Calculus A for Economics

Solutions to Exercise Number 5

1) We need to check that $\lim_{x\to x_0} f(x) = f(x_0)$.
a) $\lim_{x\to x_0} f(x) = \lim_{x\to 2}((x-3)^2 - x + 2) = 1 - 2 + 2 = 1 = f(2)$. Hence $f(x)$ is continuous at $x_0 = 2$.
b) $\lim_{x\to x_0} f(x) = \lim_{x\to -1} \frac{x-2}{2x+3} = \frac{-1-2}{-2+3} = -3 = f(-1)$. Hence $f(x)$ is continuous at $x_0 = -1$.

2) a) We will use the fact that a sum, product and composition of continuous functions is also a continuous function. The function $(x-2)^3 + 5$ is a polynomial, and hence it is continuous for all x. The function $\sqrt{x}$ is continuous for $x > 0$. Hence $\sqrt{(x-2)^3 + 5}$ is continuous at 2 if it is defined near that point. The domain of definition is $x \geq 2 + \sqrt[3]{-5}$ which clearly includes the point 2 and a neighbor of that point.
b) We use the definition of the limit as stated in exercise **1)**. To check that $\lim_{x\to 2} f(x)$ exists we check the two one side limits. First, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(x^2+4) = 8$. Then $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^3 = 8$. Thus $\lim_{x\to 2} f(x) = 8$. We have $f(2) = 2^3 = 8$. Hence $\lim_{x\to 2} f(x) = f(2)$ and the function is continuous at 2.
c) Since $f(x)$ is not defined at 2, it is clearly not continuous at that point.
d) We check the one side limits. We have $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{1}{x-2}$. This limit does not exist, and hence, eventhough the function is defined at 2, it is not continuous there.

3) a) Since $\frac{1}{x}$ is not defined at $x = 0$, then $f(x)$ is not continuous at that point. For all $x \neq 0$ the function $\frac{1}{x}$ is continuous, and 2^x is also continuous for all x. Hence $2^{\frac{1}{x}}$, being the composite function of $\frac{1}{x}$ and 2^x, is continuous for all $x \neq 0$. Clearly then $1 + 2^{\frac{1}{x}}$ is continuous for all $x \neq 0$. Since $1 + 2^{\frac{1}{x}} > 0$ for all $x \neq 0$, it follows that $f(x) = \frac{1}{1+2^{\frac{1}{x}}}$ is continuous for all $x \neq 0$.
b) The function $f(x)$ is continuous for all x. Indeed, since $|x| + 1 > 0$ for all x, and $|x|$ is continuous for all x, then $\frac{1}{|x|+1}$ is continuous for all x. The function $\frac{x^2}{2}$ is a polynomial, and hence a continuous function. Thus $f(x)$ is continuous for all x.

4) Since $x + 1$ and $3 - ax^2$ are polynomials, they are continuous for all x. Thus, the only point where $f(x)$ may not be continuous is at $x = 1$. We have $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x +$

$1) = 2$. Also, $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(3 - ax^2) = 3 - a$. Therefore, $\lim_{x\to 1} f(x)$ will exist only if we have $2 = 3 - a$, or $a = 1$. For this choice of a, we have $\lim_{x\to 1} f(x) = f(1)$. Hence $f(x)$ is continuous for all x if and only if $a = 1$.

5) The only possible problematic points are $x = 2$ and $x = 3$. Indeed, for all other values of x, the function is continuous. We have $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2-4}{x-2} = \lim_{x\to 2^-}(x+2) = 4$. Also, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(Ax+B) = 2A+B$. $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(Ax+B) = 3A+B$, and $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{x-3}{x^2-9} = \lim_{x\to 3^+} \frac{1}{x+3} = \frac{1}{6}$. Thus, $f(x)$ will be continuous at $x = 2$ if and only if $4 = 2A+B$, and at $x = 3$ if and only if $\frac{1}{6} = 3A+B$. Solving these two equations we get $A = -\frac{23}{6}$ and $B = \frac{35}{11}$.

6) The function $y = 3^x$ is continuous for all x. Therefore, if $f(x)$ is any given function then $\lim_{x\to\infty} 3^{f(x)} = 3^{\lim_{x\to\infty} f(x)}$. Hence, to compute $\lim_{x\to\infty} 3^{\frac{x^2-1}{2x^2+1}}$ we first compute $\lim_{x\to\infty} \frac{x^2-1}{2x^2+1} = \lim_{x\to\infty} \frac{1-\frac{1}{x^2}}{2+\frac{1}{x^2}} = \frac{1}{2}$. Hence $\lim_{x\to\infty} 3^{\frac{x^2-1}{2x^2+1}} = 3^{1/2}$.
b) Similarly as in **a)** we have, $\lim_{x\to 1} \ln \frac{x^2-1}{x-1} = \ln \lim_{x\to 1} \frac{x^2-1}{x-1} = \ln \lim_{x\to 1}(x+1) = \ln 2$.

