Calculus A for Economics

Solutions to Exercise Number 10

1) a) The function is defined for $x \neq 0$. Thus y can have a vertical asymptotes only at $x = 0$. We have $\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$. Hence y has a vertical asymptote from both sides at $x = 0$. As for horizontal asymptotes we compute $a = \lim_{x\to\infty} \frac{f(x)}{x} = lim_{x\to\infty} \frac{1}{x^2} = 0$. Hence $b = lim_{x\to\infty} [f(x) - ax] = lim_{x\to\infty} \frac{1}{x} = 0$. Hence $y = 0$ is a horizontal asymptote at infinity. The computation at minus infinity is the same.

b) Since $x^2 - 4x + 5$ is positive for all x, then $f(x)$ is defined for all x and hence has no vertical asymptotes. We have $a = \lim_{x\to\infty} \frac{f(x)}{x} = lim_{x\to\infty} \frac{1}{x(x^2-4x+5)} = 0$ and $b = lim_{x\to\infty} [f(x) - ax] = lim_{x\to\infty} \frac{1}{x^2-4x+5} = 0$. Hence $y = 0$ is a horizontal asymptote at infinity. The computation at minus infinity is the same.

c) Domain of definition is $x \neq 1$. Hence $x = -1$ is a possible vertical asymptote. We have $\lim_{x\to -1^+} \frac{x^3}{2(x+1)^2} = -\infty$ and $\lim_{x\to -1^-} \frac{x^3}{2(x+1)^2} = -\infty$. Hence y has a vertical asymptote from both sides at $x = 1$. Also, $a = \lim_{x\to\infty} \frac{f(x)}{x} = lim_{x\to\infty} \frac{x^3}{2x(x+1)^2} = \frac{1}{2}$ and $b = lim_{x\to\infty} [f(x) - ax] = lim_{x\to\infty} \left[\frac{x^3}{2(x+1)^2} - \frac{1}{2}x\right] = lim_{x\to\infty} \frac{x^3 - x(x+1)^2}{2(x+1)^2} = lim_{x\to\infty} \frac{-2x^2-x}{2(x+1)^2} = -1$. Hence $y = \frac{1}{2}x - 1$ is a horizontal asymptote at infinity. A similar computations shows that it is also a horizontal asymptote at minus infinity.

d) The function is defined for all x, and hence there are no vertical asymptotes. Also, $a = \lim_{x\to\infty} \frac{f(x)}{x} = lim_{x\to\infty} \frac{e^{-x^2}}{x} = 0$ and $b = lim_{x\to\infty} [f(x) - ax] = lim_{x\to\infty} e^{-x^2} = 0$. Hence, $y = 0$ is a horizontal asymptote at infinity. The computation at minus infinity is the same.

e) The function is defined for all $x > 0$. Hence $x = 0$ is the only possible point for a vertical asymptote. We have $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{L}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0$. Hence there is no vertical asymptotes at $x = 0$. Also, $a = \lim_{x\to\infty} \frac{f(x)}{x} = lim_{x\to\infty} \ln x$ which does not exist. Hence there are no horizontal asymptotes.

f) The function is defined for all x, and hence there are no vertical asymptotes. Also, $a = \lim_{x\to\infty} \frac{f(x)}{x} = lim_{x\to\infty} e^x$ which does not exist. Finally, $a = \lim_{x\to -\infty} \frac{f(x)}{x} = lim_{x\to -\infty} e^x = 0$, and $b = lim_{x\to -\infty} [f(x) - ax] = lim_{x\to -\infty} xe^x = lim_{x\to -\infty} \frac{x}{e^{-x}} \overset{L}{=} lim_{x\to -\infty} \frac{1}{-e^{-x}} =$
$= -lim_{x\to -\infty} e^x = 0$. Hence the function has no horizontal asymptote at infinity, but $y = 0$ is a horizontal asymptote at minus infinity.

