Calculus A for Economics

Exercise Number 8

1) Verify Rolle's Theorem for the function $y = x^3 + 4x^2 - 7x - 10$ in the interval $[-1, 2]$.

2) Let $f(x) = \frac{2-x^2}{x^4}$. Check that $f(-1) = f(1)$ and show that $f'(x) \neq 0$ for all $x \in [-1, 1]$ where $f'(x)$ exists. Explain why this is not a contradiction to Rolle's Theorem.

3) Prove that if the equation $a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x = 0$ has a root at $x_0 > 0$, then the equation $n a_0 x^{n-1} + (n-1) a_1 x^{n-2} + \cdots + a_{n-1} = 0$ has a positive root at x_1 where $0 < x_1 < x_0$.

4) Let $c \in \mathbf{R}$. Prove that the equation $x^3 - 3x + c = 0$ has no two distinct roots in $(0, 1)$.

5) Let $f(x) = 3\sqrt{x} - 4x$. Find $1 \leq c \leq 4$ which satisfies the mean value Theorem in $[1, 4]$.

6) Assume that $|f'(x)| \leq 1$ for all x. Prove that $|f(x_1) - f(x_2)| \leq |x_1 - x_2|$ for all x_1 and x_2.

7) Use the mean value Theorem to prove

$$\frac{a-b}{a} \leq \ln\frac{a}{b} \leq \frac{a-b}{b}$$

for all $0 < b \leq a$.

8) Let n ne a natural number. Use the mean value Theorem to prove

$$nb^{n-1}(a-b) < a^n - b^n < na^{n-1}(a-b)$$

for all $a > b$.

9) For the following functions, find possible extreme points:

a) $y = x^3 + 3x - 2$

b) $y = (1-x)^2(1+x)$

c) $y = \dfrac{1}{|x-2|}$

d) $y = |x^2 - 16|$

e) $y = x - \ln(1 + x^2)$

f) $y = (x^2 - 2x)\ln x - \dfrac{3}{2}x^2 + 4x$

g) $y = \dfrac{1}{x+1} - \dfrac{1}{x+2}$

10) Let a and b be two adjacent roots of a polynomial $f(x)$. (By adjacent we mean that there is no point c such that $a < c < b$ and that c is a root of $f(x)$). It follows from division of polynomials that we can write $f(x) = (x-a)(x-b)g(x)$ where $g(x)$ is a polynomial. You do not need to prove this. Assume that $g(a) \neq 0$ and that $g(b) \neq 0$.

a) Prove that $g(a)g(b) > 0$.

b) Prove that there is a point x such that $a < x < b$ and that $f'(x) = 0$.

11) Give an example of a function $f(x)$ such that $f'(x)$ exists for all $x > M$, and such that $\lim_{x\to\infty} f(x)$ does not exists but $\lim_{x\to\infty} f'(x)$ do exist. Here M is some nonnegative integer.

12) Without using a calculator prove that

$$\frac{1}{9} < \sqrt{66} - 8 < \frac{1}{8}$$