Solutions Linear Algebra 1 Moed A 2014

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Problem 1: Let A and B of size n be two matrices such that $AB = 0$. Prove that $\text{rank}A + \text{rank}B \leq n$.

Solution: Let $V = \{v \in F^n : Av = 0\}$. From a Theorem we proved in class we have $\text{rank}A = n - \text{dim}V$. Plugging this into $\text{rank}A + \text{rank}B \leq n$, it is equivalent to $\text{rank}B \leq \text{dim}V$. Thus it is enough to prove the last inequality. Let $v_1, \ldots, v_n \in F^n$ denote the columns of the matrix B. Then, from matrix multiplication we deduce that $AB = 0$ is equivalent to $Av_i = 0$ for all $1 \leq i \leq n$. Hence $v_i \in V$ for all $1 \leq i \leq n$. From this we obtain $C(B) = Sp\{v_1, \ldots, v_n\} \subset V$. Hence $\text{rank}B = \text{dim}C(B) \leq \text{dim}V$.

Problem 2: Let A be a matrix of size n such that rank $A = 1$ and that $n - 2$ rows of A are the zero rows. Is it true that $det(A + I) = tr(A) + 1$?

Solution: The above identity is true. To prove it, let i and k be the non zero rows of A. Since rank $A = 1$, these two rows are proportional. Assume that the $k - th$ row is β times the $i - th$ row. Thus

$$A + I = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ \alpha_1 & \ldots & \alpha_i + 1 & \ldots & & \alpha_k & & \\ & & & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & & \\ \beta\alpha_1 & \ldots & \beta\alpha_i & \ldots & & \beta\alpha_k + 1 & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \end{pmatrix}$$

In other words, all diagonal elements of $A + I$ are ones except the $i - th$ and $k - th$ rows. Expending the determinant $|A + I|$ by the first row, then by the second, then by the $i - 1$, then by the $i + 1$ and so on, we obtain that

$$|A + I| = \begin{vmatrix} \alpha_i + 1 & \alpha_k \\ \beta\alpha_i & \beta\alpha_k + 1 \end{vmatrix} = (\alpha_i + 1)(\beta\alpha_k + 1) - \alpha_k\beta\alpha_i = \alpha_i + \beta\alpha_k + 1$$

On the other hand, the diagonal entries of A are all zeros except at the $i-th$ and $k-th$ rows. In these rows the diagonal elements are α_i and $\beta\alpha_k$. Hence $\mathrm{tr}A + 1 = \alpha_i + \beta\alpha_k + 1 = |A + I|$.

Problem 3: Let V be a vector space over F. Let $T : V \mapsto V$ be a linear map. Suppose that v_1, v_2, v_3 are nonzero vectors in V. Assume that there are scalers a_1, a_2, a_3, all distinct, such that $Tv_i = a_i v_i$. Prove that $\{v_1, v_2, v_3\}$ is linearly independent.

Solution: By rearranging the order, we may assume that if one of the a_i is zero then $i = 3$. Thus, we may assume that $a_1, a_2 \neq 0$. Write

$$\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 = 0. \tag{1}$$

Apply T to this equation, and use the fact that $Tv_i = a_i v_i$ to obtain

$$a_1\beta_1 v_1 + a_2\beta_2 v_2 + a_3\beta_3 v_3 = 0. \tag{2}$$

Multiply (1) by a_1 and subtract equation (2). We obtain

$$(a_1 - a_2)\beta_2 v_2 + (a_1 - a_3)\beta_3 v_3 = 0. \tag{3}$$

Apply T to this equation,

$$a_2(a_1 - a_2)\beta_2 v_2 + a_3(a_1 - a_3)\beta_3 v_3 = 0. \tag{4}$$

Multiply (3) by a_2 and subtract equation (4). We obtain

$$(a_2 - a_3)(a_1 - a_3)\beta_3 v_3 = 0$$

Since all the a_i are distinct, we deduce that $\beta_3 = 0$. Going back to equation (3) we obtain $\beta_2 = 0$, and then, from equation (1) we obtain $\beta_1 = 0$.

Problem 4: For what values of t, the vector $v = \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}$ is in the column space of the matrix

$$A = \begin{pmatrix} 1+t & 1 & 1 \\ 1 & 1+t & 1 \\ 1 & 1 & 1+t \end{pmatrix}$$

All values are assumed to be in a fixed field F.

Solution: For those values of t such that $|A| \neq 0$, we have $C(A) = F^3$ and then $v \in C(A)$. Performing the two row operations $R_1 \to R_1 - (1+t)R_3$ and $R_2 \to R_2 - R_3$, the value of the determinant does not change and we obtain

$$|A| = \begin{vmatrix} 0 & -t & 1-(1+t)^2 \\ 0 & t & -t \\ 1 & 1 & 1+t \end{vmatrix} = \begin{vmatrix} -t & 1-(1+t)^2 \\ t & -t \end{vmatrix} = t^2(t+3)$$

Hence, if $t \neq 0, -3$ we have $v \in C(A)$.

When $t = 0$, then $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $C(A) = Sp\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\}$. So $v \notin C(A)$.

When $t = -2$, we see that the columns of A all have the property that the sum of the coordinates is zero. On the other hand, in this case $v = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$, and this vector does not have this property. Hence $v \notin C(A)$.

Problem 5: Let A, B and C be three square matrices of size n. Assume that $C(I+AB) = I$, compute the matrix $(I - BCA)(I + BA)$. Write it in the simplest form possible.

Solution: We have

$$(I - BCA)(I + BA) = I + BA - BCA - BCABA = I + BA - BCA - BC[(I + AB) - I]A =$$

$$= I + BA - BCA - BC(I + AB)A - BCA = I + BA - BCA - BIA + BCA$$

In the last equality we used the fact that $C(I + AB) = I$. Thus

$$(I - BCA)(I + BA) = I + BA - BCA - BA + BCA = I$$