens setup is that it can be used with pulses of any duration, and not just with ultrashort pulses.

In [21], we showed in an outdoor experiment that the double lens setup can be used to achieve a 20-fold delay of the filamentation distance of non-chirped 120 fs pulses propagating in air, from 16m to 330m. At 330m, the collapsing pulse was sufficiently powerful to create plasma filaments. We also showed that the scatter of the filaments at 330m can be significantly reduced by tilting the second lens (see Section 3.1). To the best of our knowledge, this is the longest distance reported in the Literature at which plasma filaments were created and controlled. Finally, we showed that the peak power at the onset of collapse is significantly higher with the double-lens setup, compared with the standard negative chirping approach.

4 Auction Theory

The mathematical theory of auctions has been, for the most part, “limited” to “exact results”[5] that can be proved rigorously. “As a result”, to these days many fundamental questions are still open. In recent years I have been involved in analyzing some of these open problems using applied math approach and techniques.

Most of auction theory has been developed for symmetric auctions, i.e., auctions in which all bidders are essentially “the same”. For example, a key result in auction theory is the Revenue Equivalence Theorem (RET), which says that in the case of symmetric auctions with risk-neutral bidders, the expected revenue for the seller is independent of the auction mechanism. Although it is known that in many real-life auctions bidders are not “the same”, analysis of asymmetric auctions is a hard problem, hence relatively little is known on asymmetric auctions. For example, while numerical simulations showed that the RET does not hold for asymmetric auctions, very little has been known on revenue ranking of asymmetric auctions.

Previously, we analyzed revenue ranking of asymmetric auctions by considering the case of a weak asymmetry. We formulated and proved an *asymptotic analogue of the RET* for private-value auctions, namely, that if ϵ is the level of asymmetry among bidders, then all asymmetric auctions are $O(\epsilon^2)$ revenue equivalent (and not only $O(\epsilon)$ revenue equivalent, as one might have expected from a continuity argument). In [25], we showed that this result can be generalized to asymmetric auctions with interdependent values, where the asymmetry can be either among the valuation functions of the bidders, or among the distribution functions of their signals.

[5] i.e., in contrast to approximate, leading-order type results

In [18], we analyzed the effect of risk aversion in all-pay auctions and showed that it is more complex than in first-price auctions. In addition, we showed that, unlike asymmetry, an $O(\epsilon)$ risk aversion does lead to an $O(\epsilon)$ revenue difference among different auction mechanisms.

In [26], we formulated and proved a different version of asymptotic revenue equivalence: We showed that "all" auctions with a large number of players ($n \gg 1$) are $O(1/n^2)$ revenue equivalent, regardless of whether bidders are risk averse or risk neutral, or whether they are symmetric or asymmetric. The results of these study were stronger than our previous studies, since we did not have to assume that asymmetry and/or risk aversion are small.

The novelty of the above studies has been twofold. First, we used applied math techniques (perturbation methods, asymptotic analysis) which have not been used before in auction theory. Second, the applied math spirit of the results that we formulated (and then proved) was also new to auction theory. Thus, we advocated the approach of obtaining approximate, leading-order results, for problems where "exact results" are hard to get.

For example, until now the research on revenue ranking of asymmetric auctions focused solely on which auction mechanism is more profitable (e.g., first- or second-price). By calculating the leading-order effect of asymmetry on the revenue, we showed analytically that the revenue differences in asymmetric auctions among different auction mechanisms are usually very small. Indeed, we observed numerically that in many cases these revenue difference can be in the fourth or fifth digit. Therefore, we argued that in such cases, revenue ranking is more of an academic interest than of practical importance.

Similarly, most studies on auctions with a large number of players ($n \gg 1$) considered the limit as $n \longrightarrow \infty$. By calculating explicitly the *leading-order deviation* from this limiting case, we were able to derive new results that are valid not only as $n \longrightarrow \infty$, but already when n is moderately large. In fact, we found out that "in most cases", our results for large auctions are already valid for auctions with as few as 6 players.

5 Optical imaging

Minimally-invasive detection of tumors is a hard task, because of the scattering properties of the human tissue. A novel approach, based on injection of fluorophore conjugated antibodies (*markers*) that selectively bind to the T-cells that are in the vicinity of the tumor, has been recently developed at the Optics in Medicine Lab at Tel Aviv University of Dr. Israel Gannot. While the preliminary experimental results were promising, it soon became clear that a real progress could only be achieved with the help of a mathematical model, since "numerical experiments" are faster, cheaper, better-controlled, and easier to interpret than in-vivo lab experiments.

