1) Give an example for a linear map \( T : F^4 \mapsto F^4 \) such that
\[ \text{Im} T = \text{Ker} T = \text{Sp}\{ (1,1,1,1); (1,1,1,0) \} \]

\textbf{Solution :} Complete the two vectors \((1,1,1,1)\) and \((1,1,1,0)\) to a basis in \( F^4 \). For example choose \((1,0,0,0)\) and \((0,1,0,0)\). Then we are looking for a map \( T \) such that
\[ T((1,1,1,1)) = T((1,1,1,0)) = 0; \quad T((1,0,0,0)) = (1,1,1,1); \quad T((0,1,0,0)) = (1,1,1,0) \]
To give an explicit formula for \( T \), let \((x,y,z,w) \in F^4\). Then a simple computation implies
\[ (x,y,z,w) = \alpha(1,1,1,1) + \beta(1,1,1,0) + (x - z)(1,0,0,0) + (y - z)(0,1,0,0) \]
Here \( \alpha \) and \( \beta \) are some elements in \( F \) which we dont care about. Hence,
\[ T((x,y,z,w)) = (x - z)T((1,0,0,0)) + (y - z)T((0,1,0,0)) = \]
\[ (x + y - 2z, x + y - 2z, x + y - 2z, x - z) \]
Clearly such a \( T \) is not unique.

2) Let \( U, V \) and \( W \) denote three vector spaces over a field \( F \). Let \( T : U \mapsto V \) and \( S : V \mapsto W \) be two linear transformation such that \( S \circ T \) is an isomorphism. Prove that \( V = \text{Im} T \oplus \text{Ker} S \).

\textbf{Solution :} To prove that the sum is direct we first prove that \( \text{Im} T \cap \text{Ker} S = \{0\} \).
Let \( v \in \text{Im} T \cap \text{Ker} S \). Then \( S(v) = 0 \), and there is \( u \in U \) such that \( T(u) = v \). But then \((S(T(u))) = 0 \). Since \( S \circ T \) is an isomorphism, we get \( u = 0 \) and hence \( v = 0 \).

We give two proofs that \( V = \text{Im} T + \text{Ker} S \). The first is based mainly on dimension considerations. Notice that since \( S \circ T \) is an isomorphism then \( T \) is one to one and \( S \) is onto. \textbf{(It is not true that \( T \) and \( S \) must be isomorphisms!!)}. For example, to show that \( T \)
is one to one, let \( Tu = 0 \) for some \( u \in U \). Then \( 0 = S(Tu) = (S \circ T)(u) \) which implies that \( u = 0 \) since \( S \circ T \) is an isomorphism. Similarly, one proves that \( S \) is onto. Hence we have

\[
\dim \ker T = 0; \quad \dim \im S = \dim W; \quad \dim U = \dim W \tag{1}
\]

The first identity follows from the fact that \( T \) is one to one. The second follows from the fact that \( S \) is onto, and the third because \( S \circ T \) is an isomorphism.

Applying the two dimension Theorems, and using identities (1) we get

\[
\dim U = \dim \im T + \dim \ker T = \dim \im T \tag{2}
\]

\[
\dim V = \dim \im S + \dim \ker S = \dim W + \dim \ker S = \dim U + \dim \ker S + \dim \im T + \dim \ker S \tag{3}
\]

where the last equality is obtained by plugging in identity (2). We obtain,

\[
\dim V = \dim \im T + \dim \ker S
\]

We also have the dimension theorem

\[
\dim(\im T + \ker S) = \dim \im T + \dim \ker S - \dim(\im T \cap \ker S) = \dim \im T + \dim \ker S = \dim V
\]

From this we deduce that \( V = \im T + \ker S \), and we are done.

In the second proof the idea is to define a certain map from \( V \) to itself. To do that, let \( K : W \rightarrow U \) denote the inverse of the map \( S \circ T \). Then \( L = T \circ K \circ S \) is a linear map from \( V \) to itself.

Let \( v \in V \). Then \( L(v) = T((K \circ S)(v)) \). Hence \( L(v) \in \im T \). Also, since \( S \circ T \circ K = I_V \) then

\[
S(v - L(v)) = S(v) - (S \circ L)(v) = S(v) - (S \circ T \circ K \circ S)(v) = S(v) - S(v) = 0
\]

Hence \( v - L(v) \in \ker S \). The identity \( v = L(v) + (v - L(v)) \) implies that \( V = \im T + \ker S \).

\[3\) Write down all the matrices \( A \) of size three such that the vector space of all the solutions to the homogeneous system \( Ax = 0 \) will be generated by the vector \((1, 2, 3)^t\).\]
Solution: We first determine all the row echelon matrices with this property. Since $V = \text{Sp}\{(1, 2, 3)^t\}$ is one dimensional, then we must have $\text{rank}(A) = 2$. So we have two possible cases for the corresponding row echelon matrix. They are

$$P = \begin{pmatrix} 1 & a \\ 1 & b \end{pmatrix} \quad Q = \begin{pmatrix} 1 & a \\ 1 & b \end{pmatrix}$$

Since $Q(1, 2, 3)^t \neq 0$, then $Q$ is not good. On the other hand $P(1, 2, 3)^t = 0$ has a unique solution which is $a = -1/3$ and $b = -2/3$. In other words we get

$$P = \begin{pmatrix} 1 & -1/3 \\ 1 & -2/3 \end{pmatrix}$$

The conclusion is that all matrices $A$ which satisfies the requirements are $EP$ where $E$ is any invertible matrix of size three.

4) Let $u, v$ and $w$ be three vectors in an inner product vector space over the Real numbers. Assume that $u + v + w = 0$ and that $||u|| = ||v|| = ||w|| = 1$. Prove that $(u, v) = -\frac{1}{2}$.

Solution: We have $w = -(u + v)$. Hence

$$1 = ||w||^2 = ||-(u + v)||^2 = ||(u + v)||^2 = (u + v, u + v) = (u, u) + (u, v) + (v, u) + (v, v) = 1 + 2(u, v) + 1$$

Here we used the fact that $||u|| = ||v|| = 1$ and that $(u, v) = (v, u)$ because it is a Real inner product space. Comparing both sides of the above equation we get $(u, v) = -\frac{1}{2}$.

5) Let $A$ be a matrix of size four whose entries are all $\pm 1$. Prove that $8|\text{det} \ A|$.

Solution: If $\text{det} \ A = 0$ the result follows. Multiplying each column by $\pm 1$, the value of $|A|$ is changed by $\pm 1$. Hence we may assume that all the entries of the first row of $A$ are all one. Multiplying the last three rows by $\pm 1$, it is enough to prove the statement for

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & \pm 1 & \pm 1 & \pm 1 \\ -1 & \pm 1 & \pm 1 & \pm 1 \\ -1 & \pm 1 & \pm 1 & \pm 1 \end{pmatrix}$$
Add the first row to each of the three last rows. Then

\[
|A| = \begin{vmatrix}1 & 1 & 1 & 1 \\ 0 & 2a_1 & 2a_2 & 2a_3 \\ 0 & 2a_4 & 2a_5 & 2a_6 \\ 0 & 2a_7 & 2a_8 & 2a_9 \end{vmatrix} = \begin{vmatrix}2a_1 & 2a_2 & 2a_3 \\ 2a_4 & 2a_5 & 2a_6 \\ 2a_7 & 2a_8 & 2a_9 \end{vmatrix}
\]

From which the claim easily follows.