Calculus A for Economics

Solutions to Exercise Number 4

1) Use the Sandwich Theorem. We have $\lim_{x\to 0}|x| = 0$ and $\lim_{x\to 0}(-|x|) = 0$. Hence by the Sandwich Theorem $\lim_{x\to 0} f(x) = 0$.

2) Use the Sandwich Theorem. For all $x \neq 0$, the condition $\frac{|f(x)|}{|x|} \leq 1$ is equivalent to $|f(x)| \leq |x|$ which is equivalent to $-|x| \leq f(x) \leq |x|$. Since $\lim_{x\to 0}(-|x|) = \lim_{x\to 0}|x| = 0$ it follows from the Sandwich Theorem that $\lim_{x\to 0} f(x) = 0$.

3) $|\sqrt{x} - \sqrt{c}| = |\frac{(\sqrt{x}-\sqrt{c})(\sqrt{x}+\sqrt{c})}{\sqrt{x}+\sqrt{c}}| = \frac{|x-c|}{|\sqrt{x}+\sqrt{c}|}$. Since $\sqrt{x}$ and $\sqrt{c}$ are both positive we may drop the absolute value and get $\frac{|x-c|}{|\sqrt{x}+\sqrt{c}|} = \frac{|x-c|}{\sqrt{x}+\sqrt{c}} < \frac{|x-c|}{\sqrt{c}}$ where the last inequality follows from the fact that $\sqrt{x} + \sqrt{c} > \sqrt{c}$, and hence $\frac{1}{\sqrt{x}+\sqrt{c}} < \frac{1}{\sqrt{c}}$. Therefore we proved that $|\sqrt{x} - \sqrt{c}| < \frac{|x-c|}{\sqrt{c}}$. Given ϵ choose $\delta = \sqrt{c}\epsilon$. In this case, if $|x - c| < \delta = \sqrt{c}\epsilon$, then $|\sqrt{x} - \sqrt{c}| < \frac{|x-c|}{\sqrt{c}} < \epsilon$.

4) a) $\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} x^2 = 1$.
b) $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} x = 2$.
c) To compute the limit at four, we consider the two one sided limits. First $\lim_{x\to 4^-} g(x) = \lim_{x\to 4^-} x = 4$. The other one is $\lim_{x\to 4^+} g(x) = \lim_{x\to 4^+}(4 - x) = 0$. Since they are not equal there is no limit at the point four.

5) a) $\lim_{x\to 2^+} \frac{x^2}{x+2} = \frac{2^2}{2+2} = 1$.
b) This limit does not exist. The function $\sqrt{x}$ is defined only for $x \geq 0$. Hence there is no sense in computing the limit from the left at zero.
c) We have

$$\sqrt{|x| - x} = \begin{cases} \sqrt{x - x}, & x \geq 0 \\ \sqrt{-x - x}, & x < 0 \end{cases} = \begin{cases} 0, & x \geq 0 \\ \sqrt{-2x}, & x < 0 \end{cases}$$

From this we get $\lim_{x\to 1^-} \sqrt{|x| - x} = \lim_{x\to 1^-} 0 = 0$.

6) a) $\lim_{x\to\infty}(\sqrt{x + a} - \sqrt{x}) = \lim_{x\to\infty} \frac{(\sqrt{x+a}-\sqrt{x})(\sqrt{x+a}+\sqrt{x})}{\sqrt{x+a}+\sqrt{x}}$. This last limit is equal to $\lim_{x\to\infty} \frac{x+a-x}{\sqrt{x+a}+\sqrt{x}} = a\lim_{x\to\infty} \frac{1}{\sqrt{x+a}+\sqrt{x}} = 0$. The last equality follows from the following. Suppose that there is a positive number M such that $f(x) > M$ ($f(x) < -M$) when $x \to \infty$

($x \to -\infty$). Then $\lim_{x\to\infty} \frac{1}{f(x)} = 0$ ($\lim_{x\to -\infty} \frac{1}{f(x)} = 0$). Directly, this can be seen as follows. We have $\lim_{x\to\infty} \frac{1}{\sqrt{x+a}+\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}(\sqrt{1+\frac{a}{x}}+1)} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} \lim_{x\to\infty} \frac{1}{(\sqrt{1+\frac{a}{x}}+1)}$. The first limit in this product is zero whereas the second one is one. Hence the value of the limit is zero.

b) $\lim_{x\to\infty}(\sqrt{x^2+1}-x) = \lim_{x\to\infty} \frac{(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x}$. This is equal to $\lim_{x\to\infty} \frac{1}{x(\sqrt{1+\frac{1}{x^2}}+1)} = 0$.

c) The limit $\lim_{x\to-\infty}(\sqrt{x^2+1}-x)$ does not exist. Indeed, when $x \to -\infty$ then $\sqrt{x^2+1}-x$ is greater than any positive number M. One may try to proceed as in part **b)**, but this will not work. Indeed, we obtain in this case $\lim_{x\to\infty} \frac{1}{x(-\sqrt{1+\frac{1}{x^2}}+1)} = \lim_{x\to\infty} \frac{1}{x} \lim_{x\to\infty} \frac{1}{1-\sqrt{1+\frac{1}{x^2}}}$. The first limit in this product is zero, but the second limit does not exist, so we cant say much.