In [8], we developed a mathematical model that takes into account markers diffusion in the tissue, and binding and dissociation of markers to the T-cells in a narrow volumetric layer near the tumor surface. Although this *volumetric model* was much simpler than the actual experimental setup, it already provided insight that helped with the design of subsequent experiments.

In [20], we presented a second mathematical model, a *surface model*, in which markers binding and dissociation occur at the tumor surface. A priori, there was a methodological problem with the surface model, since chemical kinetics relations are usually derived under the assumption that the reactions take place in a volume in which the two reactants are well mixed. Therefore, in [20] we gave both an informal and formal derivations of the surface model as the singular limit of the volumetric model, as the width of the volumetric layer (in which binding and dissociation of markers occur) shrinks to zero.

The importance of the two derivations of the surface model from the volumetric model goes beyond the specific application that we modeled. Indeed, while there are numerous problems in biology in which binding and dissociation takes place on surfaces, to the best of our knowledge, this is the *first rigorous justification for applying the laws of chemical kinetics to reactions that take place on surfaces.*

6 Biochemistry and Neurobiochemistry

Many complex biochemical processes can, in theory, be modeled by a system of coupled nonlinear ordinary differential equations. In many cases, the functional form of the equations is known from first principles to be of the form

$$y_i'(t) = \sum_{j=1}^{n} b_{i,j} y_j + \sum_{j=1}^{n} c_{i,j} y_i y_j, \qquad i = 1, \ldots, n,$$

where $y_i(t)$ is the concentration of the i'th reactant at time t. What is usually not known are the values of the coefficients $\{b_{i,j}, c_{i,j}\}_{i,j=1}^{n}$. These coefficients depend on the values of the kinetic rates of the reactions that take place, which, a-priori, can vary over several orders of magnitudes (microseconds—minutes). Clearly, knowledge of the values of the kinetic rate constants is crucial in order to be able to model these complex biochemical processes.

The method for finding the values of fast kinetic rate constants was developed at the Gutman's lab in Biochemistry at Tel Aviv University. In this method, the system under consideration is perturbed away from equilibrium by a laser pulse, and the return to equilibrium dynamics is measured. The values of the rate constants is then extracted from the experimental data, by searching for the combination of values of the kinetic rate constants that, when plugged into the ODE system, would "reproduce" the experimental data. For many years this multidimensional search was carried out at the Gutman's lab manually. Naturally, such a manual search suffers from various weaknesses (the search was prone to human bias, it was not clear whether the manual search missed other combinations of values of the kinetic rate constants that could just as well mimic the dynamics of the experimental data, etc.).

In order to eliminate the weaknesses of the manual search, in [3] we developed an automated search for the values of the kinetic rate constants using a genetics-algorithm approach. The main challenges were due to the following reasons:

1. We worked with real data that has a high noise level.

2. The concentration of only some of the reactants could be measured.

3. The parameter space had a high dimensionality.

Nevertheless, the algorithm was able to find the kinetic rate constants, without suffering from the weaknesses of the manual search discussed before, and has, by now, become a standard tool at the Gutman's lab.

In a subsequent study [13], we applied this approach to a new model of the exocytotic process. Here, again, the kinetic rate constants were recovered by the algorithm from experimental data obtained at the Asheri lab in Neurobiochemistry at Tel Aviv University. Currently, we began to address some of the more mathematical issues that arise from this problem (uniqueness of solution with clean/noisy data, number of experiments needed to achieve a prescribed accuracy, etc.).

7 Numerical Linear algebra

One of the first thing we learn (or teach) in Numerical Linear Algebra classes is that in order to solve $\mathbf{Ax} = \mathbf{b}$, we should never calculate A-inverse explicitly, as this would be inferior to solving via Gaussian elimination, both in terms of performance, and in terms of numerical accuracy. In [22], we show that this is not always the case. Indeed, we identify some common situations where the A-inverse approach is superior to the LU approach in terms of performance, and comparable in terms of the accuracy of the (forward) error.

8 Marketing (reference-price)

Quantitative marketing is a relatively young topic. It is a real challenge to anyone trying to come up with a realistic mathematical model, in part because marketing involves psychological effects for which the "first principles" are still unknown.

