Calculus A for Economics

Solutions to Exercise Number 9b

4) e) The function is defined and differentiable for all $x \in [1, e]$. We have $y' = \frac{1}{x}$ and $f(1) = 0$ and $f(e) = 1$. Hence $x = 1$ is a minimum point and $x = e$ is a maximum point.

5) Denote $h(x) = f(x) - g(x)$. Since $f(x)$ and $g(x)$ are differentiable in (a, b) then $h(x)$ is differentiable in (a, b) and since $f(x)$ and $g(x)$ are continuous in $[a, b]$ so is $h(x)$. Also, since $f(a) = g(a)$ then $h(a) = 0$. We have $h'(x) = f'(x) - g'(x)$, and since $f'(x) > g'(x)$ for all $x \in (a, b)$, it follows that $h'(x) > 0$ for all $x \in (a, b)$. Given a point $x \in (a, b)$ We apply the Mean Value Theorem to the function $h(x)$ on the interval $[a, x]$. (Verify that all the conditions of the Theorem are satisfied). Thus, there is a point $c \in (a, x)$ such that $h(x) - h(a) = h'(c)(x - a)$. Clearly $x - a > 0$ and from the above $h'(c) > 0$. Hence, for $x \in (a, b)$ we have $h(x) - h(a) > 0$ or $h(x) > h(a)$. But since $h(a) = 0$, we obtain for all $x \in (a, b)$ that $h(x) = f(x) - g(x) > 0$ which is what we needed to prove.

6) a) Denote $f(x) = 2\sqrt{x}$ and $g(x) = 3 - \frac{1}{x}$. Then for $x > 1$ these function are defined and differentiable. We have $f'(x) = \frac{1}{\sqrt{x}}$ and $g'(x) = \frac{1}{x^2}$. For all $x > 1$ we have $x^2 > \sqrt{x}$ and hence $f'(x) > g'(x)$ for all $x > 1$. Also, $f(1) = g(1)$. Therefore we may apply problem **5)** to deduce the desire inequality.

b) Consider first the case when $x > 0$. Define $f(x) = e^x$ and $g(x) = 1 + x$. Then $f'(x) = e^x$ and $g'(x) = 1$. We have $f(0) = g(0) = 1$, and for all $x > 0$ we have $e^x > 1$ or $f'(x) > g'(x)$. Therefore we may apply problem **5)** to deduce the desire inequality. When $x < 0$ we set $y = -x$. Then we need to prove that $e^{-y} > 1 - y$ for all $y > 0$. Setting $f(y) = e^{-y}$ and $g(y) = 1 - y$ and applying problem **5)** the inequality follows. (Check the conditions!)

c) Let $f(x) = \ln x$ and $g(x) = \frac{2(x-1)}{x+1}$. Both functions are defined and differentiable for all $x > 1$. We have $f(1) = g(1) = 0$. Also, $f'(x) = \frac{1}{x}$ and $g'(x) = \frac{4}{(x+1)^2}$. To check that $f'(x) > g'(x)$ we need to prove that $\frac{1}{x} > \frac{4}{(x+1)^2}$ for all $x > 1$. This is equivalent to $(x+1)^2 > 4x$ which is equivalent to $(x-1)^2 > 0$ which is true for all $x > 1$. Therefore we may apply problem **5)** to deduce the desire inequality.

d) We argue by induction on n. For $n = 0$ we get $1 < e^x$ which is true for all $x > 0$. Assume the inequality holds for $n-1$. Thus we have

$$1 + x + \frac{x^2}{2!} + \cdots \frac{x^{n-1}}{(n-1)!} < e^x \qquad (1)$$

Denote $f(x) = e^x$ and $g(x) = 1 + x + \frac{x^2}{2!} + \cdots \frac{x^n}{n!}$, and apply problem **5)** to these functions. Clearly $f(0) = g(0) = 1$. Also, $f'(x) = e^x$, and $g'(x)$ is the left hand side of equation (1). Hence, equation (1) is equivalent to $f'(x) > g'(x)$. Thus, we deduce that $f(x) > g(x)$ which is the inequality we needed to prove.

7) Denote the two numbers by a and b. Then, $a + b = 40$ or $b = 40 - a$ and their product is ab, or $a(40-a)$. Define the function $f(a) = a(40-a)$. Then we need to find the maximum of $f(a)$ when $0 \le a \le 40$. The last condition follows from the fact that outside this interval the function $f(a)$ is negative and so the product cannot be maximal. We have $f'(a) = 40 - 2a$ and $f'(a) = 0$ implies $a = 20$ which is in our interval. We have $f(0) = f(40) = 0$ and $f(20) = 400$. Hence the maximum is obtained if we take the numbers $a = b = 20$.

