Calculus A for Economics

Solutions to Exercise Number 9a

We briefly recall the notations used in the class. For studying extreme points and domains in which a function increases or decreases, we use a diagram of the sort

$$\begin{array}{c|ccccccc} y & \uparrow & & \downarrow & & \downarrow & & \uparrow \\ \hline y' & + & a & - & b & - & c & + \end{array}$$

The top row indicates the domains in which the function increases or decreases. The bottom row indicates the signs of the derivative. In the above, the function y increases in the interval $(-\infty, a)$, then decreases in the intervals (a, b) and (b, c), and increases again in the interval (c, ∞). The points a, b, c are either possible extreme points, or points where y' does not exist.

Similarly, we will use a diagram of the same shape to study the domain of concavity of the function. For example

$$\begin{array}{c|ccc} y & \cup & & \cap \\ \hline y'' & + & a & - \end{array}$$

indicates that the sign of y'' is positive in the interval $(-\infty, a)$ and hence the function concave upward. We denotes this by $\cup$. Also, the sign of y'' is negative in the interval (a, ∞) which means that the function concave downward. This is denoted by $\cap$. The point a is a point where either y'' is zero or it does not exist.

In both cases, the letters 'nd' above a certain interval indicates that the function is not defined there.

1)- 2) a) The function is defined for all $x \neq 0$. We have $y' = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2} = \frac{(x-1)(x+1)}{x^2}$. Hence the possible two extreme points are $x = \pm 1$. We have

$$\begin{array}{c|ccccccc} y & \uparrow & & \downarrow & & \downarrow & & \uparrow \\ \hline y' & + & -1 & - & 0 & - & 1 & + \end{array}$$

Thus $x = -1$ is a local maximum and $x = 1$ is a local minimum.

b) $f(x)$ is defined and differentiable for all x. We have $f'(x) = 2x = 3x^2 = -x(3x-2)$. Hence the possible two extreme points are $x = 0$ and $x = \frac{2}{3}$. We have

$$\begin{array}{c|ccccc} y & \downarrow & & \uparrow & & \downarrow \\ \hline y' & - & 0 & + & \frac{2}{3} & - \end{array}$$

Thus, $x=0$ is a local minimum and $x=\frac{2}{3}$ is a local maximum.

c) This exercise was studied in problem **9)d)** in exercise 8. The possible extreme points were $x=0$, a point where the derivative is zero, and $x=\pm 4$ where the function is defined but not differentiable. We have

$$\begin{array}{c|ccccccc} y & \downarrow & & \uparrow & & \downarrow & & \uparrow \\ \hline y' & - & -4 & + & 0 & - & 4 & + \end{array}$$

Thus, $x=0$ is a local maximum, and $x=\pm 4$ are both local minimum.

d) $f(x)$ is defined and differentiable for all $x\neq -1$. We have $f'(x)=\frac{(x+1)(-3x^2)+x^2}{(x+1)^2}=-\frac{x^2(2x+3)}{(x+1)^2}$. Thus, possible extreme points are $x=0$ and $x=-\frac{3}{2}$. We have

$$\begin{array}{c|ccccccc} y & \uparrow & & \downarrow & & \downarrow & & \downarrow \\ \hline y' & + & -\frac{3}{2} & - & -1 & - & 0 & - \end{array}$$

Thus, $x=-\frac{3}{2}$ is a local maximum and $x=0$ is not an extreme point since the function increases to the left and to the right of that point.

e) $f(x)$ is defined and differentiable for all $x>-1$. We have $f'(x)=1-\frac{1}{x+1}=\frac{x}{x+1}$. Thus $x=0$ is a possible extreme point. We have

$$\begin{array}{c|ccccc} y & \text{nd} & & \downarrow & & \uparrow \\ \hline y' & & -1 & - & 0 & + \end{array}$$

Thus $x=0$ is a local minimum.