2) Suppose that

$$f(x) = \frac{a_0 + a_1 x + \cdots + a_n x^n}{b_0 + b_1 x + \cdots + b_m x^m} \qquad a_n, b_m \neq 0$$

Then

$$\frac{f(x)}{x} = \frac{a_0 + a_1 x + \cdots + a_n x^n}{b_0 x + b_1 x^2 + \cdots + b_m x^{m+1}} = \frac{\frac{a_0}{x^{m+1}} + \cdots + \frac{a_n}{x^{m+1-n}}}{\frac{a_0}{x^m} + \cdots + b_m}$$

where the last equality is obtained by multiplying and dividing by x^{m+1}. From this we deduce that if $n > m+1$ than the limit $\lim_{x\to\infty} \frac{f(x)}{x}$ does not exist. However, since it is given that $\lim_{x\to\infty} \frac{f(x)}{x} = a$, then we must have $m+1 \geq n$. From the above equality we also deduce that if $m+1 > n$ then $a = 0$, and if $m+1 = n$ then $a = \frac{a_{m+1}}{b_m}$. It also follows from the above equality that $\lim_{x\to-\infty} \frac{f(x)}{x} = a$. Next, $f(x) - ax$ is equal to

$$\frac{a_0 + a_1 x + \cdots + a_n x^n}{b_0 + b_1 x + \cdots + b_m x^m} - ax = \frac{a_0 + a_1 x + \cdots + a_n x^n - a(b_0 x + b_1 x^2 + \cdots + b_m x^{m+1})}{b_0 + b_1 x + \cdots + b_m x^m}$$

We claim that in the right most term, the degree of the numerator is at most m. Indeed, if $a = 0$, then from the above $m \geq n$, and the claim follows. If $a \neq 0$, then $a = \frac{a_{m+1}}{b_m}$ and $m+1 = n$. This means the coefficient of x^{m+1} in the numerator is zero, and hence the claim follows. Thus, multiplying and dividing the above term by x^m, it follows that the limit $\lim_{x\to\infty}[f(x) - ax]$ exists. Thus **a)** follows. Also, when we replace the limit to infinity to the limit to minus infinity, we get the same value. Thus, **b)** follows.

3) a) The function is defined for all $x \neq \frac{1}{3}$. It intersects the axis at $(0,0)$. It is continuous at $x > \frac{1}{3}; x < \frac{1}{3}$. We have $y' = -\frac{1}{(3x-1)^2}$, and hence $y' < 0$. Thus the function decreases in its domain of definition and has no extreme points. We have $y'' = \frac{6}{(3x-1)^3}$, and hence $y'' \neq 0$ and there are no inflection points. For $x > \frac{1}{3}$, we have $y'' > 0$ and hence the function has the shape $\cup$ at that domain. For $x < \frac{1}{3}$ we have $y'' < 0$ and hence the function has the shape $\cap$ at that domain. The function has a vertical asymptote at $x = \frac{1}{3}$ and a horizontal asymptote $y = \frac{1}{3}$ at plus and minus infinity.
b) The function is defined for all $x \neq 2$. Intersection with axis at $(-3,0)$ and $(0,-\frac{3}{2})$. It is continuous at $x \neq 2$. We have $y' = -\frac{5}{(x-2)^2}$ and $y'' = \frac{10}{(x-2)^3}$. Hence it decreases for all $x \neq 2$ and has no extreme point. Since $y'' \neq 0$ it has no inflection points. In the domain $x > 2$ is has the shape $\cup$ and when $x < 2$ it has the shape $\cap$. There is a vertical asymptote at $x = 2$ and horizontal asymptote $y = 1$ at $\pm\infty$.
c) The function is defined for all x. Intersection with the axis at $(0,-2.17)$ and $(1,0)$. It is continuous for all x. We have $y' = \frac{5}{3}(x-2)^{2/3}$ and $y'' = \frac{10}{9}(x-2)^{-1/3}$. $y' = 0$ at $x = 2$. However, since $y' > 0$ for all $x \neq 2$, then the function increases for all x and has no extreme points. We have $y'' > 0$ for $x > 2$ and hence the function has the shape $\cup$ in that domain,

and for $x < 2$, we have $y'' < 0$ and hence it has the shape of $\cap$ in that domain. Even though y'' is not defined at $x = 2$ the function is defined at that point, and hence $x = 2$ is an inflection point. The function has no asymptotes.

d) Domain of definition $0 \le x \le 1$. Intersection with the axis $(0,0)$ and $(1,0)$. It is continuous in $0 \le x \le 1$. We have $y' = \frac{1-2x}{2\sqrt{x-x^2}}$ and $y'' = -\frac{1}{4(x-x^2)^{3/2}}$. Hence $y' = 0$ for $x = \frac{1}{2}$. For $\frac{1}{2} < x \le 1$ we have $y' < 0$ and hence the function decreases in that domain. For $0 \le x < \frac{1}{2}$ we have $y' > 0$ and hence the function increases in that domain. Hence $(\frac{1}{2}, \frac{1}{2})$ is a maximum point. Since $y'' \ne 0$ there are no inflection points. Also, $y'' < 0$ for all $0 \le x \le 1$, and hence the function has the shape $\cap$ in its domain of definition. There are no asymptotes.