8) Denote by a the length of the base of the box, and by b its width. Also denote by h the height of the box. Hence, if we denote its volume by V then $V = abh$, and if we denote by S the area of its faces then $S = 2(ab+ah+bh)$. It is given that $S = 200$ and that $a = 3b$. From these conditions we can write V as a function of b. Indeed, we have $V(b) = 75b - \frac{9}{4}b^3$. The equation $S = 2(ab + ah + bh)$ becomes $h = \frac{200-6b^2}{8b}$. Since $h \ge 0$, we must have $\frac{200-6b^2}{8b} \ge 0$, or $b \le \frac{10}{\sqrt{3}}$. Clearly, $b \ge 0$. Thus, we need to find the maximum of $V(b) = 75b - \frac{9}{4}b^3$ in the interval $0 \le b \le \frac{10}{\sqrt{3}}$. We have $V' = 75 - \frac{27}{4}b^2$ and the only relevant point is $b = \frac{10}{3}$ which is in our interval. We have $f(0) = f(\frac{10}{\sqrt{3}}) = 0$ and $f(\frac{10}{3}) = \frac{500}{3}$. Hence, $b = \frac{10}{3}$ is the maximal point. For this value we have $a = 3b = 10$, and $h = 5$.

9) Producing n products a week costs $600+10n+n^2$ Shekel. The amount the factory gets from selling these products is $(110-2n)n$ Shekel. If we denote the profit by P, and view it as function of n, then we have $P(n) = (110-2n)n - (600+10n+n^2) = -3n^2 + 100n - 600$. Clearly $0 \le n \le 25$. We have $P'(n) = -6n + 100$ and $P'(n) = 0$ implies $n = \frac{50}{3}$. We have $P(0) = -600; P(\frac{50}{3}) = 233\frac{1}{3}$ and $P(25) = 25$. Hence $n = \frac{50}{3} = 16\frac{2}{3}$ is the maximal point. However, the factory must produce discrete number of products, and hence, by the continuity of $P(n)$ the number of products which will produce the maximal profit is $n = 16$ or $n = 17$ or $n = 25$. We have $P(16) = 232$ and $P(17) = 233$. Hence 17 products will give the maximal profit.

10) Denote by r the radius of the half circle, by a the length of the rectangle and by h the width of the rectangle. Thus, $a = 2r$ since the half circle is to be above the rectangle. The surrounding of the window is given by $C = a + 2h + \frac{1}{2}(2\pi r) = 2r + 2h + \pi r$. The area of the window is $S = ah + \frac{1}{2}\pi r^2 = 2rh + \frac{1}{2}\pi r^2$. It is given that $C = p$. Hence $h = \frac{1}{2}(p - (2+\pi)r)$, and since $h \geq 0$ this implies that $r \leq \frac{p}{2+\pi}$. Substituting inside S, we view S as a function of r, and we have $S(r) = pr - (2 + \frac{\pi}{2})r^2$. Thus the problem is to find maximum value for $S(r) = pr - (2 + \frac{\pi}{2})r^2$ in the interval $(0, \frac{p}{2+\pi})$. We have $S'(r) = p - (4+\pi)r$ and when $S'(r) = 0$ we get $r = \frac{p}{4+\pi}$. Since $S''(r) = -(4+\pi) < 0$, then $r = \frac{p}{4+\pi}$ is a local maximum, and thats the desired radius.

11) Denote by x the length of the square. Thus,the box obtained has base length $a - 2x$, width $b - 2x$ and height x. Hence its volume is given by $V(x) = x(a-2x)(b-2x) = abx - 2(a+b)x^2 + 4x^3$. We have $0 < x < \frac{a}{2}$ and also $x < \frac{b}{2}$. Thus $0 < x < \frac{1}{2}\min\{a, b\}$. We have $V' = ab - 4(a+b)x + 12x^2$. Hence solving the quadratic equation $ab - 4(a+b)x + 12x^2 = 0$, $V'(x) = 0$ implies

$$x_{1,2} = \frac{1}{6}\left[(a+b) \pm \sqrt{a^2 - ab + b^2}\right]$$

Since $a^2 - ab + b^2 \geq a^2 - 2ab + b^2 = (a-b)^2 \geq 0$ it follows that $\sqrt{a^2 - ab + b^2}$ is a real nonnegative number. Also, $a^2 - ab + b^2 \leq a^2 + 2ab + b^2 = (a+b)^2$, and hence $\sqrt{a^2 - ab + b^2} \leq a + b$. This means that the two solutions are nonnegative numbers, and hence the two solutions are possible extreme points. Checking the signs of $V'(x)$ we get

