THE FAILURE OF DIAMOND ON A REFLECTING STATIONARY SET

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Abstract. 1. It is shown that the failure of $\diamondsuit_S$, for a set $S \subseteq \aleph_{\omega+1}$ that reflects stationarily often, is consistent with GCH and $\mathrm{AP}_{\aleph_\omega}$, relatively to the existence of a supercompact cardinal. By a theorem of Shelah, GCH and $\square^*_\lambda$ entails $\diamondsuit_S$ for any $S \subseteq \lambda^+$ that reflects stationarily often.

2. We establish the consistency of existence of a stationary subset of $[\aleph_{\omega+1}]^\omega$ that cannot be thinned out to a stationary set on which the sup-function is injective. This answers a question of König, Larson and Yoshinobu, in the negative.

3. We prove that the failure of a diamond-like principle introduced by Džamonja-Shelah is equivalent to the failure of Shelah's strong hypothesis.

0. Introduction

0.1. **Background.** Recall Jensen's diamond principle [10]: for an infinite cardinal λ and a stationary set $S \subseteq \lambda^+$, $\diamondsuit_S$ asserts the existence of a collection $\{A_\delta \mid \delta \in S\}$ such that the set $\{\delta \in S \mid A \cap \delta = A_\delta\}$ is stationary for all $A \subseteq \lambda^+$.

It is easy to see that $\diamondsuit_{\lambda^+}$ implies $2^\lambda = \lambda^+$, and hence it is natural to ask whether the converse holds. Jensen proved that for $\lambda = \aleph_0$, the inverse implication fails (see [10]), however, for $\lambda > \aleph_0$, a recent theorem of Shelah [19] indeed establishes the inverse implication, and moreover, it is proved that $2^\lambda = \lambda^+$ entails $\diamondsuit_S$ for every stationary S which is a subset of $E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)} := \{\alpha < \lambda^+ \mid \mathrm{cf}(\alpha) \neq \mathrm{cf}(\lambda)\}$.

The result of [19] is optimal: by a theorem of Shelah from [20], GCH is consistent with the failure of $\diamondsuit_{E^{\lambda^+}_\lambda}$ for a regular uncountable cardinal λ. By another theorem of Shelah, from [17, §2], GCH is consistent with the failure of $\diamondsuit_S$ for a singular cardinal λ and a stationary set $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$. However, the latter happens to be a proper subset of $E^{\lambda^+}_{\mathrm{cf}(\lambda)}$; more specifically, it is a non-reflecting stationary subset. This leads to the following question:

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Question. Suppose λ is a singular cardinal and $2^\lambda = \lambda^+$.
Must $\diamondsuit_S$ hold for every $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that reflects stationarily often?[1]

In [17, §3], Shelah answered the above question in the affirmative, provided that $\square^*_\lambda$ holds and that λ is a strong limit. Later, Zeman [21] applied ideas from [19] and eliminated the "strong limit" hypothesis. Then, in [15], the second author introduced the *Stationary Approachability Property at* λ, abbreviated SAP_λ, proved that SAP_λ is strictly weaker than $\square^*_\lambda$, and answered the above question positively in the presence of SAP_λ. It was unknown whether the hypothesis SAP_λ can be eliminated, or even whether it is possible to replace it with the usual *Approachability Property*, AP_λ.

In the present paper we answer the discussed question in the negative, and moreover, do so in the presence of AP_λ. Let $\mathrm{Refl}(T)$ denote the assertion that every stationary subset of T reflects stationarily often; then the main result of this paper reads as follows.

Theorem A. *It is relatively consistent with the existence of a supercompact cardinal that all of the following holds simultaneously:*

1. GCH;
2. $\mathrm{AP}_{\aleph_\omega}$;
3. $\mathrm{Refl}(E^{\aleph_{\omega+1}}_\omega)$;
4. $\diamondsuit_S$ *fails for some stationary* $S \subseteq E^{\aleph_{\omega+1}}_\omega$.