e) The function is defined for all x. Intersection with the axis at $(0,1)$. The function is continuous for all x. We have $y' = -\frac{2x}{(1+x^2)^2}$ and $y'' = \frac{2(3x^2-1)}{(1+x^2)^3}$. If $y' = 0$ then $x = 0$. For $x < 0$ we have $y' > 0$ and for $x > 0$ we have $y' < 0$. Hence $x = 0$ is a maximum point. If $y'' = 0$ then $x = \pm\frac{1}{\sqrt{3}}$. When $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$ then $y'' < 0$, and otherwise $y'' > 0$. Hence the function has an inflection point at $x = \pm\frac{1}{\sqrt{3}}$. There are no vertical asymptotes. The line $y = 0$ is an asymptote at $\pm\infty$.

f) The function is defined for all $x \ne 1$. Intersection with the axis at $(0,-1)$ and $(\frac{1}{2}, 0)$. The function is continuous for all $x \ne 1$. We have $y' = -\frac{2x}{(x-1)^3}$ and $y'' = \frac{4x+2}{(x-1)^4}$. Hence $y' = 0$ implies $x = 0$. For $x < 0$ we have $y' < 0$, for $0 < x < 1$ we have $y' > 0$, and for $x > 1$ we have $y' < 0$. Hence $x = 0$ is a minimum point. If $y'' = 0$ then $x = -\frac{1}{2}$. For $x < -\frac{1}{2}$ we have $y'' < 0$, for $-\frac{1}{2} < x < 1$ we have $y'' > 0$ and for $x > 1$ we have $y'' > 0$. Hence, the point $x = -\frac{1}{2}$ is an inflection point. The function has a vertical asymptote at $x = 1$ and $y = 0$ is a horizontal asymptote at $\pm\infty$.

g) The function is defined and continuous for all $x \ne \pm\sqrt{3}$. Intersection with the axis at $(0,0)$. We have $y' = \frac{x^2(9-x^2)}{(3-x^2)^2}$ and $y'' = \frac{2x(3x^2+27)}{(3-x^2)^3}$. If $y' = 0$ then $x = 0, \pm 3$. For $x < -3$ we have $y' < 0$, for $-3 < x < 3$ with $x \ne 0, \pm\sqrt{3}$ we have $y' > 0$, and for $x > 3$ we have $y' < 0$. Hence the points $x = -3$ is a minimum point and $x = 3$ is a maximum point. If $y'' = 0$ then $x = 0$. If $x < -\sqrt{3}$ then $y'' > 0$, if $-\sqrt{3} < x < 0$ then $y'' < 0$, if $0 < x < \sqrt{3}$ then $y'' > 0$, and if $x > \sqrt{3}$ then $y'' < 0$. Hence $x = 0$ is an inflection point. The function has a vertical asymptote at $x = \pm\sqrt{3}$ and $y = -x$ is a horizontal asymptote at $\pm\infty$.

h) The function is defined and continuous for all $x \ne 1$. Intersection with the axis at $(0,0)$. We have $y' = \frac{x^3(x^3-4)}{(x^3-1)^2}$ and $y'' = \frac{6x^2(x^3+2)}{(x^3-1)^3}$. If $y' = 0$ then $x = 0, \sqrt[3]{4}$. If $x < 0$ the $y' > 0$, if $0 < x < \sqrt[3]{4}$ and $x \ne 1$ then $y' < 0$, and if $x > \sqrt[3]{4}$ then $y' > 0$. Hence $x = 0$ is a maximum point and $x = \sqrt[3]{4}$ is a minimum point. If $y'' = 0$ then $x = 0, -\sqrt[3]{2}$. If $x < -\sqrt[3]{2}$ then $y'' > 0$, if $-\sqrt[3]{2} < x < 1$ and $x \ne 0$ then $y'' < 0$, and if $x > 1$ then $y'' > 0$. Hence $x = -\sqrt[3]{2}$ is an inflection point. The function has a vertical asymptote at $x = 1$ and $y = x$ is a horizontal asymptote at $\pm\infty$.

