

Calculus A for Economics

Solutions to Exercise Number 7

1) a)

$$y' = \frac{e^x - e^{-x}}{3}$$

b)

$$y' = 3 \left(x + \frac{1}{x} \right)^2 \left(1 - \frac{1}{x^2} \right)$$

c)

$$y' = 3 \left(\frac{ax+b}{cx+d} \right)^2 \frac{(cx+d) \cdot a - (ax+b) \cdot c}{(cx+d)^2} = \frac{3(ax+b)^2(ad-bc)}{(cx+d)^4}$$

d)

$$y' = (-1) \cdot \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x \right)^{-2} (x^2 + x + 1) = -\frac{x^2 + x + 1}{\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + x \right)^2}$$

e)

$$\begin{aligned} y' &= \frac{1}{2\sqrt{\frac{x+1}{x-1}}} \cdot \frac{(x-1) \cdot 1 - (x+1) \cdot 1}{(x-1)^2} = \frac{1}{2} \sqrt{\frac{x-1}{x+1}} \frac{-2}{(x-1)^2} = \\ &= -\frac{1}{(x-1)^2} \sqrt{\frac{x-1}{x+1}} = -\frac{1}{(x-1)^{3/2}(x+1)^{1/2}} \end{aligned}$$

f)

$$y' = 2 \cdot (\ln x) \cdot \frac{1}{x} = \frac{2 \ln x}{x}$$

g)

$$y' = \frac{1}{x^2 - 1} \frac{1}{\ln 3} \cdot 2x = \frac{2x}{(x^2 - 1) \ln 3}$$

h)

$$y' = \frac{1}{2\sqrt{1+e^x}} \cdot e^x = \frac{e^x}{2\sqrt{1+e^x}}$$

i)

$$y' = \frac{(1+10^x)(-10^x \ln 10) - (1-10^x)(10^x \ln 10)}{(1+10^x)^2} = \frac{-2 \cdot 10^x \ln 10}{(1+10^x)^2}$$

j)

$$y' = 2xe^{-\frac{x^2}{a^2}} + x^2 e^{-\frac{x^2}{a^2}} \cdot \frac{-2x}{a^2} = 2xe^{-\frac{x^2}{a^2}} \left(1 - \frac{x^2}{a^2} \right)$$

2) By the differentiation of composition of functions we have $(f \circ g)'(0) = f'(g(0)) \cdot g'(0) = f'(2) \cdot g'(0) = 5 \cdot 1 = 5$.

3)

$$\begin{aligned} \frac{d}{dx} \left[f \left(\frac{x-1}{x+1} \right) \right] &= f' \left(\frac{x-1}{x+1} \right) \cdot \frac{d}{dx} \left[\frac{x-1}{x+1} \right] = \\ &= f' \left(\frac{x-1}{x+1} \right) \frac{(x+1) \cdot 1 - (x-1) \cdot 1}{(x+1)^2} = f' \left(\frac{x-1}{x+1} \right) \frac{2}{(x+1)^2} \end{aligned}$$

4) By implicit differentiation we have $1 = 3y^2y' - 4y'$. Thus $y' = \frac{1}{3y^2-4}$. Here we view y as an implicit function of x .

5) Since $y = \ln(x^2 - 1)$ then $y' = \frac{dy}{dx} = \frac{2x}{x^2-1}$. If we differentiate implicitly with respect to y we get $1 = \frac{1}{x^2-1} \cdot 2x \cdot \frac{dx}{dy} = \frac{2x}{x^2-1} \frac{dx}{dy}$. Thus $\frac{dx}{dy} = \frac{x^2-1}{2x}$. Hence $\frac{dy}{dx} \frac{dx}{dy} = 1$.