Combining the preceding theorem with the results from [15], we now obtain a complete picture of the effect of weak square principles on diamond.[2]

Corollary (first three items are from [15])**.** *For a singular cardinal, λ:*

1. $\mathsf{GCH} + \mathrm{Refl}(E^{\lambda^+}_{\mathrm{cf}(\lambda)}) + \square^*_\lambda \Rightarrow \diamondsuit^*_{\lambda^+}$;
2. $\mathsf{GCH} + \mathrm{Refl}(E^{\lambda^+}_{\mathrm{cf}(\lambda)}) + \mathrm{SAP}_\lambda \not\Rightarrow \diamondsuit^*_{\lambda^+}$;
3. $\mathsf{GCH} + \mathrm{Refl}(E^{\lambda^+}_{\mathrm{cf}(\lambda)}) + \mathrm{SAP}_\lambda \Rightarrow \diamondsuit_S$ *for every stationary* $S \subseteq \lambda^+$;
4. $\mathsf{GCH} + \mathrm{Refl}(E^{\lambda^+}_{\mathrm{cf}(\lambda)}) + \mathrm{AP}_\lambda \not\Rightarrow \diamondsuit_S$ *for every stationary* $S \subseteq \lambda^+$.

Once that the effect of weak square principles to diamond is well-understood, it is natural to study which of the other combinatorial principles from [3, §4] is strong enough to impose an affirmative answer to Question 1. It turns out that even the strongest among these principles does not suffice. We prove:

Theorem B. *It is relatively consistent with the existence of a supercompact cardinal that there exists a singular cardinal λ for which all of following holds simultaneously:*

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[1] We say that S *reflects stationarily often* iff there are stationarily many $\alpha < \lambda^+$ with $\mathrm{cf}(\alpha) > \omega$ such that $S \cap \alpha$ is stationary.

[2] For the definition of $\diamondsuit^*_{\lambda^+}$, as well as Kunen's theorem that $\diamondsuit^*_{\lambda^+} \Rightarrow \diamondsuit_S$ for every stationary $S \subseteq \lambda^+$, see [12].

(1) λ *is a strong limit and* $2^\lambda = \lambda^+$;
(2) *there exists a very good scale for* λ;
(3) $\diamondsuit_S$ *fails for some* $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ *that reflects stationarily often.*

To conclude the introduction, let us say a few words about the structure of the proof of Theorem A. We start with a supercompact κ, and a singular cardinal λ above it, namely $\lambda := \kappa^{+\omega}$. We add a generic stationary subset $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$, and then kill $\diamondsuit_S$ by iteration, while preserving the stationarity of S and the supercompactness of κ. Since κ remains supercompact, $\mathrm{Refl}(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$ holds, so this already gives an example of a model of GCH on which $\diamondsuit_S$ fails for some stationary $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that *reflects stationarily often*. Nevertheless, by $\mathrm{cf}(\lambda) < \kappa < \lambda$, and a theorem Shelah from [16], AP_λ fails in our model. For this, at the final stage of the proof, we move everything down to $\aleph_\omega$, and use a method of Foreman and Magidor from [7] to insure $\mathrm{AP}_{\aleph_\omega}$.

The main problem that we address here is an iteration over the successor of a singular. More specifically, the main body of our proof is the argument that S remains stationary after the iteration for killing all diamond sequences over it.

0.2. **Organization of this paper.** In Section 1, we present a λ^+-directed-closed, λ^{++}-c.c., notion of forcing for introducing a stationary subset of λ^+ on which diamond fails. Then, in the presence of a supercompact cardinal, we appeal to this notion of forcing, and construct three models in which diamond fails on a set that reflects stationarily often. In particular, Theorem A and Theorem B are proved in this section.

In Section 2, we revisit a theorem by Džamonja and Shelah from [4] in which, starting with a supercompact cardinal, they construct a model satisfying the failure of one of the consequences of diamond. Here, we establish that this particular consequence of diamond is quite weak. We do so by reducing its consistency strength to the level of existence of a measurable cardinal κ of Mitchell order κ^{++}. In particular, its strength is lower than the one of weak square.

