Solutions to Final in Mathematics A

Moed B

1) a) We consider the two side limits

$$\lim_{x\to 2^+} \frac{x^2-4}{|x-2|} = \lim_{x\to 2^+} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2^+}(x+2) = 4$$

$$\lim_{x\to 2^-} \frac{x^2-4}{|x-2|} = \lim_{x\to 2^-} \frac{(x-2)(x+2)}{-(x-2)} = -\lim_{x\to 2^+}(x+2) = -4$$

Hence there is no limit.

b) Since we want to compute the limit at zero, we may assume that the values of x are close to zero such that $f(x) = -x$. Hence

$$\lim_{x\to 0} \frac{xe^x - x^3}{f(x)} = \lim_{x\to 0} \frac{xe^x - x^3}{-x} = -\lim_{x\to 0}(e^x - x^2) = -1$$

2) a) Assume first that $x \geq 3$. Then we need to solve $x - 3 > 3x + 2$ or $-5/2 > x$ and hence there are no solutions in this case. When $x < 3$ we have $-(x-3) > 3x+2$ or $1/4 > x$. Hence the solution is $1/4 > x$.

b) Denote $f(x) = x - e^{-2x}$. Then $f'(x) = 1 + 2e^{-2x}$. Hence $f'(x) > 0$ for all x and hence $f(x)$ increases for all x. Hence it can intersect the x axis at most once. Since $f(0) = -1 < 0$ and $f(x)$ is continuous it must intersect the x axis and hence $x = e^{-2x}$ has exactly one solution.

3) a) We have

$$g'(x) = \left(\frac{1}{f(x)} + 1\right) = -\frac{f'(x)}{f^2(x)} + 1$$

b) Since $f'(x) = (f(x))^2$, it follows from part **a)** that

$$g'(x) = -\frac{f'(x)}{f^2(x)} + 1 = -1 + 1 = 0$$

For $b > 0$ apply the mean value Theorem to the interval $[0, b]$. Then

$$\frac{g(b) - g(0)}{b - 0} = g'(c)$$

where $c \in (0, b)$. Since $g'(x) = 0$ we deduce that $g'(c) = 0$, hence $g(b) - g(0) = 0$ or $g(b) = g(0)$.

4) Since $f(x)$ has no vertical asymptote at $x = b$, then we must have $b = -2$ or $b = 2$. Indeed, if $b \neq \pm 2$ then we have

$$\lim_{x \to b^+} f(x) = \lim_{x \to b^+} \frac{(x-2)(x+2)}{(x-a)(x-b)} = \pm\infty$$

In this case $f(x)$ would have a vertical asymptote.

Assume that $b = -2$. Then

$$f(x) = \frac{(x-2)(x+2)}{(x-a)(x+2)} = \frac{x-2}{x-a}$$

We have $f'(x) = \frac{2-a}{(x-a)^2}$. Hence, $f(x)$ would have a local extreme point if and only if $a = 2$. If this is the case then $f(x) = \frac{x-2}{x-2} = 1$ if $x \neq 2$, and this function does not have a vertical asymptote at $x = a = 2$. In a similar way we handle the case when $b = 2$.

From the above we deduce that the possible values for a and b are as follows. We have $b = \pm 2$ and $a \neq \pm 2$.

5) If we denote the length of the rectangular on the axis by x, then the length of the other side is $8 - x^3$. Hence the area is given by $f(x) = x(8 - x^3) = 8x - x^4$. Thus $f'(x) = 8 - 4x^3$ and hence the derivative is zero at $x = \sqrt[3]{2}$. This is a maximum point.

6) Apply the mean value theorem to $f(x)$ in the interval $[0, 1]$. Thus we have

$$\frac{f(1) - f(0)}{1 - 0} = e^{c^2}$$

where $c \in (0, 1)$. We know that $f(0) = 10$. Hence $f(1) - 10 = e^{c^2}$. Also, e^{x^2} is an increasing function at the interval $[0, 1]$. Hence $1 = e^{0^2} < e^{c^2} < e^{1^2} = e$. Thus, from $1 < f(1) - 10 < e$ and hence $11 < f(1) < 10 + e$.

7) a) We have

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2 + x + a) = a \qquad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} ae^x = a$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} ae^x = ae \qquad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} aex = ae$$

Hence $f(x)$ is continuous for all values of a.

b) Clearly $f(x)$ is differentiable for all x except possibly at $x = 0, 1$. At zero we have

$$\lim_{h \to 0^\pm} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^\pm} \frac{f(h) - a}{h}$$

Hence

$$\lim_{h\to 0^-} \frac{h^2+h+a-a}{h} = \lim_{h\to 0^-}(h+1) = 1$$

and

$$\lim_{h\to 0^+} \frac{ae^h-a}{h} = a\lim_{h\to 0^+}\frac{e^h-1}{h} \stackrel{L}{=} a\lim_{h\to 0^+} e^h = a$$

Hence we must have $a=1$. At $x=1$ we have

$$\lim_{h\to 0^\pm} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^\pm}\frac{f(1+h)-ae}{h}$$

Hence

$$\lim_{h\to 0^-} \frac{ae^{1+h}-ae}{h} = a\lim_{h\to 0^-}\frac{e^{1+h}-e}{h} \stackrel{L}{=} a\lim_{h\to 0^-} e^{1+h} = ae$$

Also

$$\lim_{h\to 0^+} \frac{ae(1+h)-ae}{h} = ae$$

Overall we obtain that for $f(x)$ to be differentiable we must have $a=1$.