f) $f(x)$ is defined and differentiable for all x. We have $f'(x)=ae^{px}\cdot p+be^{-px}\cdot(-p)=p(ae^{px}-be^{-px})$. Thus, $f'(x)=0$ implies $ae^{px}=be^{-px}$, or $e^{2px}=\frac{a}{b}$. Hence $2px=\ln\frac{b}{a}$, or $x=\frac{1}{2p}\ln\frac{b}{a}$. Since $a,b,p>0$ all the above operations are justified. Denote $x_0=\frac{1}{2p}\ln\frac{b}{a}$. Thus x_0 is the only possible extreme point. If $x>x_0$, then $e^{px}>e^{px_0}$. This follows from the fact that the function e^x is increasing for all x, and because $p>0$. Similarly, $e^{-px}<e^{-px_0}$. Hence, if $x>x_0$ then $ae^{px}-be^{-px}>ae^{px_0}-be^{-px_0}$. Similarly, if $x<x_0$ then $ae^{px}-be^{-px}<ae^{px_0}-be^{-px_0}$. Thus we obtain

$$\begin{array}{c|ccc} y & \downarrow & & \uparrow \\ \hline y' & - & x_0 & + \end{array}$$

Hence x_0 is a local minimum.

g) $f(x)$ is defined and differentiable for all x. We have $y'=e^{-x}-xe^{-x}=(1-x)e^{-x}$. Hence $x=1$ is the unique possible extreme point. We have

$$\begin{array}{c|ccc} y & \uparrow & & \downarrow \\ \hline y' & + & 1 & - \end{array}$$

Hence $x=1$ is a local maximum.

h) $f(x)$ is defined and differentiable for all x. We have $y'=e^{x^2}+xe^{x^2}\cdot 2x=(1+2x^2)e^{x^2}$.

This function is never zero and hence there are no extreme points for the function. Since $y' > 0$ for all x, it follows that $f(x)$ increases for all x.

i) $f(x)$ is defined and differentiable for all $\frac{x}{1+x^2} > 0$, which is equivalent to $x > 0$. We have $f'(x) = \frac{1+x^2}{x}\frac{(1+x^2-x\cdot 2x}{(1+x^2)^2} = \frac{1-x^2}{x(1+x^2)}$. The points where $f'(x) = 0$ are $x = \pm 1$, but since $f(x)$ is not defined at $x = -1$, then only $x = 1$ is a possible extreme point. We have

$$\begin{array}{c|ccccc} y & \text{nd} & & \uparrow & & \downarrow \\ \hline y' & & 0 & + & 1 & - \end{array}$$

Therefore $x = 1$ is a local maximum.

j) The function $f(x)$ is defined for all x such that $\frac{x^3}{x-1} > 0$. This is satisfied in the domain $x < 0$ or $x > 1$. In this domain the function is differentiable. We have $y' = \frac{x-1}{x^3}\frac{(x-1)\cdot(3x^2)-x^3}{(x-1)^2} = \frac{2x-3}{x(x-1)}$. Thus $y' = 0$ only for $x = \frac{3}{2}$ which is in the domain of definition of the function. We have

$$\begin{array}{c|ccccccc} y & \downarrow & & \text{nd} & & \downarrow & & \uparrow \\ \hline y' & - & 0 & & 1 & - & \frac{3}{2} & + \end{array}$$

Thus $x = \frac{3}{2}$ is a local minimum.

3) a) Write $y = (1-x^2)^2$. Then $y' = 4x^3 - 4x$ and $y'' = 12x^2 - 4 = 4(x^2-3)$. The function $f(x)$ is defined and twice differentiable for all x. Hence, the only possible inflection points are $x = \pm\sqrt{3}$. We have

$$\begin{array}{c|ccccc} y & \cup & & \cap & & \cup \\ \hline y'' & + & -\sqrt{3} & - & \sqrt{3} & + \end{array}$$

Hence $x = \pm\sqrt{3}$ are both inflection points.

b) The function is defined for all x. We have $y' = \frac{1}{5}(x-3)^{-\frac{4}{5}}$ and $y'' = -\frac{4}{25}(x-3)^{-\frac{9}{5}}$. Clearly y'' is never zero. However, y'' is not defined at the point $x = 3$, but the function is defined there. So $x = 3$ is a possible inflection point. We have

$$\begin{array}{c|ccc} y & \cup & & \cap \\ \hline y'' & + & 3 & - \end{array}$$

Hence $x = 3$ is an inflection point.

c) The function is defined for $x \geq 0$, and differentiable at $x > 0$. We have $y' = -\frac{1}{\sqrt{x}(\sqrt{x}+1)^2}$. Differentiating again we get $y'' = \frac{1}{(\sqrt{x}(\sqrt{x}+1)^2)^2}[\sqrt{x}(\sqrt{x}+1)^2)^2]' = \frac{(x^{3/2}+2x+x^{1/2})'}{x(\sqrt{x}+1)^4)} = \frac{\frac{3}{2}\sqrt{x}+2+\frac{1}{2\sqrt{x}}}{x(\sqrt{x}+1)^4)}$. Thus $y'' > 0$ for all $x > 0$. Hence there are no inflection points.

