Calculus A for Economics

Solutions to Exercise Number 1

1) a) $f(0) = \frac{0-2}{0+1} = -2$; $\quad f(\sqrt{2}) = \frac{\sqrt{2}-2}{\sqrt{2}+1}$; $\quad g(4) = \frac{|4-2|}{4+1} = \frac{2}{5}$.
b) $f(t+1) - f(t) = \frac{(t+1)-2}{(t+1)+1} - \frac{t-2}{t+1} = \frac{t-1}{t+2} - \frac{t-2}{t+1} = \frac{3}{(t+1)(t+2)}$.

2) a) We have $5 - 2x \geq 0$. Hence $x \leq \frac{5}{2}$.
b) We have $|x| - x > 0$, or $|x| > x$. If $x > 0$, we get $x > x$ which is impossible. If $x < 0$, then $|x| > x$ always holds since the left hand side is positive and the right hand side is negative. Thus, the domain of definition is $x < 0$.
c) We have $x^2 - 3x + 2 > 0$. Since the roots of $x^2 - 3x + 2 = 0$ are 1 and 2, the domain of definition is $(x-1)(x-2) > 0$. This will happen if $x > 2$ or $x < 1$.
d) From the first term we have $\frac{x-2}{x+2} \geq 0$ and $x \neq -2$. Multiplying by $(x+2)^2$, the first inequality is equivalent to $(x-2)(x+2) \geq 0$ or $x^2 \geq 4$. Hence we get the domain $x < -2$ or $x \geq 2$. The right terms gives us $\frac{1-x}{1+x} \geq 0$ and $x \neq -1$. Arguing similarly, we get $x^2 \leq 1$. Thus we get $-1 < x \leq 1$. The intersection of the two domains is empty and hence the function is not defined for any x.

3) First notice that for any two *positive* numbers x and y we have $x \leq y$ if and only if $x^2 \leq y^2$. Also we have $|x|^2 = x^2$. Thus to prove the desired inequality it is enough to prove that $||a| - |b||^2 \leq |a-b|^2$ or $(|a| - |b|)^2 \leq (a-b)^2$. We have $(|a| - |b|)^2 = |a|^2 - 2|a||b| + |b|^2 = a^2 - 2|a||b| + b^2 \leq a^2 - 2ab + b^2$. The last inequality follows from the fact that $ab \leq |a||b|$ and hence $-|a||b| \leq -ab$. Thus we get $(|a| - |b|)^2 \leq a^2 - 2ab + b^2 = (a-b)^2$. Thats what we had to prove.

4) a) Suppose that $f(a) = f(b)$ where a and b are in the domain of definition of $f(x)$. Thus $\sqrt{1-a} = \sqrt{1-b}$ which clearly implies $a = b$. Thus $f(x)$ is one to one.
b) We have $g(2) = 2 + \frac{1}{2} = \frac{5}{2}$ and also $g(\frac{1}{2}) = \frac{1}{2} + \frac{1}{\frac{1}{2}} = \frac{5}{2}$. Hence $g(2) = g(\frac{1}{2})$ and the function is not one to one.

5) a) Let $f(x) = x - x^2$. Thus $f(-x) = -x - (-x)^2 = -x - x^2$ and hence $f(x)$ is neither even nor odd.
b) Let $f(x) = x - x^3 + x^5$. Then $f(-x) = -x - (-x)^3 + (-x)^5 = -(x - x^3 + x^5) = -f(x)$.

Hence $f(x)$ is odd.
c) We have $f(-x) = \frac{a^{-x}+a^{-(-x)}}{2} = \frac{a^{x}+a^{-x}}{2} = f(x)$. Hence $f(x)$ is even.

6) Let $F(x) = f(x) + f(-x)$. Then $F(-x) = f(-x) + f(-(-x)) = F(x)$. Hence $F(x)$ is even.

7) a) First notice that $F(x)$ is defined only for $-1 \le x \le 1$. Thus $0 \le x^2 \le 1$ and $0 \le 1 - x^2 \le 1$. Since the square root of a number between zero and one is also a number between zero and one, we have $0 \le y \le 1$. To show that we get all numbers between zero and one, let $0 \le c \le 1$. Choose $x = \sqrt{1-c^2}$. Then, for this value of x, we have $y = \sqrt{1-x^2} = c$. Thus the domain of $F(x)$ is all numbers $0 \le y \le 1$.
b) The domain of definition of $h(x)$ is $x < 1$. This is equivalent to $-1 < -x$, or $0 < 1 - x$, or $0 < \sqrt{1-x}$, and hence $0 < \frac{1}{\sqrt{1-x}}$. Thus $y > 0$. To show that we get all positive values in y, let $c > 0$. Choose $x = 1 - \frac{1}{c^2}$. Then $y = h(1 - \frac{1}{c^2}) = c$. Hence the range is all $y > 0$.

8) We need to solve the equation $|(x+1) + (x-2)| = |x+1| + |x-2|$ or $|2x-1| = |x+1| + |x-2|$. The points where the absolute values are zero are at $x = \frac{1}{2}, -1, 2$. Thus we need to consider 4 cases. First $x \le -1$. In this domain we have $x + 1 \le 0$ and hence $|x+1| = -(x+1)$, $2x - 1 \le 0$ and hence $|2x-1| = -(2x-1)$, $x - 2 \le 0$ and hence $|x-2| = -(x-2)$. Thus the equation we obtain is $-(2x-1) = -(x+1) - (x-2)$ or $1 - 2x = 1 - 2x$. Hence the original equation holds for all $x \le -1$. Next consider the domain $-1 \le x \le \frac{1}{2}$. In this case $x + 1 \ge 0$ and hence $|x+1| = x+1$. The other two terms are as in the domain $x \le -1$. Hence we get $-(2x-1) = (x+1) - (x-2)$. This implies that $x = -1$, which we already had in the previous case. In the domain $\frac{1}{2} \le x \le 2$ we get the equation $(2x-1) = (x+1) - (x-2)$ which gives us $x = 2$. The final case, when $x \ge 2$ we get the equation $(2x-1) = (x+1) + (x-2)$ which gives us $2x + 1 = 2x + 1$. Hence we get a solution for all $x \ge 2$. Therefore the final solution is $x \le -1$ or $x \ge 2$.

9) See separate page.

10) We have $f(x+1) - f(x) = (a(x+1)^2 + b(x+1) + 5) - (ax^2 + bx + 5) = 2ax + a + b$. Thus we have $2ax + a + b = 8x + 3$. Two polynomials are equal for all x if and only if the corresponding coefficients of each power are equal. Hence we get $2a = 8$ and $a + b = 3$. Thus $a = 4$ and $b = -1$.

11) Write $f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$. Then the first term on the right hand side is an even function and the second term is an odd function . In our case, $f(x) = x^2 + 3x + 2$. Hence, $\frac{f(x)+f(-x)}{2} = \frac{1}{2}((x^2+3x+2) + ((-x)^2 + 3(-x) + 2)) = x^2 + 2$. Similarly, $\frac{f(x)-f(-x)}{2} = \frac{1}{2}((x^2+3x+2) - ((-x)^2 + 3(-x) + 2)) = 3x$. Clearly $f(x) = (x^2+2) + 3x$.

12) See separate page.