Solution for Final in Calculus A

Moad B 2009

1) a) Using L'hopital rule we obtain

$$\lim_{x\to 1} \frac{x^{\frac{1}{3}}-1}{x^{\frac{1}{2}}-1} = \lim_{x\to 1} \frac{\frac{1}{3}x^{-\frac{2}{3}}}{\frac{1}{2}x^{-\frac{1}{2}}} = \frac{2}{3}$$

b) We have

$$\lim_{x\to 0^+} x\ln(e^x-1) = \lim_{x\to 0^+} \frac{\ln(e^x-1)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{e^x}{e^x-1}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -\frac{x^2e^x}{e^x-1} = \lim_{x\to 0^+} -\frac{x^2e^x+2xe^x}{e^x} = 0$$

where the second and fourth equality is derived from L'hopital rule.

2) Given $\epsilon > 0$ we need to find $\delta > 0$ such that $|x + \frac{1}{x} - 2| < \epsilon$ if $|x-1| < \delta$. We have

$$|x + \frac{1}{x} - 2| = \left|\frac{x^2-2x+1}{x}\right| = \frac{|x-1|^2}{|x|}$$

The equality $|x-1| < \delta$ is equivalent $1-\delta < x < 1+\delta$. Hence if we choose $\delta \le 1$ than we obtain $1 < |x|$ or $\frac{1}{|x|} < 1$. Hence for such a δ we have $\frac{|x-1|^2}{|x|} < |x-1|^2$. Choose $\delta < \sqrt{\epsilon}$. Then, if $|x-1| < \delta$ we obtain

$$|x + \frac{1}{x} - 2| < |x-1|^2 < \delta^2 < \epsilon$$

3) We use the following theorem (See exercise number 9 problem number 5): Let $f(x)$ and $g(x)$ be two functions which are differentiable in (a,b). Assume that $f(a) = g(a)$ and that $f'(x) > g'(x)$ for all $x \in (a,b)$. Then $f(x) > g(x)$ for all $x \in (a,b)$.

Let $f(x) = \frac{2x}{1-x}$ and $g(x) = \ln\frac{1+x}{1-x}$. Let $a = 0$ and $b = 1$. Then $f(0) = g(0) = 0$. We have $f'(x) = \frac{1}{(1-x)^2}$ and $g'(x) = \frac{2}{1-x^2}$. Then $f'(x) > g'(x)$ for all $0 < x < 1$.

4) Differentiating we get $2yy' = 4p$. Hence, at the point (x_0, y_0) we have $2y_0m = 4p$ or $y_0 = \frac{2p}{m}$. We have $y_0^2 = 4px_0$, or $x_0 = \frac{y_0^2}{4p}$. Plugging $y_0 = \frac{2p}{m}$ we get $x_0 = \frac{p}{m^2}$. The equation of the tangent line is $y - y_0 = m(x - x_0)$. Plugging the values for x_0 and y_0 we get $y - \frac{2p}{m} = m(x - \frac{p}{m^2})$, from which the results follows.

5) We have $f'(x) = px^{p-1}(1-x)^q - qx^p(1-x)^{q-1} = x^{p-1}(1-x)^{q-1}(p - x(p+q))$. Since $p, q \geq 2$, then $f(x)$ is differentiable for all x, and hence the only possible critical points are $x_1 = 0, x_2 = 1, x_3 = \frac{p}{p+q}$. To determine if the point is a maximum or minimum we check the sign of the derivative. In the interval $0 < x < x_3$ the derivative is always positive, and in the interval $x_3 < x < 1$ it is always negative. Hence x_3 is a maximum point. At the point $x_1 = 1$, we get a minimum point if p is even and if p is odd then it is not a minimum or a maximum point. At $x_2 = 1$ we get a minimum point if q is odd and if q is even then the point is not a minimum or a maximum point.

6) It is given that for all x $f(2x-1) = 1 - 4x^2$. Let $t = 2x - 1$. Then $x = \frac{t+1}{2}$. Hence $f(t) = 1 - 4\left(\frac{t+1}{2}\right)^2 = -t^2 - 2t$. Since this is true for all t, the function can be written as $f(x) = -x^2 - 2x$. Hence $(f \circ f)(x) = f(-x^2 - 2x) = -(-x^2 - 2x) - 2(-x^2 - 2x) = (x^2 + x)(2 - x^2 - 2x)$.

7) We have $f'(x) = 3x^2 + 2ax + b$ and $f''(x) = 6x + 2a$. Since the point $(1, -6)$ is an inflection point, and since the function has second derivative at that point, it follows that $f''(1) = 0$. Hence $6 + 2a = 0$ or $a = -3$. Clearly, $f(1) = -6$. Hence $1 - 3 + b - 4 = -6$, and hence $b = 0$.