In Section 3, we answer a question by König, Larson and Yoshinobu from [11], concerning stationary subsets of $[\lambda^+]^\omega$. We do so by linking between the diamond principle and the behavior of the sup-function on generalized stationary sets.

0.3. **Notation.** Generally speaking, we follow the notation and presentation of [12] and [3]. Let us quickly review our less standard conventions. For cardinals $\kappa < \lambda$, denote $E^\lambda_\kappa := \{\alpha < \lambda \mid \mathrm{cf}(\alpha) = \kappa\}$, and $[\lambda]^\kappa := \{X \subseteq \lambda \mid |X| = \kappa\}$. $E^\lambda_{>\kappa}$ and $[\lambda]^{<\kappa}$ are defined analogously. We let Ord denote the class of ordinals. For sets of ordinals a, b, we write $a \sqsubseteq b$ or $b \sqsupseteq a$ iff a is an initial segment of b, that is, iff $a \subseteq b$ and $b \cap \sup(a) = a$. Given a set of ordinals, a, we let $\mathrm{cl}(a) := \{\sup(a \cap \alpha) \mid \alpha \in \mathrm{Ord}\}$ denote its topological closure. For $Z \subseteq \mathrm{Ord}$ and distinct functions $g, h \in {}^Z 2$, we let $\Delta(g, h) := \min\{\alpha \in Z \mid g(\alpha) \neq h(\alpha)\}$.

Our forcing conventions are as follows. We denote by $p \geq q$ the fact that p is stronger than q. For a ground model set, x, we denote its canonical name by $\check{x}$. For arbitrary sets of the generic extension, we designate names such as $\dot{a}$ and $\underset{\sim}{a}$. If $\langle p_1, \dot{q_1}\rangle, \langle p_2, \dot{q_2}\rangle$ are conditions of a two-step iteration $\mathbb{P} * \mathbb{Q}$, and $p_2 \geq p_1 \Vdash_{\mathbb{P}} \dot{q_1} = \dot{q_2}$, then we slightly abuse notation by writing $\langle p_1, \dot{q_1}\rangle = \langle p_2, \dot{q_2}\rangle$.

1. Negation of diamond

1.1. Forcing the failure of diamond. In this subsection, we present a λ^+-directed-closed, λ^{++}-c.c., notion of forcing for introducing a stationary subset $S \subseteq \lambda^+$ on which diamond fails. For simplicity, we shall be focusing on the case $S \subseteq E^{\lambda^+}_\omega$. The general case is discussed in subsection 1.3.

Definition 1.1. For a cardinal λ, we define the forcing notion $\mathbb{S}(\lambda^+)$.

A condition s is in $\mathbb{S}(\lambda^+)$ iff s is a bounded subset of λ^+, and $\mathrm{cf}(\delta) = \omega$ for all $\delta \in s$. A condition s' is stronger than s, denoted $s' \geq s$, iff $s' \sqsupseteq s$.

Thus, $\mathbb{S}(\lambda^+)$ is simply the restricted-to-countable-cofinality version of λ^+-Cohen forcing. In particular, we have:

Lemma 1.2. *For every infinite cardinal, λ:*

1. *$\mathbb{S}(\lambda^+)$ has the $(2^\lambda)^+$-c.c.;*
2. *every increasing sequence of conditions in $\mathbb{S}(\lambda^+)$ of length $< \lambda^+$ has a least upper bound.*

For future needs, it is useful to introduce the following set-name:

$$\dot{S}(\lambda^+) := \{\langle\check{\delta}, s\rangle \mid \delta \in s \in \mathbb{S}(\lambda^+)\}.$$

Clearly, if G is $\mathbb{S}(\lambda^+)$-generic, then $\dot{S}(\lambda^+)$ is a name for $\bigcup G$. We now consider a natural forcing notion for **K**illing a given **D**iamond sequence.

Definition 1.3. For a set $S \subseteq \mathrm{Ord}$ and a sequence of sets $\vec{A}$, we define the forcing notion $\mathbb{KD}(S, \vec{A})$.

