FINAL IN KAC MOODY ALGEBRAS 2013

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Exercises

1) Let $\mathcal{G}(A)$ denote the affine Kac-Moody algebra associated with the generalized Cartan matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. Denote the simple roots by α_0, α_1.

a) Write down the set of roots of this algebra. Write down the positive roots.

b) Let W denote the Weyl group of $\mathcal{G}(A)$. Write down all the elements of W. Verify that W preserves the set Δ_+^{im}.

c) For each positive root α write a nonzero vector in $\mathcal{G}_\alpha$.

2) Let $\mathcal{G}(A)$ denote the affine Kac-Moody algebra associated with the generalized Cartan matrix $\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$. Denote the simple roots by α_1, α_2. Write down all positive roots whose height is less or equal to six. (If $\alpha = k_1\alpha_1 + k_2\alpha_2$ is a root, its height is $k_1 + k_2$).

3) Let $\mathcal{G}(A)$ denote any Kac-Moody algebra. Prove that for any two simple roots α_i and α_j we have $\dim \mathcal{G}_{\alpha_i + n\alpha_j} \leq 1$ for $n = 0, 1, 2, \ldots$. Similarly prove that $\dim \mathcal{G}_{2(\alpha_i+\alpha_j)} \leq 1$.

4) For $\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$ prove that $\dim \mathcal{G}_{2\alpha_1+3\alpha_2} = 2$ in $\mathcal{G}(A)$. Prove that for an arbitrary 2×2 matrix A, we have $\dim \mathcal{G}_{2\alpha_1+3\alpha_2} \leq 2$.

5) Let $A = (A_{i,j})$ denote a generalized Cartan matrix, and assume that for some $i \neq j$ we have $A_{i,j}A_{j,i} \geq 4$. Prove that W is not a finite group.
(**Hint:** Show that $r_{\alpha_i} r_{\alpha_j}$ has an infinite order.)

6) Let A denote a generalized Cartan matrix. We say that A is strictly hyperbolic if it is of indefinite type and if any connected proper subgraph of $D(A)$ is of finite type. We say that A is hyperbolic if it is of indefinite type and if any connected proper subgraph of $D(A)$ is of finite or affine type.

a) Prove that $A = \begin{pmatrix} 2 & a \\ b & 2 \end{pmatrix}$ is strictly hyperbolic if and only if $ab > 4$.

b) Prove that there are only finite number of hyperbolic matrices of order three.

c) Prove that the order of a strictly hyperbolic matrix is at most five. Classify them.

7) Let $\mathcal{G}$ denote a Lie algebra and let $\phi : \mathcal{G} \times \mathcal{G} \mapsto \mathbf{C}$. Let V denote a complex vector space, and let $\pi : \mathcal{G} \mapsto \mathrm{End}(V)$ satisfy

$$\pi([X, Y])v = \pi(X)\pi(Y)v - \pi(Y)\pi(X)v + \phi(X, Y)v$$

for all $X, Y \in \mathcal{G}$ and $v \in V$. Prove that ϕ is a 2-cycocle.

8) In class we mentioned that for a simple Lie algebra every derivation is inner. The aim of this exercise is to construct an example of a Lie algebra which have derivations which are not inner.

a) Let $\mathcal{H}$ denote a vector space of dimension 2 over $\mathbf{C}$. Let h_1 and h_2 be a basis for $\mathcal{H}$, and define

$$[a_1h_1 + a_2h_2, b_1h_1 + b_2h_2] = (a_1b_2 - a_2b_1)h_2$$

Prove that $\mathcal{H}$ is a non-abelian Lie algebra.

b) Prove that any two dimensional non-abelian Lie algebra is isomorphic to $\mathcal{H}$.

c) Prove that $\mathcal{H}$ is isomorphic to $\mathrm{Der}\mathcal{H}$.

d) For an arbitrary Lie algebra $\mathcal{G}$ show that the set $\mathrm{ad}(\mathcal{G}) = \{\mathrm{ad}(g) \ : \ g \in \mathcal{G}\}$ is an ideal in $\mathrm{Der}(\mathcal{G})$.

e) Prove that for the Lie algebra $\mathbf{C} \times \mathcal{H}$ there are derivations which are not inner. (Here $\mathbf{C}$ is considered as the abelian Lie algebra of dimension one.)