Calculus A for Economics

Solutions to Exercise Number 8

1) $f(x)$ is a polynomial and hence it is continuous and differentiable at any point. We have $f(-1) = f(2) = 0$. Therefore all the conditions of Rolle's Theorem are satisfied and hence there is a point $c \in (-1, 2)$ such that $f'(c) = 0$. You were not asked to find the point c, but here it is easy to do it. Indeed, we have $f'(x) = 3x^2 + 8x - 7$ and it is easy to check that one of the solutions to the quadratic equation $3x^2 + 8x - 7 = 0$ is in the interval $(-1, 2)$.

2) We have $f(\pm 1) = \frac{2-(\pm 1)^2}{(\pm 1)^4} = 1$ and hence $f(1) = f(-1)$. Also, for $x \neq 0$, $f'(x) = \frac{x^4(-2x)-(2-x^2)4x^3}{x^8} = \frac{2x^5-8x^3}{x^8} = \frac{2(x-2)(x+2)}{x^5}$. Hence $f'(x)$ is defined for all $x \neq 0$, and $f'(x) \neq 0$ for all $0 \neq x \in [-1, 1]$. There is no contradiction to Rolle's Theorem because $f(x)$ is not defined at $x = 0$ and hence clearly not continuous there.

3) Let $p(x) = a_0 x^n + \cdots + a_{n-1} x$. It is given that $p(x_0) = 0$ for some $x_0 > 0$. Also, $p(0) = 0$. Since $p(x)$ is a polynomial it is continuous and differentiable for all x. Therefore we may apply Rolle's Theorem to the interval $[0, x_0]$, to deduce that there is $x_1 \in (0, x_0)$ such that $p'(x_1) = 0$. But $p'(x) = na_0 x^{n-1} + (n-1)a_1 x^{n-2} + \cdots + a_{n-1}$. This is what we needed to prove.

4) Assume that $f(x) = x^3 - 3x + c$ vanish at the points $0 < x_1 < x_2 < 1$. That is $f(x_1) = f(x_2) = 0$. Since $f(x)$ is a polynomial it is continuous and differentiable for all x. Therefore we may apply Rolle's Theorem to the interval $[x_1, x_2]$, to deduce that there is a point $x_1 < x_3 < x_2$ such that $f'(x_3) = 0$. Since $0 < x_1 < x_2 < 1$ then $0 < x_3 < 1$. We have $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$ which is clearly nonzero for all points $0 < x < 1$. Thus we derived a contradiction, and hence $f(x)$ has no two distinct zeros in the interval $(0, 1)$.

5) The function is clearly continuous and differentiable at the interval $[1, 4]$. Hence the mean value Theorem holds. Thus, there exists a point $c \in (1, 4)$ such that $f'(c) = \frac{f(4)-f(1)}{4-1} = -3$. To find that point we compute $f'(x) = \frac{3}{2\sqrt{x}} - 4$. Thus $f'(x) = -3$ is equivalent to $\frac{3}{2\sqrt{x}} - 4 = -3$, or $3 = 2\sqrt{x}$. Hence $x = \frac{9}{4}$ and thats our point c.

6) The conditions of the mean value Theorem are satisfied for all real points. Choose $x_1 < x_2$, and apply the theorem in the interval $[x_1, x_2]$. Thus, there exists a point $c \in [x_1, x_2]$ such that $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. Hence $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Hence $|f(x_2) - f(x_1)| = |f'(c)||(x_2 - x_1)| \le |x_2 - x_1|$ where the last inequality follows from the fact that $|f'(x)| \le 1$ for all x.

7) Let $f(x) = \ln x$, and consider the interval $[b, a]$. Since $0 < b \le a$ the function $f(x) = \ln x$ is continuous and differentiable at that interval, and hence the mean value Theorem holds. Thus, there exists $b < c < a$ such that $f'(c) = \frac{f(a)-f(b)}{a-b}$, or $\frac{1}{c} = \frac{\ln a - \ln b}{a-b}$, or $\frac{a-b}{c} = \ln\frac{a}{b}$. Since $b < c < a$ then $\frac{1}{a} < \frac{1}{c} < \frac{1}{b}$, or $\frac{a-b}{a} \le \frac{a-b}{c} \le \frac{a-b}{b}$. But $\ln\frac{a}{b} = \frac{a-b}{c}$.

8) Consider the function $f(x) = x^n$. It is continuous and differentiable in any interval, and hence the mean value Theorem holds. We apply it to the interval $[b, a]$ to deduce that there is a pint $b < c < a$ such that $f'(c) = \frac{f(a)-f(b)}{a-b}$, or $nc^{n-1} = \frac{a^n-b^n}{a-b}$. Since $b < c < a$ and n is a natural number then $b^{n-1} < c^{n-1} < a^{n-1}$. From the identity $c^{n-1} = \frac{a^n-b^n}{n(a-b)}$ we obtain $b^{n-1} < \frac{a^n-b^n}{n(a-b)} < a^{n-1}$, which is clearly equivalent to the one we need to prove.

Recall that a point c is a possible extreme point for a function $f(x)$, if either $f'(x) = 0$ at $x = c$, or $f(x)$ is defined at $x = c$ but $f'(x)$ does not exist at $x = c$.

