Linear Algebra Moed A 2015

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1) Give an example for a linear map $T : F^4 \mapsto F^4$ such that

$$ImT = KerT = \text{Sp}\{(1,1,1,1);\ (1,1,1,0)\}$$

Solution : Complete the two vectors $(1,1,1,1)$ and $(1,1,1,0)$ to a basis in F^4. For example choose $(1,0,0,0)$ and $(0,1,0,0)$. Then we are looking for a map T such that

$$T((1,1,1,1)) = T((1,1,1,0)) = 0; \quad T((1,0,0,0)) = (1,1,1,1); \quad T((0,1,0,0) = (1,1,1,0))$$

To give an explicit formula for T, let $(x,y,z,w) \in F^4$. Then a simple computation implies

$$(x,y,z,w) = \alpha(1,1,1,1) + \beta(1,1,1,0) + (x-z)(1,0,0,0) + (y-z)(0,1,0,0)$$

Here α and β are some elements in F which we dont care about. Hence,

$$T((x,y,z,w)) = (x-z)T((1,0,0,0)) + (y-z)T((0,1,0,0)) =$$

$$(x+y-2z, x+y-2z, x+y-2z, x-z)$$

Clearly such a T is not unique.

2) Let U, V and W denote three vector spaces over a field F. Let $T : U \mapsto V$ and $S : V \mapsto W$ be two linear transformation such that $S \circ T$ is an isomorphism. Prove that $V = Im\ T \oplus KerS$.

Solution : To prove that the sum is direct we first prove that $Im\ T \bigcap KerS = \{0\}$. Let $v \in Im\ T \bigcap KerS$. Then $S(v) = 0$, and there is $u \in U$ such that $T(u) = v$. But then $(S(T(u)) = 0$. Since $S \circ T$ is an isomorphism, we get $u = 0$ and hence $v = 0$.

We give two proofs that $V = ImT + KerS$. The first is based mainly on dimension considerations. Notice that since $S \circ T$ is an isomorphism then T is one to one and S is onto. (**It is not true that T and S must be isomorphisms!!**). For example, to show that T

is one to one, let $Tu = 0$ for some $u \in U$. Then $0 = S(Tu) = (S \circ T)(u)$ which implies that $u = 0$ since $S \circ T$ is an isomorphism. Similarly, one proves that S is onto. Hence we have

$$\dim KerT = 0; \quad \dim ImS = \dim W; \quad \dim U = \dim W \tag{1}$$

The first identity follows from the fact that T is one to one. The second follows from the fact that S is onto, and the third because $S \circ T$ is an isomorphism.

Applying the two dimension Theorems, and using identities (1) we get

$$\dim U = \dim ImT + \dim KerT = \dim ImT \tag{2}$$

$$\dim V = \dim ImS + \dim KerS = \dim W + \dim KerS = \tag{3}$$

$$\dim U + \dim KerS + \dim ImT + \dim KerS$$

where the last equality is obtained by plugging in identity (2). We obtain,

$$\dim V = \dim ImT + \dim KerS$$

We also have the dimension theorem

$$\dim(ImT + KerS) = \dim ImT + \dim KerS - \dim(ImT \cap KerS) =$$

$$= \dim ImT + \dim KerS = \dim V$$

From this we deduce that $V = ImT + KerS$, and we are done.

In the second proof the idea is to define a certain map from V to itself. To do that, let $K : W \mapsto U$ denote the inverse of the map $S \circ T$. Then $L = T \circ K \circ S$ is a linear map from V to itself.

Let $v \in V$. Then $L(v) = T((K \circ S)(v))$. Hence $L(v) \in Im\ T$. Also, since $S \circ T \circ K = I_V$ then

$$S(v - L(v)) = S(v) - (S \circ L)(v) = S(v) - (S \circ T \circ K \circ S)(v) = S(v) - S(v) = 0$$

Hence $v - L(v) \in KerS$. The identity $v = L(v) + (v - L(v))$ implies that $V = Im\ T + KerS$.

3) Write down all the matrices A of size three such that the vector space of all the solutions to the homogeneous system $Ax = 0$ will be generated by the vector $(1, 2, 3)^t$.

Solution : We first determine all the row echelon matrices with this property. Since $V = \mathrm{Sp}\{(1,2,3)^t\}$ is one dimensional, then we must have $\mathrm{rank}(A) = 2$. So we have two possible cases for the corresponding row echelon matrix. They are

$$P = \begin{pmatrix} 1 & & a \\ & 1 & b \\ & & \end{pmatrix} \qquad Q = \begin{pmatrix} 1 & a & b \\ & & 1 \\ & & \end{pmatrix}$$

Since $Q(1,2,3)^t \neq 0$, then Q is not good. On the other hand $P(1,2,3)^t = 0$ has a unique solution which is $a = -1/3$ and $b = -2/3$. In other words we get

$$P = \begin{pmatrix} 1 & & -1/3 \\ & 1 & -2/3 \\ & & \end{pmatrix}$$

The conclusion is that all matrices A which satisfies the requirements are EP where E is any invertible matrix of size three.

4) Let u, v and w be three vectors in an inner product vector space over the Real numbers. Assume that $u + v + w = 0$ and that $||u|| = ||v|| = ||w|| = 1$. Prove that $(u, v) = -\frac{1}{2}$.

Solution : We have $w = -(u + v)$. Hence

$$1 = ||w||^2 = ||-(u+v)||^2 = ||(u+v)||^2 = (u+v, u+v) =$$

$$= (u,u) + (u,v) + (v,u) + (v,v) = 1 + 2(u,v) + 1$$

Here we used the fact that $||u|| = ||v|| = 1$ and that $(u, v) = (v, u)$ because it is a Real inner product space. Comparing both sides of the above equation we get $(u, v) = -\frac{1}{2}$.

5) Let A be a matrix of size four whose entries are all ± 1. Prove that $8 | \det A$.

Solution : If $\det A = 0$ the result follows. Multiplying each column by ± 1, the value of $|A|$ is changed by ± 1. Hence we may assume that all the entries of the first row of A are all one. Multiplying the last three rows by ± 1, it is enough to prove the statement for

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & \pm 1 & \pm 1 & \pm 1 \\ -1 & \pm 1 & \pm 1 & \pm 1 \\ -1 & \pm 1 & \pm 1 & \pm 1 \end{pmatrix}$$

Add the first row to each of the three last rows. Then

$$|A| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2a_1 & 2a_2 & 2a_3 \\ 0 & 2a_4 & 2a_5 & 2a_6 \\ 0 & 2a_7 & 2a_8 & 2a_9 \end{vmatrix} = \begin{vmatrix} 2a_1 & 2a_2 & 2a_3 \\ 2a_4 & 2a_5 & 2a_6 \\ 2a_7 & 2a_8 & 2a_9 \end{vmatrix}$$

From which the claim easily follows.