# Solutions to Final in Mathematics A

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**1) a)** Find the inverse function of

$$y = \frac{x-1}{x+3}$$

**Solution:** Replacing $x$ with $y$ and $y$ with $x$ we obtain $x = \frac{y-1}{x+3}$. Hence $x(y+3) = y-1$ and $xy + 3x = y - 1$. Hence the inverse function is $y = \frac{3x+1}{1-x}$.

**b)** Differentiate

$$y = \frac{(x+1)(x+2)(x+3)}{(x^2+1)(x^2+2)(x^2+3)}$$

**Solution:** Taking logarithm in both sides we obtain

$$\ln y = \ln \frac{(x+1)(x+2)(x+3)}{(x^2+1)(x^2+2)(x^2+3)} = \ln(x+1) + \ln(x+2) + \ln(x+3) -$$

$$-\ln(x^2+1) - \ln(x^2+2) - \ln(x^2+3)$$

Differentiating we obtain

$$\frac{y'}{y} = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{2x}{x^2+1} - \frac{2x}{x^2+2} - \frac{2x}{x^2+3}$$

Hence

$$y' = \frac{(x+1)(x+2)(x+3)}{(x^2+1)(x^2+2)(x^2+3)} \left( \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{2x}{x^2+1} - \frac{2x}{x^2+2} - \frac{2x}{x^2+3} \right)$$

**2) a)** Using the definition only, find the derivative of

$$y = 4 - \sqrt{x+3}$$

**Solution:** We have

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(4-\sqrt{x+h+3})-(4-\sqrt{x+3})}{h} = \lim_{h\to 0}\frac{\sqrt{x+3}-\sqrt{x+h+3}}{h} =$$

$$= \lim_{h\to 0}\frac{(\sqrt{x+3}-\sqrt{x+h+3})(\sqrt{x+3}-\sqrt{x+h+3})}{h(\sqrt{x+3}-\sqrt{x+h+3})} = \lim_{h\to 0}\frac{-h}{h(\sqrt{x+3}-\sqrt{x+h+3})} =$$

$$= \lim_{h\to 0}\frac{-1}{\sqrt{x+3}-\sqrt{x+h+3}} = -\frac{1}{2\sqrt{x+3}}$$

**b)** Compute the limit $\lim_{x\to 1}\frac{2^x-2}{|x-1|^2}$.

**Solution:** We have

$$\lim_{x\to 1}\frac{2^x-2}{|x-1|^2} = \lim_{x\to 1}\frac{2^x-2}{(x-1)^2} = \lim_{x\to 1}\frac{2^x \ln 2}{2(x-1)}$$

The last equality follows from L'hospital rule. The right most limit does not exist since it is of the type one over zero.

**3)** Find the equation of the normal to $y^3(2-y)=x^2$ at $(-1,1)$.

**Solution:** Write the equation as $2y^3-y^4=x^2$. Differentiating, we get $6y^2y'-4y^3y'=2x$. Plugging the point $(-1,1)$ we get $2y'|_{(-1,1)}=-2$ or $y'|_{(-1,1)}=-1$. Hence, the equation of the normal is $(y-1)=(x+1)$ or $y=x+2$

**4)** Define

$$f(x)=\begin{cases}3, & x \text{ is odd}\\ x^2+3x, & \text{otherwise}\end{cases}$$

Determine where $f(x)$ is continuous.

**Solution:** The function is not continuous. For example, at $x=1$ we have $\lim_{x\to 1} f(x) = \lim_{x\to 1}(x^2+3x)=4\neq f(1)$.

**5)** Prove that for $0\le x\le 1$ we have

$$1+\ln(x+\sqrt{1+x^2})\ge\sqrt{1+x^2}$$

Define $f(x)=1+\ln(x+\sqrt{1+x^2})$ and $g(x)=\sqrt{1+x^2}$. We have $f(0)=g(0)$. Hence, it is enough to prove that for all $0\le x\le 1$ we have $f'(x)\ge g'(x)$. We have

$$f'(x)=\frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{x}{\sqrt{1+x^2}}\right)=\frac{1}{x+\sqrt{1+x^2}}\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}=\frac{1}{\sqrt{1+x^2}}$$

Also, we have

$$g'(x) = \frac{1}{\sqrt{1+x^2}}$$

Since $0 \leq x \leq 1$, we obtain $f'(x) \geq g'(x)$.

**6)** Define

$$f(x) = \frac{x^2-9}{(x-a)(x-b)}$$

It is given that $a > 0$ and that $\lim_{x\to a} f(x) = 3$. Find all local extreme points for $f(x)$.

**Solution:** Since the limit exists at $x = a$ then $x = \pm 3$. Since $a > 0$ then $x = 3$. Hence, for all $x \neq 3$ we have $f(x) = \frac{x+3}{x-b}$. Hence $f'(x) = \frac{-b-3}{(x-b)^2}$. This is zero only if $b = 3$. But since $\lim_{x\to a} f(x) = 3$, then $b \neq 3$. So $f(x)$ has no local extreme points.

**7)** Assume $a > 0$. For what values of $a$ the function

$$f(x) = x^2 - 2a\ln x - a$$

has exactly one root.

**Solution:** We have $f'(x) = \frac{2(x^2-a)}{x}$. The function is defined for all $x > 0$. Since $a > 0$, we see that $x = a$ is a local minimum. Therefore, if $f(a) >=$, the function has no roots, and if $f(a) < 0$, the function has 2 roots. Hence, it will have one root only if $f(a) = 0$. This happens if $a = 1$.