Solution of Moed A in Linear Algebra 2 2013

David Ginzburg

1) Find a 3×3 matrix over $\mathbf{R}$, which is not diagonizable and which commutes with the matrix $A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Solution: The characteristic polynomial of A is $x(x-1)^2$. Thus, we have $A = P^{-1}RP$ where $R = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$ or $R = \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 0 \end{pmatrix}$. (In fact if you solve the equation $(A-I)v = 0$, then you will see that A is diagonizable.) In both cases, one can check that $L = \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix}$ commutes with R. Hence $P^{-1}LP$ commutes with A.

2) Let $f\ :\ \mathbf{R}^2 \times \mathbf{R}^2 \mapsto \mathbf{R}$ denote a symmetric bi-linear form. Let $g \in Mat_2(\mathbf{R})$ satisfy $f(gu, gv) = f(u, v)$ for all $u, v \in \mathbf{R}^2$. What are the possible values for $\det g$?

Solution: Write $f(u, v) = u^tAv$ and A is symmetric. Then $f(gu, gv) = f(u, v)$ implies $u^tg^tAgv = u^tAv$ for all $u, v \in \mathbf{R}^2$. This last equation holds for all $u, v \in \mathbf{R}^2$ if and only if $g^tAg = A$. Since A is symmetric, there is an invertible matrix P such that $A = P^tDP$ and where $D = \pm I$ or $D = \begin{pmatrix} \pm 1 & \\ & 0 \end{pmatrix}$. In both cases, plugging this into the equality $g^tAg = A$, we obtain $g^tP^tDPg = P^tDP$, or $(PgP^{-1})^tD(PgP^{-1}) = D$. Let $h = PgP^{-1}$. Then $h^tDh = D$ and $\det g = \det h$. If $D = \pm I$, then $h^tDh = D$ is the same as $h^th = I$ and h is orthogonal. Hence $\det g = \det h = \pm 1$. If $D = \begin{pmatrix} \pm 1 & \\ & 0 \end{pmatrix}$ and $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $h^tDh = D$ is equivalent to $\begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Thus $a = \pm 1$ and $b = 0$, and $h = \begin{pmatrix} \pm 1 & 0 \\ c & d \end{pmatrix}$. This implies that in this case $\det g$ can obtain any value.

3) Let V denote an inner product vector space over $\mathbf{C}$. Let $T\ :\ V \mapsto V$ denote a linear map with the property that every eigenvector of $T + T^*$ is also an eigenvector of $T - T^*$. Prove that T is normal.

Solution: By choosing a base of V, we may assume that T is represented by a matrix A with the property that every eigenvector of $A+A^*$ is also an eigenvector of $A-A^*$. We need to prove that A is normal. Since $(A+A^*)^* = A+A^*$, we deduce that $A+A^*$ is self adjoint. Hence there is a unitary matrix U such that $A+A^* = U^*D_1U$ and D_1 is diagonal. The columns of U are all eigenvectors for $A+A^*$. Since every eigenvector for $A+A^*$ is also an eigenvector for $A-A^*$, it follows that that the columns of U are also a base of eigenvectors for $A-A^*$. Hence, there exists a diagonal matrix D_2 such that $A-A^* = U^*D_2U$. Hence

$$A = \frac{1}{2}(A+A^*) + \frac{1}{2}(A-A^*) = \frac{1}{2}(U^*D_1U + U^*D_2U) = \frac{1}{2}U^*(D_1+D_2)U$$

Since D_1+D_2 is a diagonal matrix, it is normal, and hence A is normal.

4) Let V denote an an inner product vector space over $\mathbf{C}$. Let $T, S : V \mapsto V$ be linear maps such that S and S^*T are positive definite. Prove that the eigenvalues of T are all positive real numbers.

Solution: Since S^*T is positive definite, it is self adjoint. Hence, $S^*T = (S^*T)^* = T^*S$. For all $v \in V$ we have

$$0 < (S^*Tv, v) = (T^*Sv, v) = (Sv, Tv)$$

If v is an eigenvector for T with eigenvalue λ, then $Tv = \lambda v$. Hence,

$$0 < (Sv, Tv) = (Sv, \lambda v) = \bar{\lambda}(Sv, v)$$

Since S is positive definite, then $(Sv, v) > 0$, and hence λ must be a real positive number.