Previously, we analyzed the effect of reference-price on the equilibrium strategies. The main mathematical difficulty was due to the fact that reference price effects are believed to be asymmetric between the perception of losses and gains. Therefore, the resulting optimization problem is nonsmooth. Nevertheless, we were able to solve explicitly for the equilibrium strategies, both for a monopolist retailer, and under various forms of competition (open loop, closed-loop).

One of the consequences of that study (and of studies of other researchers) was that under asymmetric reference price effects, the equilibrium strategies reach a steady stage when loss effects are larger than gain effects, but have a highly oscillatory pattern ("chattering") when loss effects are smaller than gain effects. However, there was no good *intuitive explanation* of *why* the behavior is so different between these two cases. In [19], we provided such an intuitive explanation, by introducing a toy model in which we calculated the overall profitability of a single price promotion.

In [5], we analyzed the effect of reference-price on price elasticity of demand. This study showed that large errors can occur if one determines price elasticity of demand from real-life data without taking into account reference-price effects, and pointed out the relevance of reference-price to the difference between short-term and long-term elasticities.

9 Recent Publications

1. G. Fibich, B. Ilan
   Optical light bullets in a pure Kerr medium
   Optics Letters 29 (2004), 887-889.

2. A. Dubietis, G. Tamošauskas, G. Fibich, B. Ilan
   Multiple filamentation induced by input-beam ellipticity
   Optics Letters 29 (2004), 1126-1128.

3. D. Moscovitch, O. Noivirt, A. Mezer, E. Nachliel, T. Mark, M. Gutman, G. Fibich
   Determination of a unique solution to parallel proton transfer reactions using the genetic algorithm
   Biophysical Journal 87 (2004), 47-57.

4. G. Fibich, S. Eisenmann, B. Ilan, A. Zigler
   Control of multiple filamentation in air
   Optics Letters 29 (2004), 1772-1774.

5. G. Fibich, A. Gavious, O. Lowengart
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   Journal of the Academy of Marketing Science 33 (2005), 66-78.

6. G. Fibich, E. Eisenmann, B. Ilan, Y. Erlich, M. Fraenkel, Z. Hennis, A.L. Gaeta, A. Zigler
   Self focusing distance of very high-power laser pulses
   Optics Express 13 (2005), 5897-5903.

7. G. Fibich and S. Tsynkov
   Numerical Solution of the Nonlinear Helmholtz Equation Using Nonorthogonal Expansions
   Journal of Computational Physics 210 (2005), 183-224.

8. G. Fibich, A. Hammer, G. Gannot, A. Gandjbakhche, I. Gannot
   Modeling and simulations of the pharmacokinetics of fluorophore conjugated antibodies in tumor vicinity for the optimization of fluorescence based optical imaging
   Lasers in Surgery and Medicine 37 (2005), 155-160.

9. G. Fibich, N. Gavish, X.P. Wang
   New singular solutions of the nonlinear Schrödinger equation
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    Collapse dynamics of super-Gaussian Beams
    Optics Express 14 (2006) 5468–5475.

11. G. Baruch, G. Fibich, S. Tsynkov
High-Order Numerical Solution of the Nonlinear Helmholtz Equation with Axial Symmetry
Journal of Computational and Applied Mathematics - accepted

12. G. Fibich, Y. Sivan, M.I. Weinstein
Bound states of nonlinear Schrödinger equations with a periodic nonlinear microstructure
Physica D 217 (2006), 31-57.

13. A. Mezer, U. Ashery, M. Gutman, E. Project, G. Fibich, and E. Nachliel
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Biochimica et Biophysica Acta 1763 (2006), 345–355.

14. L.T. Vuong, T.D. Grow, A. Ishaaya, A.L. Gaeta, G.W. Hooft, E.R. Eliel and G. Fibich
Collapse of Optical Vortices
Physical Review Letters 96 (2006), 133901.

15. G. Fibich, Y. Sivan, Y. Ehrlich, E. Louzon, M. Fraenkel, S. Eisenmann, Y. Katzir, A. Zigler
Control of the collapse distance in atmospheric propagation
Optics Express 14 (2006) 4946–4957.

16. G. Fibich, F.Merle, P. Raphaël
Proof of a spectral property related to the singularity formation for the L^2 critical nonlinear Schrödinger equation
Physica D 220 (2006), 1–13.

17. Y. Sivan, G. Fibich and M.I. Weinstein
Waves in nonlinear lattices: Ultrashort optical pulses and Bose-Einstein condensates
Physical Review Letters 97 (2006), 193902

18. G. Fibich, A. Gavious, A. Sela
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International Journal of Game Theory 34 (2006), 583-599

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