d) $\lim_{x\to\infty}(\sqrt{x^2-2x-1}-\sqrt{x^2-7x+3}) = \lim_{x\to\infty} \frac{(\sqrt{x^2-2x-1}-\sqrt{x^2-7x+3})(\sqrt{x^2-2x-1}+\sqrt{x^2-7x+3})}{\sqrt{x^2-2x-1}+\sqrt{x^2-7x+3}}$
$= \lim_{x\to\infty} \frac{5x-4}{\sqrt{x^2-2x-1}+\sqrt{x^2-7x+3}} = \lim_{x\to\infty} \frac{x\left(5-\frac{4}{x}\right)}{x\sqrt{1-\frac{2}{x}-\frac{1}{x^2}}+x\sqrt{1-\frac{7}{x}+\frac{3}{x^2}}} =$
$= \lim_{x\to\infty} \frac{5-\frac{4}{x}}{\sqrt{1-\frac{2}{x}-\frac{1}{x^2}}+\sqrt{1-\frac{7}{x}+\frac{3}{x^2}}} = \frac{5}{1+1} = \frac{5}{2}$

e)

$$\lim_{x\to\infty} \frac{1+x-3x^3}{1+x^2+3x^3} = \lim_{x\to\infty} \frac{x^3(\frac{1}{x^3}+\frac{1}{x^2}-3)}{x^3(\frac{1}{x^3}+\frac{1}{x}+3)} = \lim_{x\to\infty} \frac{\frac{1}{x^3}+\frac{1}{x^2}-3}{\frac{1}{x^3}+\frac{1}{x}+3} = \frac{0+0-3}{0+0+3} = -1$$

f) $\lim_{x\to\infty}\left(\frac{x^3}{x^2+1}-x\right) = \lim_{x\to\infty} \frac{x^3-x(1+x^2)}{1+x^2} = \lim_{x\to\infty} \frac{-x}{1+x^2} = \lim_{x\to\infty} \frac{x^2\left(\frac{-x}{x^2}\right)}{x^2\left(\frac{1}{x^2}+1\right)} =$ $\lim_{x\to\infty} \frac{-\frac{1}{x}}{\left(\frac{1}{x^2}+1\right)} = \frac{0}{0+1} = 0$.

g) $\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4} = \lim_{x\to 0} \frac{(\sqrt{x^2+1}-1)(\sqrt{x^2+1}+1)(\sqrt{x^2+16}+4)}{(\sqrt{x^2+16}-4)(\sqrt{x^2+1}+1)(\sqrt{x^2+16}+4)} = \lim_{x\to 0} \frac{x^2(\sqrt{x^2+16}+4)}{x^2(\sqrt{x^2+1}+1)} = \frac{8}{2} = 4$

h)

$$\lim_{x\to\infty} \frac{x^4-5x}{x^2-3x+1} = \lim_{x\to\infty} \frac{x^4(1-\frac{5}{x^3})}{x^4(\frac{1}{x^2}-\frac{3}{x^3}+\frac{1}{x^4})} = \lim_{x\to\infty} \frac{1-\frac{5}{x^3}}{\frac{1}{x^2}-\frac{3}{x^3}+\frac{1}{x^4}}$$

As $x \to \infty$ the top part tends to one, but the bottom part tends to zero. Hence the limit does not exist.

i)

$$\lim_{x\to 1} \frac{(x-1)\sqrt{2-x}}{x^2-1} = \lim_{x\to 1} \frac{(x-1)\sqrt{2-x}}{(x-1)(x+1)} = \lim_{x\to 1} \frac{\sqrt{2-x}}{x+1} = \frac{1}{2}$$

j) We consider two cases. First, assume that $0 < a < 1$. In this case $\lim_{x\to\infty} a^x = 0$. Hence $\lim_{x\to\infty} \frac{a^x}{a^x+1} = \frac{0}{0+1} = 0$. If $a > 1$ then $\lim_{x\to\infty} a^{-x} = 0$ and hence $\lim_{x\to\infty} \frac{a^x}{a^x+1} =$ $\lim_{x\to\infty} \frac{a^x}{a^x(1+\frac{1}{a^x})} = \lim_{x\to\infty} \frac{1}{1+a^{-x}} = \frac{1}{1+0} = 1$.

k) Assume first that $0 < a < 1$. Then $\lim_{x\to\infty} \frac{a^x-a^{-x}}{a^x+a^{-x}} = \lim_{x\to\infty} \frac{a^{-x}(a^{2x}-1)}{a^{-x}(a^{2x}+1)} = \lim_{x\to\infty} \frac{a^{2x}-1}{a^{2x}+1} =$ $\frac{0-1}{0+1} = -1$. If $a > 1$ then $\lim_{x\to\infty} \frac{a^x-a^{-x}}{a^x+a^{-x}} = \lim_{x\to\infty} \frac{a^x(1-a^{-2x})}{a^x(1+a^{-2x})} = \frac{1-0}{1+0} = 1$.

7) The condition $-1 \leq f(x) \leq 1$ is equivalent to $0 \leq |f(x)| \leq 1$. Assuming that $x \neq 0$ we divide by $|x|$ and we get $0 \leq \frac{|f(x)|}{|x|} \leq \frac{1}{|x|}$. Since $\lim_{x \to \infty} \frac{1}{|x|} = 0$, the result follows from the Sandwich Theorem.