A condition p is in $\mathbb{KD}(S, \vec{A})$ iff $p = (x, c)$, where c is a closed set of ordinals, $x \subseteq \max(c) < \sup(S)$, and for all $\delta \in c \cap S \cap \mathrm{dom}(\vec{A})$, $x \cap \delta \neq \vec{A}(\delta)$.

A condition (x, c) is stronger than (x', c'), denoted $(x, c) \geq (x', c')$, iff $x \sqsupseteq x'$ and $c \sqsupseteq c'$.

Notice that indeed if $\vec{A} = \langle A_\delta \mid \delta \in S\rangle$ is a $\diamondsuit_S$ sequence in V, then $\vec{A}$ will cease to be so in $V^{\mathbb{KD}(S,\vec{A})}$.

For a strong limit singular cardinal λ with $2^\lambda = \lambda^+$ and a subset $S \subseteq \lambda^+$, let $\mathbb{KAD}(S)$ denote the forcing notion for **K**illing **A**ll **D**iamond sequences over S. That is, $\mathbb{KAD}(S)$ is a $(\leq \lambda)$-support iteration

$$(\langle \mathbb{P}_\alpha \mid \alpha \leq \lambda^{++}\rangle, \langle \dot{\mathbb{R}}_\alpha \mid \alpha < \lambda^{++}\rangle),$$

such that $\mathbb{P}_0$ is a trivial forcing, and for all $\alpha < \lambda^{++}$, $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{R}}_\alpha$ is a name for the forcing $\mathbb{KD}(\check{S}, \underset{\sim}{\vec{A}_\alpha})$, whereas $\underset{\sim}{\overrightarrow{A_\alpha}}$ is a $\mathbb{P}_\alpha$-name for a sequence chosen by a book-keeping function in such a way that all potential diamond sequences are handled at some stage. The existence of such a function follows from cardinal arithmetic hypothesis and the λ^{++}-c.c. of $\mathbb{KD}(S, \vec{A})$.

Definition 1.4. Let $\mathbb{Q}(\lambda^+) := \mathbb{S}(\lambda^+) * \mathbb{KAD}(\dot{S}(\lambda^+))$.

We also define $\mathbb{Q}'(\lambda^+)$. A condition $\langle s, k\rangle$ is in $\mathbb{Q}'(\lambda^+)$ iff all of the following holds.

(1) $\langle s, k\rangle \in \mathbb{Q}(\lambda^+)$;
(2) s decides the support of k to be, say, $\mathrm{supp}(k)$;
(3) for all $\alpha \in \mathrm{supp}(k)$, $k(\alpha)$ is an $\mathbb{S}(\lambda^+) * \mathbb{P}_\alpha$-canonical name for a pair (x^k_α, c^k_α);
(4) for all $\alpha \in \mathrm{supp}(k)$ and $\delta \in s \cap c^k_\alpha$, $\langle s, k\rangle \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\alpha$ decides $\vec{A}_\alpha(\delta)$;
(5) $\sup(x^k_\alpha) = \max(c^k_\alpha) \geq \sup(s)$ for all $\alpha \in \mathrm{supp}(k)$.

Lemma 1.5. *every increasing sequence of conditions in $\mathbb{Q}'(\lambda^+)$ of length $< \lambda^+$ has a least upper bound.*

Proof. Suppose $\langle\langle s_\beta, k_\beta\rangle \mid \beta < \theta\rangle$ is an increasing sequence of conditions in $\mathbb{Q}'(\lambda^+)$, with $\theta < \lambda^+$. Define $\langle s, k\rangle$ by letting:

- $s := \bigcup_{\beta<\theta} s_\beta$;
- $\mathrm{supp}(k) := \bigcup_{\beta<\theta} \mathrm{supp}(k_\beta)$;
- for all $\alpha \in \mathrm{supp}(k)$:

$$x^k_\alpha := \bigcup\{x^{k_\beta}_\alpha \mid \beta < \theta \wedge \alpha \in \mathrm{supp}(k_\beta)\},$$
$$c^k_\alpha := \mathrm{cl}(\bigcup\{c^{k_\beta}_\alpha \mid \beta < \theta \wedge \alpha \in \mathrm{supp}(k_\beta)\}).$$

Note that, by definition, $\sup(s)$ is not necessarily a member of s.

