Calculus A for Economics

Solutions to Exercise Number 6

1) a)

$$f'(-1)=\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}=\lim_{h\to 0}\frac{(3(-1+h)+5)-(3(-1)+5)}{h}=\lim_{h\to 0}\frac{3h}{h}=3$$

b)

$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=\lim_{h\to 0}\frac{(x_0+h-1)^2-(x_0-1)^2}{h}=$$

$$=\lim_{h\to 0}\frac{h(2x_0-2+h)}{h}=2(x_0-1)$$

c)

$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=\lim_{h\to 0}\frac{\frac{1}{\sqrt{x_0+h-1}}-\frac{1}{\sqrt{x_0-1}}}{h}=\lim_{h\to 0}\frac{\sqrt{x_0-1}-\sqrt{x_0+h-1}}{h\sqrt{x_0-1}\sqrt{x_0+h-1}}=$$

$$=\lim_{h\to 0}\frac{(\sqrt{x_0-1}-\sqrt{x_0+h-1})(\sqrt{x_0-1}+\sqrt{x_0+h-1})}{h(\sqrt{x_0-1}\sqrt{x_0+h-1})(\sqrt{x_0-1}+\sqrt{x_0+h-1})}=$$

$$=\lim_{h\to 0}\frac{x_0-1-x_0-h+1}{h(\sqrt{x_0-1}\sqrt{x_0+h-1})(\sqrt{x_0-1}+\sqrt{x_0+h-1})}=$$

$$=\lim_{h\to 0}\frac{-1}{(\sqrt{x_0-1}\sqrt{x_0+h-1})(\sqrt{x_0-1}+\sqrt{x_0+h-1})}=-\frac{1}{2(x-1)\sqrt{x-1}}=-\frac{1}{\sqrt[3]{(x-1)^2}}$$

2) a) We have

$$f(x)=|2x-5|=\begin{cases}2x-5, & x\geq \frac{5}{2}\\ -(2x-5), & x<\frac{5}{2}\end{cases}$$

It is clear that $f(x)$ is differentiable for all $x\neq\frac{5}{2}$. To check $x=\frac{5}{2}$ we compute the two one side derivatives. First $\lim_{h\to 0^+}\frac{f(\frac{5}{2}+h)-f(\frac{5}{2})}{h}=\lim_{h\to 0^+}\frac{2h-0}{h}=2$ where the first equality follows from the fact that when $h>0$ then $\frac{5}{2}+h>\frac{5}{2}$ and hence $f(\frac{5}{2}+h)=2(\frac{5}{2}+h)-5=2h$. Similarly, $\lim_{h\to 0^-}\frac{f(\frac{5}{2}+h)-f(\frac{5}{2})}{h}=\lim_{h\to 0^-}\frac{-2h-0}{h}=-2$. From this we deduce that the two side derivatives are not equal and hence $f(x)$ is not differentiable at $x=\frac{5}{2}$.

b) The function $f(x)$ is defined for all $x\geq -1$. When $x>-1$, we have

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h+1}-\sqrt{x+1}}{h} = \lim_{h\to 0} \frac{(\sqrt{x+h+1}-\sqrt{x+1})(\sqrt{x+h+1}+\sqrt{x+1})}{h(\sqrt{x+h+1}+\sqrt{x+1})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h+1}+\sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}$. At the point $x = -1$ we can check if $f(x)$ is differentiable from the right. To do that we compute $\lim_{h\to 0^+} \frac{f(x+h)-f(h)}{h} = \lim_{h\to 0^+} \frac{\sqrt{h}-0}{h} = \lim_{h\to 0^+} \frac{1}{\sqrt{h}}$ and this limit does not exists. Thus $f(x)$ is not differentiable from the right at $x = -1$.

3) We have $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f(x)-f(0)}{g(x)-g(0)}$. This follows from the fact that $f(0) = g(0) = 0$. Hence, multiplying and dividing by $\frac{1}{x}$, when $x \neq 0$, we get $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\frac{f(0+x)-f(0)}{x}}{\frac{g(0+x)-g(0)}{x}} = \frac{f'(0)}{g'(0)}$.

