Solution for Final in Calculus A

Moad A 2009

1) The function $f(x)$ is clearly continuous for all $x \neq 0, 1$. Indeed, for $x \neq 0, 1$, $f(x)$ is a sum, product or a quotient of continuous functions whose denominator is nonzero. To check what happens in $x = 0$, we compute

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^{ax} - e^{-ax}}{bx} = \lim_{x\to 0^-} \frac{ae^{ax} + ae^{-ax}}{b} = \frac{2a}{b}$$

Here, the second equality follows from L'hopital's rule, since the limit is of type $\frac{0}{0}$. To compute the limit $\lim_{x\to 0^+} f(x)$ we notice that $|x| = x$ for $x > 0$. Hence

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(\frac{x}{x} + \frac{x^3 - 1}{x - 1}\right) = \lim_{x\to 0^+} \left(1 + \frac{x^3 - 1}{x - 1}\right) = 2$$

For $f(x)$ to be continuous at $x = 0$ we need to have $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$. Hence we get the equation $\frac{2a}{b} = c = 2$. To check the point $x = 1$, we compute

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \left(\frac{x}{x} + \frac{x^3 - 1}{x - 1}\right) = \lim_{x\to 1^-} \left(1 + \frac{(x-1)(x^2 + x + 1)}{x - 1}\right) = \lim_{x\to 1^-} (1 + x^2 + x + 1) = 4$$

Since $\lim_{x\to 1^+} f(x) = b$, then we must have $b = 4$. Hence the answer is $a = b = 4$ and $c = 2$.

2) Write $f(x) = Ax^{-1/2} + Bx^{1/2}$. Then $f'(x) = -\frac{1}{2}Ax^{-3/2} + \frac{1}{2}Bx^{-1/2} = \frac{x^{-3/2}}{2}(-A + Bx)$. It is given that $f(x)$ has a local minimum at the point $x = 9$. Hence $f'(9) = 0$. From this we deduce that $-A + 9B = 0$ or $A = 9B$. Thus $f(x) = 9Bx^{-1/2} + Bx^{1/2}$. Since the value of the function at the minimum point is 6, this means that the point $(9, 6)$ is on the graph of the function. Hence $6 = 9B9^{-1/2} + B9^{1/2}$, from which we deduce that $B = 1$. Hence $A = 9$.

3) a) Denote $y = (3^x + 4^x)^{1/x}$. Taking logarithm in both sides we obtain $\ln y = \frac{\ln(3^x + 4^x)}{x}$. We have

$$\lim_{x\to\infty} \frac{\ln(3^x + 4^x)}{x} = \lim_{x\to\infty} \frac{3^x \ln 3 + 4^x \ln 4}{3^x + 4^x}$$

where the equality is obtained using L'hopital's rule. We can use this rule since the limit on the left is of type $\frac{\infty}{\infty}$. We also used the formula $(a^x)' = a^x \ln a$. Pulling out the factor 4^x we obtain

$$\lim_{x\to\infty} \frac{3^x\ln3 + 4^x\ln4}{3^x + 4^x} = \lim_{x\to\infty} \frac{4^x((\frac{3}{4})^x \ln3 + \ln4)}{4^x((\frac{3}{4})^x + 1)} = \lim_{x\to\infty} \frac{((\frac{3}{4})^x \ln3 + \ln4)}{((\frac{3}{4})^x + 1)}$$

Since $\lim_{x\to\infty} \left(\frac{3}{4}\right)^x = 0$, we obtain

$$\lim_{x\to\infty} \frac{((\frac{3}{4})^x \ln3 + \ln4)}{((\frac{3}{4})^x + 1)} = \frac{0 + \ln4}{0+1} = \ln4$$

To summarize, we obtain $\lim_{x\to\infty} \ln y = \ln4$. Since the logarithm is continuous, we obtain $\ln \lim_{x\to\infty} y = \ln4$. Hence $\lim_{x\to\infty} y = 4$, or $\lim_{x\to\infty}(3^x + 4^x)^{1/x} = 4$.

b) Close to zero, for example you can take the interval $(-\frac{1}{4}, \frac{1}{4})$, the values of $2x - 1$ are negative, and the values of $2x+1$ are positive. Hence, close to zero we have $|2x-1| = -(2x-1)$ and $|2x+1| = 2x+1$. Hence,

$$\lim_{x\to0} \frac{|2x-1| - |2x+1|}{x} = \lim_{x\to0} \frac{-(2x-1) - (2x+1)}{x} = \lim_{x\to0} \frac{-4x}{x} = -4$$

4) To find inflection points we compute the second derivative. Notice that the function is defined for all $x > 0$. We have $y' = \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}$. Hence $y'' = -\frac{1}{x^2} + \frac{1}{(x+1)^2} + \frac{2}{(x+1)^3}$. Thus $y'' = \frac{-(x+1)^3 + x^2(x+1) + 2x^2}{x^2(x+1)^3} = -\frac{3x+1}{x^2(x+1)^3}$. Hence, the only possible inflection point is $x = -\frac{1}{3}$. However, since the function is defined for $x > 0$, it is clearly not an inflection point. Thus, y has no inflection points.