To establish that $\langle s, k\rangle \in \mathbb{Q}'(\lambda^+)$, let us show that $\langle s, k\rangle \in \mathbb{Q}(\lambda^+)$. So, suppose $\alpha \in \mathrm{supp}(k)$, and $\delta \in s \cap c^k_\alpha$. Pick $\beta < \theta$ such that $\delta \in s_\beta$ and $\alpha \in \mathrm{supp}(k_\beta)$. By property (5), we have $\max(c^{k_\beta}_\alpha) \geq \sup(s_\beta) \geq \delta$, and hence $\delta \in c^{k_\beta}_\alpha$. By property (4), $\langle s_\beta, k_\beta\rangle \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\alpha$ decides $\vec{A}_\alpha(\delta)$, and hence $x^{k_\beta}_\alpha \cap \delta \neq \vec{A}_\alpha(\delta)$. Finally, by property (5), we have $\sup(x^{k_\beta}_\alpha) = \max(c^{k_\beta}_\alpha) \geq \delta$, and hence $x^{k_\beta}_\alpha \cap \delta = x^k_\alpha \cap \delta$, so $x^k_\alpha \cap \delta \neq \vec{A}_\alpha(\delta)$. $\square$

Lemma 1.6. *$\mathbb{Q}'(\lambda^+)$ is λ^+-directed closed.*

Proof. Virtually the same proof as the preceding. $\square$

Lemma 1.7. *Assume λ is a strong limit singular cardinal, and $2^\lambda = \lambda^+$. Then:*

(1) $|\mathbb{Q}(\lambda^+)| = \lambda^{++}$;
(2) $\mathbb{Q}(\lambda^+)$ *has the* λ^{++}*-c.c.;*
(3) $\mathbb{Q}'(\lambda^+)$ *is dense in* $\mathbb{Q}(\lambda^+)$.

Proof. (1) is obvious, and (2) follows from a standard Δ-system argument.

(3) To simplify the notation, for all $\beta \leq \lambda^{++}$, let $\mathbb{Q}'_\beta := \mathbb{Q}'(\lambda^+) \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta$. Note that the proof of Lemma 1.5 shows that every increasing sequence of conditions in $\mathbb{Q}'_\beta$ of length $< \lambda^+$ has a least upper bound. We now prove by induction on $\beta \leq \lambda^{++}$ that $\mathbb{Q}'_\beta$ is dense in $\mathbb{Q}_\beta := \mathbb{S}(\lambda^+) * \mathbb{P}_\beta$.

Induction base: For $\beta = 0$, we have $\mathbb{Q}'_\beta = \mathbb{S}(\lambda^+) * \mathbb{P}_0 = \mathbb{Q}_\beta$.

Induction step: Suppose the claim holds for α, and $\langle s, k\rangle$ is a given element of $\mathbb{Q}_\beta$, for $\beta = \alpha + 1$. We would like to find a condition in $\mathbb{Q}'_\beta$ which is stronger than $\langle s, k\rangle$.

Since $\mathbb{S}(\lambda^+)$ is λ^+-closed, and $\mathbb{P}_\beta$ is a $(\leq \lambda)$-support iteration, we may assume that s already decides the support of k. To avoid trivialities, we may also assume that $\alpha \in \mathrm{supp}(k)$.

Since $\mathbb{Q}'_\alpha$ is a λ^+-closed, dense subset of $\mathbb{Q}_\alpha$, and $k(\alpha)$ is a $\mathbb{Q}_\alpha$-name for a pair of bounded subsets of λ^+, let us pick a condition $\langle s_0, k_0\rangle \geq \langle s, k\rangle$ and a pair (x_0, c_0) such that $\langle s_0, k_0\rangle \restriction \mathbb{Q}_\alpha \in \mathbb{Q}'_\alpha$, $k_0(\alpha)$ is the $\mathbb{Q}_\alpha$-canonical name for (x_0, c_0), and:

$$\langle s_0, k_0\rangle \restriction \mathbb{Q}_\alpha \Vdash k(\alpha) = k_0(\sigma).$$