V	$\uparrow$		$\downarrow$		$\uparrow$
V'	$+$	x_1	$-$	x_2	$+$

Hence $x_1 = \frac{1}{6}\left[(a+b) - \sqrt{a^2 - ab + b^2}\right]$ is a local maximum. Finally, we need to check that $0 < x_1 < \frac{1}{2}\min\{a, b\}$. Assume for simplicity that $b \leq a$. From the above we deduce that $a^2 - ab + b^2 \geq (a-b)^2$ or $\sqrt{a^2 - ab + b^2} \geq a - b$ where here we used the assumption that $b \leq a$. Hence $x_1 = \frac{1}{6}\left[(a+b) - \sqrt{a^2 - ab + b^2}\right] \leq \frac{1}{6}\left[(a+b) - (a-b)\right] = \frac{b}{3} < \frac{b}{2} \leq \frac{a}{2}$ where the last inequality follows from the assumption that $b \leq a$.

12) Assume that m is the slope of the line which passes through the point $(1, 2)$. Then clearly $m < 0$, for otherwise there will be no triangular in the first quadrant. Hence the equation of the line is $y - 2 = m(x-1)$ or $y = mx + (2 - m)$. Therefore the intersection points of this line with the axes are the points $(0, 2-m)$ and $(\frac{m-2}{m}, 0)$. Hence, the area of the triangular is $S(m) = \frac{1}{2}(2-m)\left(\frac{m-2}{m}\right) = -\frac{(m-2)^2}{2m}$. Hence $S'(m) = -\frac{1}{2}\frac{2m(m-2)-(m-2)^2}{m^2} = -\frac{(m-2)(m+2)}{2m^2}$. Hence $S'(m) = 0$ implies $m = \pm 2$. Since we are only interested in negative values of m, then we need only to consider $m = -2$. By checking the signs of the first derivative, we see that $S(m)$ decreases for $m < -2$ and increases when $m > -2$. Hence $m = -2$ is a local minimum, and thats the slope of the desired line. Thus, the equation of the line is $y = -2x + 4$ and its three vertices are in $(0, 0); (0, 2)$ and $(4, 0)$.

13) Denote the two numbers by a and b. Thus $ab = 36$. We want to minimize the term $a^2 + b^2$. Clearly $a, b > 0$. Plugging $b = \frac{36}{a}$ we obtain the function $f(a) = a^2 + \frac{36^2}{a^2}$. We have $f'(a) = 2a + \frac{36^2 \cdot (-2)}{a^3} = \frac{2}{a^3}(a^4 - 36^2)$. Thus $f'(a) = 0$ implies $a^4 - 36^2 = 0$ or $a = \pm 6$. the relevant point is $a = 6$. We have $f''(a) = 2 + \frac{6 \cdot 36^2}{a^4} > 0$. Hence $a = 6$ is a local minimum. Also, this is the only extreme point in the interval $(0, \infty)$, and hence the minimum is given when $a = b = 6$.

14) We need to consider three cases. First, assume that $x \geq a$. For these values of x we have

$$f(x) = \frac{1}{1+x} + \frac{1}{1+x-a}$$

Differentiating, we get

$$f'(x) = -\frac{1}{(1+x)^2} - \frac{1}{(1+x-a)^2}$$

which is clearly always negative. Hence the function decreases for all $x \geq a$, and hence obtains its maximal value at $x = a$. We have $f(a) = \frac{2+a}{1+a}$. Next, consider the domain $0 < x < a$. In this domain,

$$f(x) = \frac{1}{1+x} + \frac{1}{1-x+a}$$

and hence

$$f'(x) = -\frac{1}{(1+x)^2} + \frac{1}{(1-x+a)^2} = \frac{(2+a)(a-2x)}{(1+x)^2(1-x+a)^2}$$

hence $f'(x) = 0$ implies $x = \frac{a}{2}$. We have $f(\frac{a}{2}) = \frac{4}{a+2} < \frac{2+a}{1+a}$ and hence $x = \frac{a}{2}$ is not a maximal value for the function. Finally, in the domain $x \leq 0$ we have

$$f(x) = \frac{1}{1-x} + \frac{1}{1-x+a}$$

and hence

$$f'(x) = \frac{1}{(1-x)^2} + \frac{1}{(1-x+a)^2}$$

which is always positive. Hence $f(x)$ increases in this domain and hence the maximal value is obtained in $x = 0$. Since $f(0) = \frac{2+a}{1+a}$, we proved that indeed $\frac{2+a}{1+a}$ is the maximal value that $f(x)$ obtains.