d) The function is defined for all $x > 0$. We have $y' = 1 - \frac{1}{x}$ and $y'' = \frac{1}{x^2}$. Hence $y'' > 0$ for all $x > 0$, and hence there are no inflection points. Thus

$$\begin{array}{c|ccc} y & \text{nd} & & \cup \\ \hline y'' & & 0 & + \end{array}$$

e) The function is defined and differentiable for all x. We have $y' = 2xe^{-x} + x^2e^{-x}(-1) = (2x - x^2)e^{-x}$, and $y'' = (2 - 2x)e^{-x} + (2x - x^2)e^{-x}(-1) = (x^2 - 4x + 2)e^{-x}$. Thus $y'' = 0$ if and only if $x^2 - 4x + 2 = 0$, or $x_1 = 2 - \sqrt{2}$ and $x_2 = 2 + \sqrt{2}$. We have

$$\begin{array}{c|ccccc} y & \cup & & \cap & & \cup \\ \hline y'' & + & x_1 & - & x_2 & + \end{array}$$

Therefore x_1 and x_2 are inflection points.

f) $f(x)$ is defined and differentiable for all $x > 0$. We have $y' = -\frac{a}{x^2}\ln\frac{x}{a} + \frac{a}{x}\frac{1}{\frac{x}{a}}\frac{1}{a} = \frac{a}{x^2}(1 - \ln\frac{x}{a})$, and $y'' = -\frac{2a}{x^2}(1 - \ln\frac{x}{a}) + \frac{a}{x^2}(-\frac{1}{x}) = -\frac{2a}{x^3}(2(1 - \ln\frac{x}{a}) + 1)$. Thus, $y'' = 0$ if and only if $2(1 - \ln\frac{x}{a}) + 1 = 0$, or $\ln\frac{x}{a} = \frac{3}{2}$ or $x = ae^{3/2}$.

$$\begin{array}{c|ccccc} y & \text{nd} & & \cap & & \cup \\ \hline y'' & & 0 & - & ae^{3/2} & + \end{array}$$

Hence, $x = ae^{3/2}$ is an inflection point.

g) The function is defined and differentiable for all x. We have $y' = -2xe^{-x^2}$ and $y'' = 2e^{-x^2}(2x^2 - 1)$. Hence $y'' = 0$ if and only if $x_1 = -\frac{1}{\sqrt{2}}$ or $x_2 = \frac{1}{\sqrt{2}}$. We have

$$\begin{array}{c|ccccc} y & \cup & & \cap & & \cup \\ \hline y'' & + & x_1 & - & x_2 & + \end{array}$$

Hence both x_1 and x_2 are inflection points.

4) a) The function is defined for all $x \geq 0$ and is differentiable for all $x > 0$. We have $y' = 1 + \frac{1}{\sqrt{x}}$, and clearly $y' > 0$ for all $x \in (0, 4]$. hence the extreme points are the end points. We have $f(0) = 0$ and $f(4) = 8$. Hence $x = 0$ is the minimum point and $x = 4$ is the maximum point in the interval.

b) The function is defined and differentiable for all x. We have $y' = 3x^2 - 6x + 5$ which is positive for all x. Hence the extreme points are the end points. We have $f(-1) = -12$ and $f(1) = 2$. Hence $x = -1$ is the minimum point and $x = 1$ is the maximum point in the interval.

c) The function is defined and differentiable for all $x \in [0, 1]$. We have $y' = \frac{4x-2}{(1+x-x^2)^2}$. Hence $y' = 0$ if $x = \frac{1}{2}$ which is a point in the interval. We have $f(0) = 1$; $f(\frac{1}{2}) = \frac{3}{5}$ and $f(1) = 1$. Hence $x = \frac{1}{2}$ is a minimum point and $x = 0, 1$ are both maximum points.

d) The function is defined and differentiable for all $x \in [0, 3]$. We have $y' = -2xe^{-x^2}$ which is not positive for all $x \in [0, 3]$. Thus the extreme points are the end points. We have $f(0) = 1$ and $f(3) = e^{-9}$. Hence $x = 0$ is a maximum point and $x = 3$ is a minimum point.