i) The function is defined and continuous for all x. Intersection with the axis at $(0,0)$. We have $y' = e^{-x}(1-x)$ and $y'' = e^{-x}(x-2)$. If $y' = 0$ then $x = 1$. If $x > 1$ the $y' < 0$, and if $x < 1$ then $y' > 0$. Hence $x = 1$ is a maximum point. If $y'' = 0$ then $x = 2$. If $x < 2$ then $y'' < 0$ and if $x > 2$ then $y'' > 0$. Hence $x = 2$ is an inflection point. The function has no vertical asymptotes, and $y = 0$ is an asymptote at ∞. At $-\infty$ there is no asymptote.

j) The function is defined and continuous for all $x > 1$. Intersection with the axis at $(0,0)$. We have $y' = \frac{x}{x+1}$ and $y'' = \frac{1}{(x+1)^2}$. If $y' = 0$ then $x = 0$. If $-1 < x < o$ then $y' < 0$ and if $x > 0$ then $y' > 0$. Hence $x = 0$ is a minimum point. Since $y'' \neq 0$ for all x, there are no inflection points. For all $x > -1$ we have $y'' > 0$. The function has a vertical asymptote at $x = -1$, and there are no horizontal asymptotes.

k) The function is defined and continuous for all x. Intersection with the axis at $(0,0)$. We have $y' = \frac{2x}{x^2+1}$ and $y'' = \frac{2(1-x^2)}{(x^2+1)^2}$. If $y' = 0$ then $x = 0$. If $x < 0$ then $y' < 0$ and if $x > 0$ then $y' > 0$. Hence $x = 0$ is a minimum point. If $y'' = 0$ then $x = \pm 1$. If $x < -1$ then $y'' < 0$, if $-1 < x < 1$ then $y'' > 0$ and if $x > 1$ then $y'' < 0$. Hence $x = \pm 1$ are both inflection points. There are no asymptotes.

l) The function is defined and continuous for all x. Intersection with the axis at $(0,0)$. We have $y' = 2xe^{-x^2}(1-x^2)$ and $y'' = 2e^{-x^2}(2x^4 - 5x^2 + 1)$. If $y' = 0$ then $x = 0, \pm 1$. If $x < -1$ then $y' > 0$, if $-1 < x < 0$ then $y' < 0$, if $0 < x < 1$ then $y' > 0$ and if $x > 1$ then $y' < 0$. Hence $x = \pm 1$ are maximum points and $x = 0$ is a minimum point. If $y'' = 0$ then $x \cong \pm 1.51, \pm 0.47$. If $x < -1.51$ then $y'' > 0$, if $-1.51 < x < -0.47$ then $y'' < 0$, if $-0.47 < x < 0.47$ then $y'' > 0$, if $0.47 < x < 1.51$ then $y'' < 0$ and if $x > 1.51$ then $y'' > 0$. Hence all the four points are inflection points. There are no vertical asymptotes. The line $y = 0$ is a horizontal asymptote at $\pm\infty$.

m) The function is defined and continuous for all $x > 0$. Intersection with the axis at $(1,0)$. We have $y' = \frac{1-\ln x}{x^2}$ and $y'' = \frac{2\ln x - 3}{x^3}$. If $y' = 0$ then $x = e$. If $x < e$ then $y' > 0$ and if $x > e$ then $y' < 0$. Hence $x = e$ is a maximum point. If $y'' = 0$ then $x = e^{3/2}$. If $x < e^{3/2}$ then $y'' < 0$ and if $x > e^{3/2}$ then $y'' > 0$. Hence $x = e^{3/2}$ is an inflection point. The line $x = 0$ is a vertical asymptote, and $y = 0$ is a horizontal asymptote at ∞.

n) The function is defined and continuous for all x. Intersection with the axis at $(0,0)$. We have $y' = x^2e^{-x}(3-x)$ and $y'' = xe^{-x}(x^2 - 6x + 6)$. If $y' = 0$ then $x = 0, 3$. If $x < 3$ then $y' > 0$, and if $x > 3$ then $y' < 0$. Hence, $x = 3$ is a maximum point. If $y'' = 0$ then $x = 0, 3 \pm \sqrt{3}$. If $x < 0$ then $y'' < 0$, if $0 < x < 3 - \sqrt{3}$ then $y'' > 0$, if $3 - \sqrt{3} < x < 3 + \sqrt{3}$ then $y'' < 0$ and if $x > 3 + \sqrt{3}$ then $y'' > 0$. Hence, all three points $x = 0, 3 \pm \sqrt{3}$ are inflection points. There are no vertical asymptotes, and $y = 0$ is an asymptote at ∞.