Calculus A for Economics

Solutions to Exercise Number 2

1) a) $(f \circ g)(x) = f(g(x)) = f(x^2) = 2x^2+5$; $(g \circ f)(x) = g(f(x)) = g(2x+5) = (2x+5)^2$.
b) $(f \circ g)(x) = f(g(x)) = \frac{1}{g(x)} = \frac{1}{\frac{1}{x}} = x$; $(g \circ f)(x) = g(f(x)) = \frac{1}{f(x)} = \frac{1}{\frac{1}{x}} = x$.
c) $(f \circ g)(x) = f(g(x)) = e^{g(x)+1} = e^{\ln x+1} = e \cdot e^{\ln x} = ex$; $(g \circ f)(x) = g(f(x)) = \ln(f(x)) = \ln(e^{x+1}) = x + 1$.

2) a) $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt{x})) = f(\frac{\sqrt{x}}{4}) = 4\frac{\sqrt{x}}{4} - 8 = \sqrt{x} - 8$.
b) $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt{x})) = f(e^{\sqrt{x}}) = \frac{1}{e^{\sqrt{x}}}$.
c) $(f \circ g \circ h)(x) = f(g(h(x))) = \ln(g(h(x))) = \ln((h(x))^2 + 3) = \ln(\frac{1}{x^2} + 3)$.

3) a) Write $y = x^2+1$. Interchange x and y to get $x = y^2+1$. Hence $y^2 = x-1$. Since we cannot recover y as a function of x, the function $f^{-1}(x)$ does not exists for all x. However, we can write $y = \sqrt{x-1}$ which is valid for $x \geq 1$. Thus in the domain $x \geq 1$ the function $y = \sqrt{x-1}$ is the inverse of $f(x)$. We can do a similar construction with $y = -\sqrt{x-1}$.
b) Write $y = \sqrt[3]{x^2+1}$. Consider $x = \sqrt[3]{y^2+1}$. This can be written as $y^2 = x^3 - 1$, or $y = \sqrt{x^3-1}$. This is defined only for $x \geq 1$. Thus $f^{-1}(x)$ does not exists for all x, but only when $x \geq 1$.
c) Write $y = \frac{2x+3}{x-1}$. Thus $x = \frac{2y+3}{y-1}$. Recovering y we get $y = \frac{x+3}{x-2}$. Thus $f^{-1}(x)$ exists for all $x \neq 1, 2$.
d) Write $y = 10^{x+1}$ and $x = 10^{y+1}$. Thus $y + 1 = \log_{10} x$ and $y = \log_{10} x - 1$. Thus $f^{-1}(x)$ exists for $x > 0$.
e) Write $y = 1+\ln(x+2)$. This is valid for $x > -2$. In this domain we have $x = 1+\ln(y+2)$ or $x - 1 = \ln(y + 2)$. Hence $y + 2 = e^{x-1}$ or $y = e^{x-1} - 2$.
f) Write $y = \frac{2^x}{1+2^x}$. Since 2^x is always positive, the function is defined for all x. Write $x = \frac{2^y}{1+2^y}$ or $x(1 + 2^y) = 2^y$. From this we get $x = 2^y(x - 1)$ or $2^y = \frac{x}{x-1}$. Hence $y = \log_2 \frac{x}{x-1}$. This function is defined for $x > 1$.

4) $f(f(x)) = \sqrt[n]{a - (f(x))^n} = \sqrt[n]{a - (\sqrt[n]{a - x^n})^n} = \sqrt[n]{a - (a - x^n)} = \sqrt[n]{x^n} = x$. From this we deduce that in the domain of definition of $f(x)$ we get $f^{-1}(x) = f(x)$. The domain of definition depends on n. For example if n is odd, it is all $x > 0$. If n is even, we have $0 < x < a$.

5) **a)** We must have $\frac{x}{4} > 0$. Hence $x > 0$ is the domain of definition.
b) We have $\frac{1-2x}{4} > 0$. Thus $1 - 2x > 0$ or $x < \frac{1}{2}$ is the domain of definition.
c) First we have $4 - x^2 \neq 0$ or $x \neq \pm 2$. Then we need $x^3 - x > 0$. This is the same as $(x-1)x(x+1) > 0$. The product of three numbers is positive if all numbers are positive, or if two of them are negative and the third is positive. Going over all possibilities we get $-1 < x < 0$ or $1 < x$. From this domain we need to eliminate $x = \pm 2$. Thus, the domain of definition is $-1 < x < 0$ or $1 < x < 2$ or $x > 2$.

6) See separate page.

7) See separate page.

8) We have

$$f(f(x)) = f\left(\frac{ax+b}{cx+d}\right) = \frac{a\left(\frac{ax+b}{cx+d}\right)+b}{c\left(\frac{ax+b}{cx+d}\right)+d} = \frac{(a^2+bc)x+b(a+d)}{c(a+d)x+cb+d^2}$$

This expression must equal to x. Thus we get the equality $(a^2+bc)x + b(a+d) = x(c(a+d)x + cb + d^2)$. Since this holds for all x then the coefficients of the corresponding powers of x must equal in both sides. Thus we get the three equations $b(a+d) = 0,\ c(a+d) = 0$ and $a^2 = d^2$. From the last equation we get two possibilities. First $a = -d$. In this case the other two equations are satisfied and we get as a solution the function $f(x) = \frac{ax+b}{cx-a}$. The second possibility is $a = d$. We may assume that they are not equal to zero, because this case was included in the case $a = -d$. Thus to satisfy the other two equations we get $b = c = 0$. Thus we get another option which is $f(x) = x$.

9) The equation $f(g(x)) = g(f(x))$ is equivalent to $g(x) + 1 = g(x+1)$. The function $g(x) = x$ is an example to a function which satisfies this last equation.

10) First notice that $f(x)$ is one to one. Indeed, suppose that $f(x_1) = f(x_2)$. If $x_1 \geq 0$ and $x_2 < 0$, then $-x_1^2 = 1 - x_2^3$ which is impossible since the left hand side is negative and the right hand side is positive. Therefore x_1 and x_2 are of the same sign. If both positive, then we get $-x_1^2 = -x_2^2$ which implies $x_1 = \pm x_2$. But since both numbers are of the same sign, we get $x_1 = x_2$. Similarly, if both numbers are negative we get $x_1 = x_2$. Hence $f^{-1}(x)$ exist. To find its inverse, consider $y = -x^2$. Interchanging x and y, we get $x = -y^2$, or $y = \sqrt{-x}$ which is well defined if $x \leq 0$. Similarly, let $y = 1 - x^3$. Interchanging x and y we get $x = 1 - y^3$ or $y = \sqrt[3]{1-x}$. This is defined for all x, however since the range of $1 - x^3$, when $x < 0$, is all $x > 1$, then this is the domain we consider for $y = \sqrt[3]{1-x}$. To summarize, we have

$$f^{-1}(x) = \begin{cases} \sqrt[3]{1-x}, & x > 1 \\ \sqrt{-x}, & x \leq 0 \end{cases}$$