6) a) $f'(x) = \frac{(1+x) \cdot 1 - x \cdot 1}{(1+x)^2} = \frac{1}{(1+x)^2} = (1+x)^{-2}$. Hence $f''(x) = (-2)(1+x)^{-3} \cdot 1 = -\frac{2}{(1+x)^3}$.
b) We have $y' = 4ax^3$; $y'' = 12ax^2$; $y^{(3)} = 24ax$; $y^{(4)} = 24a$.
c) Let $y = x^3 + x^{-3}$. Then $y' = 3x^2 - 3x^{-4}$; $y'' = 6x + 12x^{-5} = 6x + \frac{12}{x^5}$.

7) Write $y = (2x - x^2)^{1/2}$. Then $y' = \frac{1}{2}(2x - x^2)^{-1/2}(2 - 2x) = (1 - x)(2x - x^2)^{-1/2}$. Then $y'' = -1 \cdot (2x - x^2)^{-1/2} + (1 - x)[- \frac{1}{2}(2x - x^2)^{-3/2}(2 - 2x)] = -\frac{1}{\sqrt{2x-x^2}} - \frac{(1-x)^2}{\sqrt{(2x-x^2)^3}} = \frac{1}{(\sqrt{2x-x^2})^3}$.
Hence $y^3y'' - 1 = (\sqrt{2x-x^2})^3 \frac{1}{(\sqrt{2x-x^2})^3} - 1 = 1 - 1 = 0$.

8) a) Applying the logarithm to both sides of $y = x^{(x+2)}$, we obtain $\ln y = (x+2)\ln x$. Differentiating implicitly we obtain $\frac{y'}{y} = \ln x + (x+2)\frac{1}{x}$. Thus, $y' = y(\ln x + \frac{x+2}{x}) = x^{(x+2)}(\ln x + \frac{x+2}{x})$.

b) Applying the logarithm to both sides we obtain $\ln y = \ln(x+2) + \ln(x^2+1) + \ln(x-3) - \ln(3x+1) - 2\ln(x-5)$. Differentiating we obtain $\frac{y'}{y} = \frac{1}{x+2} + \frac{2x}{x^2+1} + \frac{1}{x-3} - \frac{3}{3x+1} - \frac{2}{x-5}$. Thus $y' = \frac{(x+2)(x^2+1)(x-3)}{(3x+1)(x-5)^2} \left(\frac{1}{x+2} + \frac{2x}{x^2+1} + \frac{1}{x-3} - \frac{3}{3x+1} - \frac{2}{x-5} \right)$.

c) Applying the logarithm to both sides we obtain $\ln y = x \ln(\ln x)$. Hence $y' = y(1 \cdot \ln(\ln x) + x \cdot \frac{1}{\ln x} \cdot \frac{1}{x}) = (\ln x)^x (\ln(\ln x) + \frac{1}{\ln x})$.

d) Applying the logarithm to both sides we obtain $\ln y = x \ln \left(\frac{x}{1+x} \right) = x(\ln x - \ln(1+x))$. Hence $y' = y(\ln x - \ln(1+x) + x(\frac{1}{x} - \frac{1}{1+x})) = y \left(\ln \left(\frac{x}{1+x} \right) + x \frac{1+x-x}{x(1+x)} \right) = \left(\frac{x}{1+x} \right)^x \left(\ln \left(\frac{x}{1+x} \right) + \frac{1}{1+x} \right)$.

In each of the following cases we use L'Hospital's rule. The symbol $\stackrel{L}{=}$ indicates that at this equality we apply the rule.

9) a)

$$\lim_{x \rightarrow a} \frac{x - a}{x^n - a^n} \stackrel{L}{=} \lim_{x \rightarrow a} \frac{1}{nx^{n-1}} = \frac{1}{na^{n-1}}$$

b)

$$\lim_{x \rightarrow 0} \frac{1 + x - e^x}{x(e^x - 1)} \stackrel{L}{=} \lim_{x \rightarrow 0} \frac{1 - e^x}{(e^x - 1) + xe^x} \stackrel{L}{=} \lim_{x \rightarrow 0} \frac{-e^x}{e^x + e^x + xe^x} = -\frac{1}{2}$$

c)

