# Calculus A for Economics

## Solutions to Exercise Number 3

**1)** Recall the definition of $\lim_{x \to x_0} f(x) = l$. Given an $\epsilon > 0$ we look for $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - l| < \epsilon$.
**a)** $|f(x) - l| = |(3x - 1) - 5| = 3|x - 2|$. Thus, the condition $|f(x) - l| < \epsilon$ is equivalent to $3|x - 2| < \epsilon$ or $|x - 2| < \frac{\epsilon}{3}$. Hence, if we choose $\delta = \frac{\epsilon}{3}$ then $|x - 3| < \delta = \frac{\epsilon}{3}$ implies $|f(x) - 5| < \epsilon$.
**b)** $|f(x) - l| = |(6x - 7) - 11| = 6|x - 3|$. Choose $\delta = \frac{\epsilon}{6}$. Then $|x - 3| < \delta = \frac{\epsilon}{6}$ implies $|f(x) - 11| < \epsilon$.
**c)** $|f(x) - l| = |x^2 - 4| = |x + 2||x - 2| \leq (2 + |x|)|x - 2|$ where the last inequality is obtained by the triangular inequality. Given $\epsilon$, we are looking for $\delta$ such that when $|x - 2| < \delta$, then $(2 + |x|)|x - 2| < \epsilon$. The condition $|x - 2| < \delta$ is equivalent to $-\delta < x - 2 < \delta$ or $2 - \delta < x < 2 + \delta$. Therefore, if we choose $\delta \leq 1$ then we have that $1 < x < 3$. Hence we have $|x| < 3$ if $2 - \delta < x < 2 + \delta$. In this case we would have $(2 + |x|)|x - 2| < 5|x - 2|$. If we further assume that $\delta \leq \frac{\epsilon}{5}$ then $|x - 2| < \delta = \frac{\epsilon}{5}$. Hence, if we choose $\delta \leq min\{1, \frac{\epsilon}{5}\}$ then if $|x - 2| < \delta$ then $|f(x) - 4| < \epsilon$.

**2) a)** $\lim_{x \to c}(f(x))^2 = \lim_{x \to c} f(x) \cdot \lim_{x \to c} f(x) = 2 \cdot 2 = 4$.
**b)** $\lim_{x \to c} \frac{h(x)}{f(x)} = \frac{\lim_{x \to c} h(x)}{\lim_{x \to c} f(x)} = \lim_{x \to c} \frac{h(x)}{f(x)} = \frac{0}{2} = 0$.
**c)** $\lim_{x \to c} \frac{1}{f(x) - g(x)} = \frac{1}{\lim_{x \to c} f(x) - \lim_{x \to c} g(x)} = \frac{1}{2 - (-1)} = \frac{1}{3}$.

**3)** Assume that $\lim_{x \to c} f(x)$ exists and is equal to $m$. Then, we have $\lim_{x \to c} f(x)g(x) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) = m \cdot 0 = 0$. On the other hand $\lim_{x \to c} f(x)g(x) = \lim_{x \to c} 1 = 1$. Thus we derived a contradiction to the assumption that $\lim_{x \to c} f(x)$ exists.

**4) a)**

$$\lim_{x \to 2} \frac{x^2 + x + 1}{x^2 + 2x} = \frac{2^2 + 2 + 1}{2^2 + 2 \cdot 2} = \frac{7}{8}$$

**b)**

$$\lim_{x \to 2} \frac{x - 2}{x^2 - 4} = \lim_{x \to 2} \frac{x - 2}{(x - 2)(x + 2)} = \lim_{x \to 2} \frac{1}{x + 2} = \frac{1}{2 + 2} = \frac{1}{4}$$

**c)**

$$\lim_{x \to 1} \frac{x^2 - 2x + 1}{x^3 - x} = \lim_{x \to 1} \frac{(x - 1)^2}{x(x^2 - 1)} = \lim_{x \to 1} \frac{x - 1}{x(x + 1)} = 0$$

**d)**

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to 0}\frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{x(\sqrt{1+x}+1)}=\lim_{x\to 0}\frac{(\sqrt{1+x})^2-1^2}{x(\sqrt{1+x}+1)}=$$
$$\lim_{x\to 0}\frac{x}{x(\sqrt{1+x}+1)}=\lim_{x\to 0}\frac{1}{(\sqrt{1+x}+1)}=\frac{1}{2}$$