4) $\lim_{x\to 0} \frac{(x+a)f(a)-af(x+a)}{x} = \lim_{x\to 0} \frac{xf(a)-a(f(x+a)-f(a))}{x} = \lim_{x\to 0} \left(f(a) - a\frac{f(x+a)-f(a)}{x}\right) = f(a) - a\lim_{x\to 0} \frac{f(x+a)-f(a)}{x} = f(a) - af'(a)$.

5) We need to prove that if $f(-x) = f(x)$ then $f'(-x) = -f'(x)$ We use the definition of the derivative. $f'(-x) = \lim_{h\to 0} \frac{f(-x+h)-f(-x)}{h} = \lim_{h\to 0} \frac{f(x-h)-f(x)}{h}$. Here we used the fact that $f(x)$ is even, and hence $f(-x+h) = f(x-h)$ and $f(-x) = f(x)$. Next we define $t = -h$. Clearly if $h \to 0$ then $t \to 0$. Therefore we get $f'(-x) = \lim_{h\to 0} \frac{f(x-h)-f(x)}{h} = \lim_{t\to 0} \frac{f(x+t)-f(x)}{-t} = -\lim_{t\to 0} \frac{f(x+t)-f(x)}{t} = -f'(x)$.
Remark: This exercise is easier to solve if you know how to differentiate composition of functions.

6) a) $y' = 15x^2 - 3x^2$.
b) $y' = (x^2+1)'(x^5-3) + (x^2+1)(x^5-3)' = 2x(x^5+1) + 5x^4(x^2+1) = 7x^6 + 5x^4 - 6x$.
c) Write $y = (\sqrt{x}+1)(\frac{1}{\sqrt{x}} - 1) = 1 - \sqrt{x} + \frac{1}{\sqrt{x}} - 1 = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$. Hence $y' = \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}} = \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x^3}}$.
d) Write $y = x^2 - \frac{1}{x^{1/2}} + \frac{\sqrt[3]{x^2}}{3x} = x^2 - x^{-1/2} + \frac{1}{3}x^{-1/3}$. Then $y' = 2x + \frac{1}{2}x^{-3/2} - \frac{1}{9}x^{-4/3} = 2x + \frac{1}{2\sqrt{x^3}} - \frac{1}{9\sqrt[3]{x^4}}$.
e)

$$y' = \frac{(x^3-2x)'(x^2+x+1) - (x^3-2x)(x^2+x+1)'}{(x^2+x+1)^2} =$$

$$= \frac{(3x^2-2)(x^2+x+1) - (x^3-2x)(2x+1)}{(x^2+x+1)^2} = \frac{x^4+2x^3+5x^2-2}{(x^2+x+1)^2}$$

f) We differentiate by the quotient rule. $y' = \frac{0\cdot(x^2+x+1)-1(x^2+x+1)'}{(x^2+x+1)^2}$. Here the zero comes from the derivative of the constant function 1. Thus, $y' = -\frac{2x+1}{(x^2+x+1)^2}$.
g) Write $y = \frac{a}{am+bm^2}x + \frac{b}{am+bm^2}x^2$. Then $y' = \frac{a}{am+bm^2} + \frac{2b}{am+bm^2}x$.
h)

$$y' = \frac{(3-x)'(1-x^2)(1-2x^3) - [(1-x^2)(1-2x^3)]'(3-x)}{(1-x^2)^2(1-2x^3)^2} =$$

$$= \frac{-(1-x^2)(1-2x^3) - [-2x(1-2x^3) - 6x^2(1-x^2)](3-x)}{(1-x^2)^2(1-2x^3)^2} =$$
$$= \frac{8x^5 - 30x^4 - 4x^3 + 17x^2 + 6x - 1}{(1-x^2)^2(1-2x^3)^2}$$

7) If $f(x) = x + \frac{1}{x}$ then $f'(x) = 1 - \frac{1}{x^2}$. Hence $f(5) = \frac{26}{5}$ and $f'(5) = 1 - \frac{1}{25} = \frac{24}{25}$. Also $-\frac{1}{f'(5)} = -\frac{25}{24}$. Hence, the equation of the tangent line at $x_0 = 5$ is $y - \frac{26}{5} = \frac{24}{25}(x-5)$ and the equation of the normal line is $y - \frac{26}{5} = -\frac{25}{24}(x-5)$.

