Calculus A for Economics

Exercise Number 5

1) Use the definition of a continuous function at a point x_0, to prove that the following functions are continuous at the specified point x_0:

$a)\ f(x) = (x-3)^2 - x + 2 \qquad x_0 = 2 \qquad\qquad b)\ f(x) = \frac{x-2}{2x+3} \qquad x_0 = -1$

2) Determine if the following functions are continuous at the point $x = 2$:

$a)\ \ f(x) = \sqrt{(x-2)^3+5} \qquad\qquad b)\ \ f(x) = \begin{cases} x^2+4, & x < 2, \\ x^3, & x \geq 2 \end{cases}$

$c)\ \ f(x) = \sqrt{1-x} \qquad\qquad d)\ \ f(x) = \begin{cases} \frac{1}{x-2}, & x \neq 2 \\ 0, & x = 2 \end{cases}$

3) Determine the points in which the following functions are not continuous:

$a)\ \ f(x) = \frac{1}{1+2^{\frac{1}{x}}} \qquad\qquad b)\ \ f(x) = \frac{1}{|x|+1} - \frac{x^2}{2}$

4) Find all the values of a such that the following function will be continuous for all x.

$$f(x) = \begin{cases} x+1, & x \leq 1 \\ 3 - ax^2, & x > 1 \end{cases}$$

5) Find A and B such that the following function will be continuous for all x.

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x < 2 \\ Ax+B, & 2 \leq x \leq 3 \\ \frac{x-3}{x^2-9} & x > 3 \end{cases}$$

6) Compute the following limits:

$a)\ \ \lim_{x\to\infty} 3^{\frac{x^2-1}{2x^2+1}} \qquad\qquad b)\ \ \lim_{x\to 1} \ln\frac{x^2-1}{x-1}$

Consider the following

Theorem: *Let $f(x)$ be a continuous function in the closed interval $[a, b]$. Assume that*

$f(a)f(b) \leq 0$. *Then there exist a point* $a \leq c \leq b$ *such that* $f(c) = 0$.

Use this Theorem to solve exercises **7)-9)**.

7) Prove that the equation $x^4 - 2x - 3 = 0$ has a root in the interval $[0, 2]$.

8) Let n be an odd number. Prove that the polynomial $f(x) = x^n - 3x + 1$ has at least one real root.

9) Let $f(x)$ be a continuous function in the interval $[a, b]$. Given a number d between $f(a)$ and $f(b)$ prove that there is a number c such that $a \leq c \leq b$ such that $f(c) = d$.
Hint: Consider the function $g(x) = f(x) - d$.

10) Give an example of a function $f(x)$ defined in the interval $[a, b]$, satisfies $f(a) < 0$ and $f(b) > 0$, and such that $f(x) \neq 0$ for all $x \in [a, b]$.

11) Let $f(x)$ be a function which is defined in the interval $[a, b]$. Assume that $f(x)$ is continuous and one to one in this interval. Prove that $f(x)$ obtain its maximal and minimal values in $[a, b]$ at the end points of the interval.

12) Let $f(x)$ be a function which satisfies $f(x+y) = f(x)+f(y)$ for all x and y. Assume that $f(x)$ is continuous at $x = 0$. Prove that it is continuous for all x.