**e)**

$$\lim_{x\to 5}\frac{\sqrt{x-1}-2}{x-5}=\lim_{x\to 5}\frac{(\sqrt{x-1}-2)(\sqrt{x-1}+2)}{(x-5)(\sqrt{x-1}+2)}\lim_{x\to 5}\frac{(\sqrt{x-1})^2-2^2}{(x-5)(\sqrt{x-1}+2)}=$$
$$\lim_{x\to 5}\frac{x-5}{(x-5)(\sqrt{x-1}+2)}=\lim_{x\to 5}\frac{1}{\sqrt{x-1}+2}=\frac{1}{4}$$

**5)** Suppose that $\lim_{x\to c}\frac{f(x)}{g(x)}$ exists and is equal to $m$. Then $\lim_{x\to c} f(x)=\lim_{x\to c}\frac{f(x)}{g(x)}g(x)$ $=\lim_{x\to c}\frac{f(x)}{g(x)}\lim_{x\to c} g(x)=m\cdot 0=0$. On the other hand, we are given that $\lim_{x\to c} f(x)=l$ and $l\neq 0$. From the uniqueness of the limit we get a contradiction. Thus, $\lim_{x\to c}\frac{f(x)}{g(x)}$ does not exist.

**6)** Let $f(x)=x$ and $g(x)=x^2-1$. Then $\lim_{x\to 1} f(x)=\lim_{x\to 1} x=1\neq 0$. Also, $\lim_{x\to 1} g(x)=\lim_{x\to 1}(x^2-1)=0$. Therefore, it follows from exercise **5)** that $\lim_{x\to 1}\frac{x}{x^2-1}$ does not exist.

**7) a)** $|f(x)-l|=|x^4-2^4|=|x^2-2^2||x^2+2^2|=|x-2||x+2|(x^2+4)$. Given $\epsilon$ we look for a $\delta$ such that if $|x-2|<\delta$, then $|x-2||x+2|(x^2+4)<\epsilon$. The condition $|x-2|<\delta$ is equivalent to $2-\delta<x<2+\delta$. Therefore, if we require that $\delta<1$ then we get $|x|<3$ if $2-\delta<x<2+\delta$ . In this domain, using the triangular inequality we get $|x-2||x+2|(x^2+4)\leq|x-2|(|x|+2)(x^2+4)\leq 65|x-2|$. Thus if we choose $\delta=min\{1,\frac{\epsilon}{65}\}$ we get the result.

**b)** $|f(x)-l|=|\frac{1}{x}-\frac{1}{2}|=\frac{|x-2|}{2|x|}$. Given $\epsilon$ we look for a $\delta$ such that if $|x-2|<\delta$, then $\frac{|x-2|}{2|x|}<\epsilon$. The condition $|x-2|<\delta$ is equivalent to $2-\delta<x<2+\delta$. Assume that $\delta\leq 1$. Then $1<|x|$ if $2-\delta<x<2+\delta$. Hence, for these values of $x$ we get $\frac{1}{|x|}<1$. Hence $\frac{|x-2|}{2|x|}<\frac{|x-2|}{2}$. Choosing $\delta\leq min\{1,2\epsilon\}$, we get the result.

**8)** An example: $f(x)=\frac{1}{x}$ and $g(x)=-\frac{1}{x}$.

**9)** It follows from a Theorem proved in class, that given two functions $F(x)$ and $G(x)$ such that $\lim_{x\to 0} F(x)$ and $\lim_{x\to 0} G(x)$ exist then the limit $\lim_{x\to 0}(F(x)-G(x))$ exist. Suppose that $\lim_{x\to 0}(f(x)+g(x))$ does exist. Apply the above to the case when $F(x)=f(x)+g(x)$ and $G(x)=f(x)$. Hence the limit $\lim_{x\to 0}(F(x)-G(x))=\lim_{x\to 0}(f(x)+g(x)-f(x))=$ $\lim_{x\to 0} g(x)$ exists. This is a contradiction. Hence, the limit $\lim_{x\to 0}(f(x)+g(x))$ does not exist.