Evidently, $\langle s_0, k_0\rangle \geq \langle s, k\rangle$. Next, suppose $n < \omega$ and $\langle s_n, k_n\rangle, (c_n, k_n)$ are defined. For all $\delta < \lambda^+$, let D^n_δ denote the collection of all conditions $\langle s', k'\rangle \in \mathbb{Q}_\beta$ such that all of the following holds:

- $\langle s', k'\rangle \geq \langle s_n, k_n\rangle$;
- $k'(\alpha) = k_n(\alpha)$;
- $\langle s', k'\rangle \restriction \mathbb{Q}_\alpha \in \mathbb{Q}'_\alpha$;
- $\langle s', k'\rangle \restriction \mathbb{Q}_\alpha$ decides $\vec{A}_\alpha(\delta)$.

Since $\mathbb{Q}'_\alpha$ is a λ^+-closed, dense subset of $\mathbb{Q}_\alpha$, we get that D^n_δ is dense in $\mathbb{Q}_\beta$ above $\langle s_n, k_n\rangle$, so let us pick a condition $\langle s'_n, k'_n\rangle \in \bigcap_{\delta \in c_n} D^n_\delta$.

Pick a limit $\gamma_{n+1} \in E^{\lambda^+}_{\neq\omega}$ such that $\gamma_{n+1} > \sup(x_n \cup c_n \cup c^{k'_n}_i)$ for all $i \in \mathrm{supp}(k'_n)$. Let $\langle s_{n+1}, k_{n+1}\rangle$ be the condition in $\mathbb{Q}_\beta$ satisfying:

- $\langle s_{n+1}, k_{n+1}\rangle \restriction \mathbb{Q}_\alpha = \langle s'_n \cup \{\gamma_{n+1} + \omega\}, k'_n\rangle$;
- $k_{n+1}(\alpha)$ is the $\mathbb{Q}_\alpha$-canonical name for the following pair:

$$(x_{n+1}, c_{n+1}) := (x_n \cup \{\alpha \mid \max(c_n) < \alpha < \gamma_{n+1}\}, c_n \cup \{\gamma_{n+1}\}).$$

Note that $\mathrm{supp}(k_{n+1}) = \mathrm{supp}(k'_n)$, that $s_{n+1} \cap c^{k_{n+1}}_i = s'_n \cap c^{k'_1}_i$ and $x^{k_{n+1}}_i = x^{k'_n}_i$ for all $i \in \mathrm{supp}(k_{n+1}) \cap \alpha$, and that $x_{n+1} \cap \delta = x_n \cap \delta$ for all $\delta \in s_{n+1} \cap c_{n+1} = s'_n \cap c_n$. Since, for all $i \in \mathrm{supp}(k'_n) \cap \alpha$ and all $\delta \in s'_n \cap c^{k'_n}_i$,

$\langle s'_n, k'_n\rangle \restriction \mathbb{Q}_i$ decides $\vec{A}_i(\delta)$, and since $\langle s'_n, k'_n\rangle$ decides $\vec{A}_\alpha(\delta)$ for all $\delta \in c_n$, we conclude that $\langle s_{n+1}, k_{n+1}\rangle$ is indeed well-defined.

Now, suppose $\langle\langle(s_n, k_n), (x_n, c_n)\rangle \mid n < \omega\rangle$ has already been defined. Let $\langle s^*, k^*\rangle$ be the condition satisfying:

- $s^* := \bigcup_{n<\omega} s_n$;
- $\mathrm{supp}(k^*) := \bigcup_{n<\omega} \mathrm{supp}(k_n)$;
- $k^*(\alpha)$ is the canonical name for the pair $\big(\bigcup_{n<\omega} x_n, \mathrm{cl}(\bigcup_{n<\omega} c_n)\big)$;
- for all $i \in \mathrm{supp}(k^*) \cap \alpha$, $k^*(i)$ is the canonical name for the pair $(x_i^{k^*}, x_i^{k^*})$, where:

$$x_i^{k^*} := \bigcup\{x_i^{k_n} \mid n < \omega \wedge i \in \mathrm{supp}(k_n)\},$$
$$c_i^{k^*} := \mathrm{cl}(\bigcup\{c_i^{k_n} \mid n < \omega \wedge i \in \mathrm{supp}(k_n)\}).$$