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} \stackrel{L}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0$$

d)

$$\lim_{x \rightarrow 1} \frac{\sqrt[n]{x} - 1}{\sqrt[m]{x} - 1} = \lim_{x \rightarrow 1} \frac{x^{1/n} - 1}{x^{1/m} - 1} \stackrel{L}{=} \lim_{x \rightarrow 1} \frac{\frac{1}{n}x^{\frac{1}{n}-1}}{\frac{1}{m}x^{\frac{1}{m}-1}} = \frac{\frac{1}{n}}{\frac{1}{m}} = \frac{m}{n}$$

e) We first compute the limit $\lim_{x \rightarrow \infty} \ln \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x}\right) = \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{x}\right)}{\frac{1}{x}} \stackrel{L}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \frac{-1}{x^2}}{\frac{-1}{x^2}} = \lim_{x \rightarrow \infty} \frac{1}{1+\frac{1}{x}} = 1$. Using that we now compute $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e^{\lim_{x \rightarrow \infty} \ln \left(1 + \frac{1}{x}\right)^x} = e^{\lim_{x \rightarrow \infty} \ln \left(1 + \frac{1}{x}\right)^x}$. The last step follows from the fact that the logarithm function is continuous in its domain of definition. Using our first computation we deduce that $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e^1 = e$.

f) We start with $\lim_{x \rightarrow 0} \ln \left[(e^x + x)^{\frac{1}{x}}\right] = \lim_{x \rightarrow 0} \frac{\ln(e^x + x)}{x} \stackrel{L}{=} \lim_{x \rightarrow 0} \frac{\frac{1}{e^x + x}(e^x + 1)}{1} = \lim_{x \rightarrow 0} \frac{e^x + 1}{e^x + x} = \frac{2}{1} = 2$. Hence $\lim_{x \rightarrow 0} (e^x + x)^{\frac{1}{x}} = e^{\lim_{x \rightarrow 0} \ln \left[(e^x + x)^{\frac{1}{x}}\right]} = e^2$.

g) We start with $\lim_{x \rightarrow 1} \ln \left(x^{\frac{1}{x-1}}\right) = \lim_{x \rightarrow 1} \frac{\ln x}{1-x} \stackrel{L}{=} \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{-1} = -1$. Hence $\lim_{x \rightarrow 1} x^{\frac{1}{x-1}} = e^{\lim_{x \rightarrow 1} \ln x^{\frac{1}{x-1}}} = e^{-1} = \frac{1}{e}$.

h)

$$\lim_{x \rightarrow 1} \frac{x^{1/2} - x^{1/4}}{x - 1} \stackrel{L}{=} \lim_{x \rightarrow 1} \frac{\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{4}x^{-\frac{3}{4}}}{1} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

10) No they are not. Indeed, by the chain rule it follows that $(f \circ f)'(x) = f'(f(x))f'(x)$. Hence, we will have equality only if $f'(x) = 1$ for all x . Taking for example, $f(x) = x^2$ we can easily show that the two functions are not the same.

11) We have

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{f(h)}{h} = \lim_{h \rightarrow 0} \frac{g(h)}{h^2}$$

Here we used the fact that $f(0) = 0$. Since it is given that $g(0) = 0$ we can apply L'Hospital rule. Thus we obtain $f'(0) = \lim_{h \rightarrow 0} \frac{g(h)}{h^2} \stackrel{L}{=} \lim_{h \rightarrow 0} \frac{g'(h)}{2h}$. It is also given that $g'(0) = 0$, hence we can use L'Hospital rule once again. We have

$$f'(0) = \lim_{h \rightarrow 0} \frac{g'(h)}{2h} \stackrel{L}{=} \lim_{h \rightarrow 0} \frac{g''(h)}{2} = \frac{g''(0)}{2} = \frac{17}{2}$$

Notice that we used here the fact that $g''(x)$ is continuous at zero.