7) Let $f(x) = x^4 - 2x - 3$. The function is continuous for all x. Also $f(0)f(2) = (-3)\cdot 9 < 0$. Therefore by the stated Theorem $f(x)$ has a root in the interval $[0, 2]$.

8) The function $f(x)$ is continuous for all x. Therefore to obtain the result, we need to find an interval $[a, b]$ such that $f(a)f(b) \le 0$. Choose $a = -1$ and $b = 1$. Then $f(-1)f(1) = 3 \cdot (-1) < 0$. Apply the Theorem.

9) Given $f(a) < d < f(b)$ define $g(x) = f(x) - d$. It is given that $f(x)$ is continuous in the interval $[a, b]$. Hence $g(x)$ is also continuous in that interval. Since $f(a) < d$ then $g(a) = f(a) - d < 0$. Since $d < f(b)$ then $g(b) = f(b) - d > 0$. hence $g(a)g(b) < 0$. Thus, it follows from the Theorem that there is a point $c \in [a, b]$ such that $g(c) = 0$. Hence $f(c) = d$.

10) Consider the function

$$f(x) = \begin{cases} -x - 2, & 0 \le x < 1 \\ x, & 1 \le x \le 2 \end{cases}$$

Let $[a, b] = [0, 2]$. Then $f(0) = -2 < 0$ and $f(2) = 2 > 0$. Also, for any $x \in [0, 2]$ we have $f(x) \ne 0$. There is no contradiction to the Theorem because it is easy to check that $f(x)$ is not continuous at $x = 1$.

11) Assume that the claim about the maximal point is not true. This means that there is a point $a < c < b$ such that $f(c)$ is the maximal value that the function $f(x)$ obtains in $[a,b]$. Since $f(x)$ is one to one, it follows that $f(a) < f(c)$ and $f(b) < f(c)$. Clearly the function $f(x)$ is continuous in both intervals $[a,c]$ and $[c,b]$. Choose a point d such that $f(a) < d < f(c)$ and $f(b) < d < f(c)$. Such a d exists because $f(a) < f(c)$ and $f(b) < f(c)$. Applying exercise **9)** twice, once for the interval $[a,c]$ and then for $[c,b]$, we can find $a \leq x_1 < c$ and $c < x_2 \leq b$ such that $f(x_1) = d$ and $f(x_2) = d$. Since $x_1 \neq x_2$ we obtained a contradiction to the fact that $f(x)$ is one to one. Thus, the maximal value of $f(x)$ is obtained at one of the end points a or b. The argument for the minimal value is the same.

12) We need to prove that for any point x_0 we have $\lim_{x\to x_0} f(x) = f(x_0)$. We know that $f(x+y) = f(x) + f(y)$ for all x and y. Plug $x = y = 0$ to get $f(0) = 2f(0)$ and hence $f(0) = 0$. Also, plug $y = -x$ to get $f(0) = f(x) + f(-x)$ and since $f(0) = 0$ we get $f(-x) = -f(x)$. In other words $f(x)$ is an odd function. Plug $y = -x_0$ to get $f(x - x_0) = f(x) + f(-x_0) = f(x) - f(x_0)$, where the last equality follows from the fact that $f(x)$ is odd. Hence $f(x) = f(x - x_0) + f(x_0)$. Taking the limit as x goes to x_0 we have $\lim_{x\to x_0} f(x) = \lim_{x\to x_0}(f(x - x_0) - f(x_0))$. Notice that we need to prove that $\lim_{x\to x_0} f(x)$ exists. We will prove that once we prove that the two limits $\lim_{x\to x_0} f(x - x_0)$ and $\lim_{x\to x_0} f(x_0)$ exists. The last one clearly exists and is equal to $f(x_0)$. As for the first one, we have $\lim_{x\to x_0} f(x - x_0) = \lim_{z\to 0} f(z) = f(0) = 0$. The first equality follows from the change of variables $z = x - x_0$. Clearly then $x \to x_0$ implies $z \to 0$. The second equality follows from the fact that $f(x)$ is continuous at zero. The last equality follows from $f(0) = 0$. Hence, the identity $\lim_{x\to x_0} f(x) = \lim_{x\to x_0}(f(x - x_0) - f(x_0))$, implies that $\lim_{x\to x_0} f(x)$ exists, and also that $\lim_{x\to x_0} f(x) = f(x_0)$. Hence $f(x)$ is continuous at $x = x_0$.