Solution for Final in Calculus A

Moad C 2009

1) Suppose that $h \neq 0$. Dividing by h we obtain

$$\frac{f(x+h)-f(x)}{h} = 2x + h - 1$$

Taking the limit $h \to 0$ we obtain that it exists and equals to $2x-1$. Hence $f'(x) = 2x - 1$, and so $f'(2) = 3$. Also, $f''(x) = 2$, and hence $f''(2) = 2$.

2) From left to right, the polynomial $p(x)$ has a maximum at $x = -2$ and this maximum point is above the x axis since it is given that $p(-2) = 4$. Also $x = 1$ is a minimum point and it also located above the x axis since $p(1) = 1$. Similarly the maximum point at $x = 5$ is located above the x axis. Hence, to the left of $x = -2$ the function $p(x)$ increases. Since polynomials has no asymptotes it follows that $p(x)$ intersects the x axis exactly once at a point left to $x = -2$. Similarly, $p(x)$ decreases to the right of $x = 5$, and hence intersects the x axis exactly once at a point right to $x = 5$. From the data it follows that $p(x)$ cannot intersect the x axis at $-2 \leq x \leq 5$. Hence $p(x)$ has two roots.

3) a) For the function to be continuous at $x = a$ we need that $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$. Hence we get $\lim_{x\to a^-}(x+1) = \lim_{x\to a^-} x^2$ or $a + 1 = a^2$. Therefore $f(x)$ is continuous if and only if a is a root of the equation $x^2 - x - 1 = 0$. This equation has two roots.
b) From the data it follows that

$$\lim_{x\to b}(\frac{1}{2}g(x) + \frac{1}{2}h(x)) = 1 \qquad \lim_{x\to b}(\frac{1}{2}g(x) - \frac{1}{2}h(x)) = \frac{1}{2}$$

Hence

$$\frac{3}{2} = \lim_{x\to b}(\frac{1}{2}g(x) + \frac{1}{2}h(x)) + \lim_{x\to b}(\frac{1}{2}g(x) - \frac{1}{2}h(x)) = \lim_{x\to b} g(x)$$

Similarly we get $\lim_{x\to b} h(x) = \frac{1}{2}$. From this we deduce that $\lim_{x\to b} g(x)h(x) = \frac{3}{4}$.

4) For the term $\sqrt{1-x^2}$ to be well defined we need $-1 \leq x \leq 1$. Since square roots are always positive, it follows that $\sqrt{1+\sqrt{1-x^2}}$ is well defined. For $-1 \leq x \leq 1$ we have $0 \leq \sqrt{1-x^2} \leq 1$. Hence $\sqrt{1+\sqrt{1-x^2}} \leq 2$. From this we conclude that the domain of definition of $f(x)$ is $-1 \leq x \leq 1$.

5) Let $y = ax + b$ be the line which tangent the parabola. It intersects the x axis at $(-\frac{b}{a}, 0)$ and the y axis at $(0, b)$. Hence the area of the triangular is $S = -\frac{b^2}{2a}$. Notice that a is negative and hence S is positive. Let (x_0, y_0) denote the tangent point. Then $f'(x_0) = a$. Hence, we have the following three equations

$$y_0 = ax_0 + b; \qquad -2x_0 = a; \qquad y_0 = 4 - x_0^2$$

From these equation we obtain the relation $b = \frac{16+a^2}{16}$. Hence, if we plug this into S we obtain the function $S(a) = -\frac{(16+a^2)^2}{32a}$, and we look for a point which will give us a minimum. We have

$$S'(a) = -\frac{1}{32a^2}(4a^2(16+a^2) - (16+a^2)^2)$$

Solving the equation $S'(a) = 0$ we obtain $3a^2 = 16$, and since a is negative we obtain $a = -\frac{4}{\sqrt{3}}$.

6) a) We have

$$\lim_{x\to-\infty}(\sqrt{x^2+2x}+x) = \lim_{x\to-\infty}\frac{(\sqrt{x^2+2x}+x)(\sqrt{x^2+2x}-x)}{(\sqrt{x^2+2x}-x)} = \lim_{x\to-\infty}\frac{2x}{\sqrt{x^2(1+\frac{2}{x})}-x}$$

Since x is negative, we have $-x = \sqrt{x^2}$. Hence the above is equal to

$$\lim_{x\to-\infty}\frac{2x}{-x\sqrt{1+\frac{2}{x}}-x} = \lim_{x\to-\infty}\frac{2}{-\sqrt{1+\frac{2}{x}}-1} = -1$$

b) Let $y = x^{x+1}$. Then $\ln y = (x+1)\ln x$. Hence $\frac{y'}{y} = \ln x + \frac{x+1}{x}$. Thus, $y' = y(\ln x + \frac{x+1}{x}) = x^{x+1}(\ln x + \frac{x+1}{x})$.

7) The function is differentiable for all x, and hence the only possible extreme points are points which satisfy $f'(x) = 0$. We have $f'(x) = 1 - x - (e^{-x} - xe^{-x}) = 1 - x - (1-x)e^{-x} = (1-x)(1-e^{-x})$. Hence, either $1 - x = 0$ or $1 - e^{-x} = 0$. The possible solutions are $x_1 = 0$ and $x_2 = 1$. To the left of x_1 and to the right of x_2 we have $f'(x) < 0$, in $x_1 < x < x_2$ we have $f'(x) > 0$. Hence $x_1 = 0$ is a minimum point and $x_2 = 1$ is a maximum point.