9) a) We have $y' = 3x^2 + 3 = 3(x^2 + 1)$. Thus $f'(x)$ is positive for all x. Also, since $f(x)$ is polynomial it is differentiable for all x. Thus $f(x)$ has no extreme points.
b) We have $y' = 2(1-x)(-1)(1+x) + (1-x)^2 \cdot 1 = -(1-x)(1+3x)$. Thus $f'(x) = 0$ implies $x = 1$ or $x = -\frac{1}{3}$. Since $f(x)$ is a polynomial it is differentiable for all x. Thus, these are the only possible extreme points for $f(x)$.
c) We have

$$f(x) = \frac{1}{|x-2|} = \begin{cases} \frac{1}{x-2}, & x > 2 \\ -\frac{1}{x-2}, & x < 2 \end{cases} \qquad f'(x) = \begin{cases} -\frac{1}{(x-2)^2}, & x > 2 \\ \frac{1}{(x-2)^2}, & x < 2 \end{cases}$$

Hence $f'(x)$ is never zero. Clearly, $f(x)$ is not differentiable at $x = 2$, but it is also not defined at that point. Hence $x = 2$ is not a possible extreme point. Thus, $f(x)$ has no extreme points.
d) We have

$$f(x) = |x^2 - 16| = \begin{cases} x^2 - 16, & x \le -4, x \ge 4 \\ -(x^2 - 16), & -4 < x < 4 \end{cases} \qquad f'(x) = \begin{cases} 2x, & x < -4, x > 4 \\ -2x, & -4 < x < 4 \end{cases}$$

Hence $f'(x) = 0$ if and only if $x = 0$. Also, $f(x)$ is not differentiable at $x = \pm 4$. However, $f(x)$ is defined at these two points. Hence the possible extreme points for $f(x)$ are $x = 0, \pm 4$.

e) The domain of definition is $x^2+1>0$ and hence the function is defined and differentiable for all x. We have $f'(x)=1-\frac{1}{x^2+1}\cdot 2x$. The equation $f'(x)=0$ is equivalent to $\frac{2x}{1+x^2}=1$ or $(1-2x+x^2)=0$ or $(x-1)^2=0$. Hence there is only one possible extreme point and it is $x=1$.

f) The function is defined for all $x>0$ and is differentiable at that domain. We have $f'(x)=(2x-2)\ln x+(x^2-2x)\frac{1}{x}-\frac{3}{2}\cdot 2x+4=2(x-1)\ln x-2x+2=2(x-1)(\ln x-1)$. Thus $f'(x)=0$ has two possible solutions. First $x=1$ and second $\ln x=1$ or $x=e$. Thus the two possible extreme points are $x=1,e$.

g) We have

$$f(x)=\frac{1}{x+1}-\frac{1}{x+2}=\frac{1}{(x+1)(x+2)}$$

This is a rational function and hence it is defined and differentiable for all points except $x=1,2$. We have

$$f'(x)=\frac{-1}{(x+1)^2}+\frac{1}{(x+2)^2}=-\frac{2x+3}{(x+1)^2(x+2)^2}$$

Therefore $f'(x)=0$ is equivalent to $2x+3=0$ or $x=-\frac{3}{2}$ which is the only possible extreme point. Clearly $f'(x)$ is not defined at $x=1,2$, but these are not possible extreme points because the function $f(x)$ is not defined at these points.

10) a) This follows from the following Theorem which was stated in Exercise **5**.
Theorem: *Let $f(x)$ be a continuous function in the closed interval $[a,b]$. Assume that $f(a)f(b)\le 0$. Then there exist a point $a\le c\le b$ such that $f(c)=0$.*
Hence, if $g(a)g(b)\le 0$ then there would be a point $a<c<b$ such that $g(c)=0$. But then $f(c)=0$, and this is a contradiction to the fact that the points a and b are adjacent.
b) Apply Roll's Theorem to the interval $[a,b]$. Indeed, since a and b are roots of $f(x)$, then $f(a)=f(b)=0$ and since $f(x)$ is a polynomial, all the conditions of the Theorem hold.

11) The function $f(x)=\sqrt{x}$ is an example. Indeed, $f'(x)=\frac{1}{2\sqrt{x}}$.

12) Apply the mean value Theorem to the function $f(x)=\sqrt{x}$ in the interval $[64,66]$. Clearly all the conditions of the Theorem hold. Since $f'(x)=\frac{1}{2\sqrt{x}}$ we deduce that there is a point $64<c<66$ such that

$$\frac{1}{2\sqrt{c}}=\frac{f(66)-f(64)}{66-64}=\frac{\sqrt{66}-8}{2}$$

Hence $\frac{1}{\sqrt{c}}=\sqrt{66}-8$. Since $64<c$, then $8<\sqrt{c}$, or $\frac{1}{\sqrt{c}}<\frac{1}{8}$. Hence $\sqrt{66}-8<\frac{1}{8}$. Since $c<66<81$ then $\frac{1}{9}<\frac{1}{\sqrt{c}}$ and hence $\frac{1}{9}<\sqrt{66}-8$.