Solutions to Final in Mathematics A

Moed A

1) a) Since $1 - x + e^x$ equals two when $x = 0$, this is a limit of the type $2/0$ which does not exist.
b) We have

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+12}-4}{x+2} = \lim_{x\to-\infty} \frac{\sqrt{x^2(1+\frac{12}{x^2})}-4}{x+2} = \lim_{x\to-\infty} \frac{-x\sqrt{(1+\frac{12}{x^2})}-4}{x(1+\frac{2}{x})}$$

Here, the minus sign follows from the fact that for negative values of x, we have $\sqrt{x^2} = -x$. Thus, we obtain,

$$\lim_{x\to-\infty} \frac{-x(\sqrt{(1+\frac{12}{x^2})}-\frac{4}{x})}{x(1+\frac{2}{x})} = \lim_{x\to-\infty} \frac{-(\sqrt{(1+\frac{12}{x^2})}-\frac{4}{x})}{1+\frac{2}{x}} = -1$$

2) The function is defined for all $x > 0$. Write $f(x) = -1 - \frac{\ln x}{\sqrt{x}}$. Differentiating we obtain

$$f'(x) = \frac{\frac{1}{x}\sqrt{x} - \ln x \cdot \frac{1}{2x\sqrt{x}}}{x} = \frac{\ln x - 2}{2x\sqrt{x}}$$

The derivative is zero if $\ln x - 2 = 0$ or $x = e^2$. Because the logarithm is an increasing function, we have $f'(x) > 0$ for all $x > e^2$ and $f'(x) < 0$ for all $0 < x < e^2$. Hence $x = e^2$ is a minimum point.

The function can have a vertical asymptote at $x = 0$ and a horizontal one at infinity. At $x = 0$ we have

$$\lim_{x\to0^+} \left(-1 - \frac{\ln x}{\sqrt{x}}\right) = \infty$$

Indeed, this follows from $\lim_{x\to0^+} \ln x = -\infty$ and $\lim_{x\to0^+} \frac{1}{\sqrt{x}} = \infty$. Hence $x = 0$ is a vertical asymptote.

At infinity we have

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \left(\frac{-1}{x} - \frac{\ln x}{x\sqrt{x}}\right) \overset{L}{=} 0 - \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{3}{2}x^{1/2}} = 0$$

Hence

$$b = \lim_{x\to\infty} f(x) = -1 - \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \stackrel{L}{=} -1 - \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = -1 - \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = -1$$

Hence $y = -1$ is a horizontal asymptote at infinity.

3) As a composite function of continuous functions, it is clear that $f(x)$ is continuous for all $x \neq 4$. We check at $x = 4$. We clearly have $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} a = a$. To compute $\lim_{x\to 4^+} f(x)$ we define

$$y = (x-3)^{\frac{1}{x-4}}$$

and taking the logarithm in both sides we obtain

$$\ln y = \frac{\ln(x-3)}{x-4}$$

Hence

$$\lim_{x\to 4^+} \ln y = \lim_{x\to 4^+} \frac{\ln(x-3)}{x-4} \stackrel{L}{=} \lim_{x\to 4^+} \frac{\frac{1}{x-3}}{1} = 1$$

Hence $\ln \lim_{x\to 4^+} y = 1$ or $\lim_{x\to 4^+} y = e$. Thus, for $f(x)$ to be continuous at $x = 4$ we need $a = e$.

4) The function $f(x) = \ln(x^2+1)$ is differentiable for all x. If $x > 0$ we apply the mean value theorem to the interval $[0, x]$, and if $x < 0$ we apply it to $[x, 0]$. Since $f'(x) = \frac{2x}{x^2+1}$, it follows from the mean value theorem that there is a point c between zero and x such that

$$f'(c) = \frac{2c}{c^2+1} = \frac{f(x) - f(0)}{x - 0} = \frac{\ln(x^2+1)}{x}$$

Here we used the fact that $f(0) = 0$. Thus the result will follow if we prove that $-1 \leq \frac{2c}{c^2+1} \leq 1$ for all c. Multiplying by c^2+1 this is equivalent to $-(c^2+1) \leq 2c \leq c^2+1$ or $-(c^2+2c+1) \leq 0 \leq c^2-2c+1$ or $-(c+1)^2 \leq 0 \leq (c-1)^2$ which is clearly true.

5) Define $f(x) = x^3 - 3ax + 2$. Then $f'(x) = 3x^2 - 3a = 3(x^2 - a)$. Suppose that $a < 0$. Then $f'(x) > 0$ for all x, which means that $f(x)$ is an increasing function. We proved in class that a polynomial with odd degree has at least one real root. Hence, if $a < 0$, then $f(x)$ has exactly one real root. If $a = 0$, then $x^3 - 2$ has one real root, so we may assume that $a > 0$. In this case, $f'(x) = 0$ if and only if $x = \pm\sqrt{a}$. The function $f'(x)$ is positive if $x < -\sqrt{a}$ or $x > \sqrt{a}$ and negative if $-\sqrt{a} < x < \sqrt{a}$. Hence $f(x)$ has a maximum point at $x = -\sqrt{a}$ and a minimum point at $x = \sqrt{a}$. We have $f(-\sqrt{a}) = -a^{3/2} - 3a(-a^{1/2} + 2 = 2(a^{3/2} + 2) > 0$. Since $f(x)$ is an increasing function in the interval $(-\infty, -\sqrt{a})$, and since the maximum point gives us a positive number, it follows that $f(x)$ has one real root at the interval $(-\infty, -\sqrt{a})$. Thus, for $f(x)$ to have a unique real root we must have that $f(\sqrt{a}) > 0$. Hence $a^{3/2} - 3a^{3/2} + 2 = 2(1 - a^{3/2}) > 0$. Hence $a < 1$. We conclude that $f(x)$ has a unique real root if and only if $a < 1$.

6) Differentiating $g(x)$ gives us $g'(x) = f'(x)f'(f(x))$ and $g''(x) = f''(x)f'(f(x)) + (f'(x))^2 f''(f(x))$ where we obtained the last equality by using the formula for the derivative of a product of two functions. Hence, $g'(1) = f'(1)f'(f(1)) = f'(1)f'(2) = 1 \cdot 1 = 1$. Also, $g''(1) = f''(1)f'(f(1)) + (f'(1))^2 f''(f(1)) = f''(1)f'(2) + (f'(1))^2 f''(2) = (-1) \cdot 1 + 1^2 \cdot (-1) = -2$.

7) Plug $x = 1000$ into the equality $f(x)f(f(x)) = 1$ and get $f(1000)f(f(1000)) = 1$ or $999f(999) = 1$. Hence $f(999) = \frac{1}{999}$. Since $f(x)$ is continuous and obtains the values $\frac{1}{999}$ and 999 then by a theorem (see problem **9** in Exercise Number **5**) it obtains any value between $\frac{1}{999}$ and 999. Hence there is a number a such that $f(a) = 500$.