To study the asymptotes, we notice that since the function is defined for $x > 0$, then the only possibilities are at $x = 0$ and at ∞. At $x = 0$ we have $\lim_{x\to0^+}(\ln x - \ln(x+1) + \frac{1}{x+1}) = -\infty$. This follows from the fact that $\lim_{x\to0^+} \ln(x+1) = 0$ and $\lim_{x\to0^+} \frac{1}{x+1} = 1$ and $\lim_{x\to0^+} \ln x = -\infty$.

At ∞ we have

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \left(\frac{\ln x}{x} - \frac{\ln(x+1)}{x} + \frac{1}{x(x+1)} \right)$$

We have $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$ where the first equality follows using L'hopital's rule. Similarly, $\lim_{x\to\infty} \frac{\ln(x+1)}{x} = \lim_{x\to\infty} \frac{1}{x+1} = 0$. Finally, $\lim_{x\to\infty} \frac{1}{x(x+1)} = 0$. Thus $a = 0$. Hence, using the identity $\ln x - \ln(x+1) = \ln\frac{x}{x+1}$ we obtain

$$b = \lim_{x\to\infty}(f(x) - ax) = \lim_{x\to\infty} f(x) = \lim_{x\to\infty}(\ln\frac{x}{x+1} + \frac{1}{x+1}) = \lim_{x\to\infty} \ln\frac{x}{x+1} + \lim_{x\to\infty} \frac{1}{x+1}$$

The right most limit is zero, and $\lim_{x\to\infty} \ln\frac{x}{x+1} = \ln \lim_{x\to\infty} \frac{x}{x+1} = \ln 1 = 0$. Thus $b = 0$. Hence, the line $y = 0$ is an asymptote for $f(x)$ at ∞.

5) a) To show that $f(x)$ is continuous at $x = c$, we need to prove that $\lim_{x\to c} f(x) = f(c)$. It is given that for all x we have $|f(x) - f(c)| \leq 2009|x - c|$. This is equivalent to

$$-2009|x - c| \leq f(x) - f(c) \leq 2009|x - c|$$

Applying the Sandwich theorem, since $\lim_{x\to c} |x - c| = 0$, we deduce that $\lim_{x\to c}(f(x) - f(c)) = 0$. Since $f(c)$ is constant with respect to x, this is the same as $\lim_{x\to c} f(x) = f(c)$. Hence $f(x)$ is continuous at $x = c$.
b) The function $f(x)$ need not have a derivative at $x = c$. As an example, choose $f(x) = |x|$ and $c = 0$. Then the inequality $|f(x) - f(c)| \leq 2009|x - c|$ reduces to $|x| \leq 2009|x|$ which is clearly true. However, as shown in class $f(x) = |x|$ has no derivative at $x = 0$.

6) Assume that the function $f(x) = 6x^4 - 7x + 1$ has at least three roots. Let $a < b < c$ be three roots of $f(x)$. Apply Rolle's theorem twice. First apply it to the interval $[a, b]$. Since $f(a) = f(b) = 0$ and since $f(x)$ is a polynomial, we deduce from Rolle's theorem that there is a point $x_1 \in (a, b)$ such that $f'(x_1) = 0$. Similarly, applying Rolle's theorem to $[b, c]$, we obtain a point $x_2 \in (b, c)$ such that $f'(x_2) = 0$. Clearly $x_1 \neq x_2$. In other words, $f'(x)$ has at least two distinct roots. However, $f'(x) = 24x^3 - 7$ and hence $f'(x) = 0$ has only one solution, which means that $f'(x)$ has only one root. Thus, we derived a contradiction, and hence $f(x)$ has at most two roots.

7) Let $-2 < x < 4$, and apply the Mean Value Theorem to the function $f(x)$ in the interval $[-2, x]$. Thus, there exists a point $c \in (-2, x)$ such that

$$\frac{f(x) - f(-2)}{x - (-2)} = f'(c)$$

This is equivalent to $f(x) - 1 = f'(c)(x + 2)$ Taking absolute value, we obtain $|f(x) - 1| = |f'(c)||x + 2| \leq 5(x + 2)$. The last inequality follows from the fact that $|f'(x)| \leq 5$ for all $x \in [-2, 4]$, and from the fact that $x + 2$ is positive for all $x \in [-2, 4]$. The inequality $|f(x) - 1| \leq 5(x + 2)$ is equivalent to $-5(x + 2) \leq f(x) - 1 \leq 5(x + 2)$ or to $-5x - 10 \leq f(x) - 1 \leq 5x + 10$ or $-5x - 9 \leq f(x) \leq 5x + 11$.