Linear Algebra Moed b 2015

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1) Consider the following system of equations over the Real numbers.

$$ax + y + bz = 1$$

$$-x + 3y - z = 1$$

$$x - y + 3z = 1$$

For what values of a and b, the system has infinite number of solutions?

Solution : Write the extended matrix of the system. We have

$$\begin{pmatrix} a & 1 & b & 1 \\ -1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 3 & 1 \\ -1 & 3 & -1 & 1 \\ a & 1 & b & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 3 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 1+a & b-3a & 1-a \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1+a & b-3a & 1-a \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & b-4a-1 & -2a \end{pmatrix}$$

If $b - 4a - 1 \neq 0$, then the system has a unique solution. Hence $b - 4a - 1 = 0$. If $-2a \neq 0$, the system has no solution. Hence $a = 0$ and then $b = 1$. In this case the system has infinite number of solutions.

2) Let $f(x)$ be a polynomial of degree n over the Real numbers. Prove that for all polynomial $g(x)$ whose degree is at most n, there are Real numbers α_i for $0 \leq i \leq n$ such that

$$g(x) = \alpha_0 f(x) + \alpha_1 f^{(1)}(x) + \cdots + \alpha_{n-1} f^{(n-1)}(x) + \alpha_n f^{(n)}(x)$$

Here $f^{(i)}(x)$ is the i-th derivative of $f(x)$.

Solution : This can be proved by induction on n. Its easy to check for $n = 1$. Assume its true for $n - 1$, and prove for n. Assume that

$$f(x) = a_n x^n + \cdots + a_1 x + a_0 \qquad\qquad g(x) = b_n x^n + \cdots + b_1 x + b_0$$

where we know that $a_n \neq 0$. Define the polynomial $h(x) = g(x) - \frac{b_n}{a_n} f(x)$. Then the degree of $h(x)$ is at most $n-1$. By the induction hypothesis, applied to $f^{(1)}(x)$, which is a polynomial of degree $n-1$, we have constants β_i such that

$$h(x) = \beta_1 f^{(1)}(x) + \beta_2 f^{(2)}(x) + \cdots + \beta_{n-1} f^{(n-1)}(x)$$

Plugging the definition of $h(x)$ into the last equation, the result follows for $g(x)$.

3) Let A be a matrix of size 2 over the Real numbers, and assume that there is a Real number $a > 0$ such that $|A^2 + aI| = 0$.
a) Prove that there is a nonzero vector $v \in \mathbf{R}^2$ such that $(A^2 + aI)v = 0$.
b) Prove that the set $\{v, Av\}$ is a linearly independent set.
c) Express $|A|$ in terms of a only.

Solution : a) This follows immediately from the fact that $|A^2 + aI| = 0$. Indeed, if the homogeneous system $(A^2 + aI)x = 0$ has only the trivial solution then $|A^2 + aI| \neq 0$.
b) Suppose that the set $\{v, Av\}$ is a linearly dependent set. Then there is a nonzero Real number λ such that $Av = \lambda v$. Hence

$$0 = (A^2 + aI)v = (A^2 v + av) = A(Av) + av = A(\lambda v) + av = \lambda^2 v + av = (\lambda^2 + a)v$$

By assumption $v \neq 0$, and since $a > 0$ is a Real number, then $\lambda^2 + a \neq 0$. Hence we obtain a contradiction, and hence the set $\{v, Av\}$ is a linearly independent set.
c) It follows from the first two parts that $\{v, Av\}$ is a base for $\mathbf{R}^2$. Also, we have

$$(A^2 + aI)Av = A(A^2 + aI)v = 0$$

Hence v and Av is a basic for the solution space of the system $(A^2 + aI)x = 0$. This implies that $\text{rank}(A^2 + aI) = 0$ or that $A^2 + aI = 0$. Hence $A^2 = -aI$ and $|A|^2 = a^2$. Since $a > 0$, then $|A| = a$.

4) Let $T : Mat_{n\times n}(F) \to Mat_{n\times n}(F)$ denote the map defined by $T(A) = A + aA^t$. Here F is a field and $a \in F$.
a) If $a \neq \pm 1$, prove that T is an isomorphism.
b) Give an explicit formula for T^{-1}.

Solution : a) To prove that T is linear we have

$$T(\alpha A + \beta B) = (\alpha A + \beta B) + a(\alpha A + \beta B)^t = \alpha(A + aA^t) + \beta(B + aB^t) = \alpha T(A) + \beta T(B)$$

To prove that it is one to one we consider its kernel. Assume that $T(A) = 0$. Then $A+aA^t = 0$ or $A = -aA^t$. Taking transpose on this last equation we obtain $A^t = -aA$. Plugging this into the first equation we obtain $A = a^2 A$. Since $a \neq \pm 1$ we obtain that $A = 0$. Hence $\ker T = 0$. Since every one to one linear map from a vector space to itself is an isomorphism, the first part follows.

b) Assume that $T^{-1}(B) = A$. Then $B = T(A) = A + aA^t$. Taking transpose we obtain $B^t = A^t + aA$. Hence $A^t = B^t - aA$. Plugging into the first equation we obtain $B = A + a(B^t - aA)$, or $A = \frac{1}{1-a^2}(B - aB^t)$.

5) Let $v \cdot u$ denote the standard inner product on $\mathbf{R}^n$. In other words, if $v = (x_1, \ldots, x_n)$ and $u = (y_1, \ldots, y_n)$, then $v \cdot u = x_1 y_1 + \cdots + x_n y_n$. Let A denote a matrix of order n with entries in $\mathbf{R}$. Prove that $< v, u >= (Av) \cdot (Au)$ defines an inner product on $\mathbf{R}^n$ if and only if $\text{rank} A = n$.

Solution : We apply the definition. First, linearity. We have

$$< \alpha v + \beta w, u >= (A(\alpha v + \beta w)) \cdot (Au) = (\alpha Av + \beta Aw) \cdot (Au) =$$

$$= \alpha (Av) \cdot (Au) + \beta (Aw) \cdot (Au) = \alpha < v, u > + \beta < v, w >$$

where the third equality follows from the fact that $v \cdot u$ is a linear map. Next we prove that $< v, u >=< u, v >$. Indeed,

$$< v, u >= (Av) \cdot (Au) = (Au) \cdot (Av) =< v, u >$$

Finally, we need to prove that if $< v, v >= 0$, then $v = 0$. By definition $< v, v >= 0$ is equivalent to $(Av) \cdot (Av) = 0$. Since $v \cdot u$ is an inner product, then $(Av) \cdot (Av) = 0$ is equivalent to $Av = 0$. If $\text{rank} A = n$, then the only solution to the system $Ax = 0$ is $x = 0$. This follows from the theorem we proved that the dimension of the space of all solutions to $Ax = 0$ is equal to $n - \text{rank} A$.