Solution of Moed B in Linear Algebra 2 2013

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1) a) Let V be a vector space over $\mathbf{C}$ and let $T : V \mapsto V$ denote a linear map satisfying $T^m = T^n$ for some positive integers $m \neq n$. Find conditions on T, m and n such that $T^2 = T$.

Solution: Let A be the matrix representing T with respect to any base. Then $A^m = A^n$. If $A = PRP^{-1}$ and R is the Jordan matrix corresponding to A, then $R^m = R^n$. Hence, if $J_r(\lambda)$ is any Jordan block appearing in R, then we require that $J_r(\lambda)^2 = J_r(\lambda)$. By matrix multiplication this can happen only if $r = 1$ and $\lambda = 0, 1$. Thus the conditions are: first, T must be diagonalizable and $m = n \pm 1$. The last condition guarantee that the eigenvalues of A are zeros and ones.

b) Let V be a vector space over $\mathbf{Q}$, and let f denote a bi-linear form defined on V. Prove that $f(u, u) = 0$ for all $u \in V$ if and only if $f(v, w) = -f(w, v)$ for all $v, w \in V$.

Solution: Assume that $f(u, u) = 0$ for all $u \in V$. Then, for $u = v + w$ we have

$$0 = f(v + w, v + w) = f(v, v) + f(v, w) + f(w, v) + f(w, w) = f(v, w) + f(w, v)$$

Hence $f(v, w) = -f(w, v)$. Conversely, assume that $f(v, w) = -f(w, v)$ for all $v, w \in V$. Choosing $v = w = u$ we obtain $f(u, u) = -f(u, u)$, and hence $f(u, u) = 0$.

2) Let n be an even positive integer, and let $A, B \in Mat_2(\mathbf{R})$ such that $A = B^2$. Let $p(x)$ denote the characteristic polynomial of A, and let $q(x)$ denote the characteristic polynomial of B. Prove that $p(x^2) = q(x)q(-x)$.

Solution: We may assume that A and B are in Jordan form (viewing $\mathbf{R}$ as a subfield of $\mathbf{C}$). Over $\mathbf{C}$ we can write $q(x) = (x - \lambda_1) \ldots (x - \lambda_n)$. Here $\lambda_i \in \mathbf{C}$ and need note be distinct. Thus, the condition $A = B^2$ implies that $p(x) = (x - \lambda_1^2) \ldots (x - \lambda_n^2)$. Hence,

$$p(x^2) = (x^2 - \lambda_1^2) \ldots (x^2 - \lambda_n^2) = (x - \lambda_1) \ldots (x - \lambda_n)(x + \lambda_1) \ldots (x + \lambda_n) =$$

$$= q(x)(-1)^n(-x - \lambda_1) \ldots (-x - \lambda_n) = q(x)q(-x)$$

where we used the fact that n is even to write $(-1)^n = 1$.

3) Assume that $A \in Mat_n(\mathbf{R})$ satisfies $A + A^t = I$. Prove that λ is an eigenvalue of A then $\lambda = \frac{1}{2} + i\alpha$ where $\alpha \in \mathbf{R}$.

Solution: We view A as a matrix over $\mathbf{C}$. Let $(,)$ denote the standard inner product over $\mathbf{C}$, then $A^* = A^t$. Assume that $Av = \lambda v$. Then

$$(v, v) = ((A + A^t)v, v) = (Av, v) + (A^t v, v) = (Av, v) + (A^* v, v) = (Av, v) + (v, Av) =$$

$$= (Av, v) + \overline{(Av, v)} = (\lambda v, v) + \overline{(\lambda v, v)} = (\lambda + \bar{\lambda})(v, v)$$

Since $(v, v) \neq 0$, then $\lambda + \bar{\lambda} = 1$ which implies that $\lambda = \frac{1}{2} + i\alpha$.

4) Let $A \in Mat_n(\mathbf{C})$ whose rank is r. Let $m(t)$ denote its minimal polynomial. Prove that $\deg m(t) \leq r + 1$. Find conditions when $\deg m(t) = r + 1$.

Solution: We may assume that A is in its Jordan form. Let λ denote an eigenvalue for A. Let $J_m(\lambda)$ denote the *largest* Jordan block which appears in A and is associated with λ. In other words, if $J_k(\lambda)$ also appears in A, then $k \leq m$. Thus, this eigenvalue will contribute a factor of $(t - \lambda)^m$ to the minimal polynomial, and hence it will contribute the factor m to the degree of $m(t)$. Notice that the rank of the matrix $J_m(\lambda)$ is m if $\lambda \neq 0$ and is equal to $m - 1$ if $\lambda = 0$. Thus, each eigenvalue λ contributes to the degree of $m(t)$ a factor which is less or equal to the number rank $J_m(\lambda) + 1$, where $J_m(\lambda)$ is as above. Summing over all eigenvalues, the result follows.

The find conditions when there is an equality, assume first that $J_m(\lambda)$ is the largest block appearing in A and that $\lambda \neq 0$. Then this is the only Jordan block associated with λ which appears in A. Indeed, if $J_k(\lambda)$ also appears in A, then λ will contribute a factor of m to the degree of $m(t)$, but at least a factor of $m + k$ to the rank of A. Hence there can not be an equality. Therefore, we can have at most one block associated with each nonzero eigenvalue. The only way we can increase the degree of $m(t)$ by a factor which is greater then the contribution to the rank is by having at least one block Jordan of the form $J_m(0)$. However, to obtain the equality $\deg m(t) = r + 1$ we cannot have more than one Jordan block of the form $J_m(0)$ with $m > 1$.

To summarize the equality $\deg m(t) = r + 1$ holds if and only if A has the following

Jordan decomposition

$$\begin{pmatrix} J_{m_1}(\lambda_1) & & & & \\ & \ddots & & & \\ & & J_{m_k}(\lambda_k) & & \\ & & & J_{m_{k+1}}(0) & \\ & & & & 0_l \end{pmatrix}$$

Here $\lambda_i \neq \lambda_j$ for all $i \neq j$, and for all $1 \leq i \leq k$ we have $\lambda_i \neq 0$. Also, $m_{k+1} \geq 1$ and $l \geq 0$. By 0_l we mean the zero matrix of size l. For A as above we have $m(t) = t^{m_{k+1}}(t-\lambda_1)^{m_1} \dots (t-\lambda_k)^{m_k}$ and $\operatorname{rank} A = m_1 + \dots + m_k + m_{k+1} - 1$. So $\deg m(t) = r + 1$.