8) Since the tangent line to $f(x) = (x-1)^3$ and the line $y = x$ are parallel, they have the same slope. Hence $f'(x) = 1$, or $3(x-1)^2 = 1$. Therefore we get two possible solutions $x_1 = 1 + \frac{1}{\sqrt{3}}$ and $x_2 = 1 - \frac{1}{\sqrt{3}}$. For these values of x we have $f(x_1) = \frac{1}{3\sqrt{3}}$ and $f(x_2) = -\frac{1}{3\sqrt{3}}$. Thus, the two points are $(1 + \frac{1}{\sqrt{3}}, \frac{1}{3\sqrt{3}})$ and $(1 - \frac{1}{\sqrt{3}}, -\frac{1}{3\sqrt{3}})$.

9) Since the tangent to $f(x) = \frac{1}{x}$ is perpendicular to $y = x$ we get the equation $-\frac{1}{f'(x)} = 1$, or $-\frac{1}{\frac{-1}{x^2}} = 1$, or $x^2 = 1$. Hence we get the solutions $x_1 = 1$ and $x_2 = -1$. Thus, the two points are $(1,1)$ and $(-1,-1)$.

10) First, it is given that the point $(1,3)$ is on $f(x) = Ax^2 + Bx + C$. Hence $3 = A + B + C$. Second, it is given that $y = 8 - 4x$ is the tangent line to $f(x)$ at the point $(2,0)$. This tells us two things. First, that the point $(2,0)$ is on $f(x)$. Hence $0 = 4A + 2B + C$. Second, we have $f'(2) = -4$, or $4A + B = -4$. We got three equations: $A + B + C = 3$; $\quad 4A + 2B + C = 0$; $\quad 4A + B = -4$. Solving this system we get $A = -1,\ B = 0$ and $C = 4$. Thus the equation of the parabola is $y = -x^2 + 4$.

11) $F'(x) = 1 \cdot (x-2)(x-3) + (x-1) \cdot 1 \cdot (x-3) + (x-1)(x-2) \cdot 1$. When we plug $x = 1$ the last two terms are zero, and so $F'(1) = (1-2)(1-3) = 2$.

12) $y' = \frac{1}{2\sqrt{x}}(1 + \sqrt{2x})(1 + \sqrt{3x}) + \frac{2}{2\sqrt{2x}}(1 + \sqrt{x})(1 + \sqrt{3x}) + \frac{3}{2\sqrt{3x}}(1 + \sqrt{x})(1 + \sqrt{2x})$.

13) $g'(x) = \frac{(-1)\cdot(1+x) - (a-x)\cdot 1}{(1+x)^2} = -\frac{(1+a)}{(1+x)^2}$. Hence $g'(1) = -\frac{1+a}{4}$.

14) Since the two functions should have the same tangent, the slope of this tangent should be equal. Hence, we are looking for a point x where the derivative at that point of the two function will be equal. Thus we have the equation $2x = 4x + 2a$, or $x = -a$. If the point which the tangent line is tangent to the two functions is (a, y_0), then we must have $y_0 = a^2 - 2$ and $y_0 = 2a^2 - 2a^2 + b$. Thus we get $a^2 - 2 = b$ or $b = a^2 - 2$. This means that for any fix value of a, the two functions will have the same tangent at the point $(a, a^2 - 2)$.

15) Plugging $x = 0$ in the inequality $|f(x)| \leq x^2$ we get $f(0) = 0$. Also, the above inequality is equivalent to $-x^2 \leq f(x) \leq x^2$. For all $x \neq 0$ we have $-x \leq \frac{f(x)}{x} \leq x$. Since $\lim_{x\to 0} x = \lim_{x\to 0}(-x) = 0$, it follows from the Sandwich Theorem that $\lim_{x\to 0} \frac{f(x)}{x} = 0$. We have $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = 0$ where the first equality follows from the fact that $f(0) = 0$ and the second is the same limit we computed above it.

16) Since $g(x)$ is continuous at $x = 0$ it is defined there, and hence $f(0) = 0 \cdot g(0) = 0$. We have $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$ where the last equality follows from the fact that $f(0) = 0$. Moreover, $\lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \frac{hg(h)}{h} = \lim_{h\to 0} g(h) = g(0)$, where the last equality follows from the fact that $g(x)$ is continuous at zero. Hence $f'(0)$ exists and is equal to $g(0)$.