Assume indirectly that $\langle s^*, k^*\rangle$ is not a legitimate condition. Since $\langle s^*, k^*\rangle \restriction \mathbb{Q}_\alpha$ is just the least upper bound of $\langle\langle s'_n, k'_n\rangle \restriction \mathbb{Q}_\alpha \mid n < \omega\rangle$, then it must be the case that the "problem" is with the α_{th}-coordinate, that is, there exists some $\delta \in s^* \cap \mathrm{cl}(\bigcup_{n<\omega} c_n)$ for which $(\bigcup_{n<\omega} x_n) \cap \delta = \vec{A}_\alpha(\delta)$. Fix such δ. Put $\gamma := \sup(s^*)$. Then $\gamma = \sup_{n<\omega}(\gamma_{n+1} + \omega) = \sup_{n<\omega} \max(c_{n+1})$, and hence $\mathrm{cl}(\bigcup_{n<\omega} c_n) = (\bigcup_{n<\omega} c_n) \cup \{\gamma\}$. As $\gamma \notin \sup(s^*)$, let us pick some $n < \omega$ such that $\delta \in c_n \cap s_n$. Then, by the choice of $\langle s'_n, k'_n\rangle$, we know that s'_n decides $\vec{A}_\alpha(\delta)$, and that $k'_n(\alpha)$ is the canonical name for the pair (x_n, c_n). In particular, $x_n \cap \delta \neq \vec{A}_\alpha(\delta)$. As $\delta \in c_n$, the definition of x_{n+1} yields that $x_n \cap \delta \sqsubseteq x_{n+1}$ with $\sup(x_{n+1}) = \gamma_{n+1} > \delta$. Consequently, $(\bigcup_{n<\omega} x_n) \cap \delta \neq \vec{A}_\alpha(\delta)$. A contradiction.

Thus, $\langle s^*, k^*\rangle \in \mathbb{Q}_\beta$. Recalling that $\langle s^*, k^*\rangle \restriction \mathbb{Q}_\alpha \in \mathbb{Q}'_\alpha$, and the definition of $\langle(x_n, c_n, s_n) \mid n < \omega\rangle$, we now conclude that $\langle s^*, k^*\rangle \in \mathbb{Q}'_\beta$.

Limit step: Suppose β is a limit ordinal, and $\langle s, k\rangle \in \mathbb{Q}_\beta$.

Clearly, we may assume that s decides the support of k. To avoid trivialities, we may also assume that $\sup(\mathrm{supp}(k)) = \beta$. In particular, $\mathrm{cf}(\beta) < \lambda$.

Let $\langle\beta_\alpha \mid \alpha < \mathrm{cf}(\beta)\rangle$ be an increasing sequence of ordinals converging to β. Evidently, this sequence may be chosen in such a way that $\mathrm{cf}(\beta_\alpha) < \mathrm{cf}(\beta) < \lambda$ for all $\alpha < \mathrm{cf}(\beta)$.

Denote $\beta_{\mathrm{cf}(\beta)} := \beta$. Recursively define an increasing sequence of conditions in $\mathbb{Q}$, $\langle\langle s_\alpha, k_\alpha\rangle \mid \alpha \leq \mathrm{cf}(\beta)\rangle$, in such a way that:

- $\langle s_0, k_0\rangle = \langle s, k\rangle$;
- s_α decides the support of k_α for all $\alpha \leq \mathrm{cf}(\beta)$;
- $\langle s_\alpha, k_\alpha\rangle \restriction \mathbb{Q}_{\beta_\alpha} \in \mathbb{Q}'_{\beta_\alpha}$ for all $\alpha \leq \mathrm{cf}(\beta)$;
- $k_\alpha \restriction (\lambda^{++} \setminus \beta_\alpha) = k \restriction (\lambda^{++} \setminus \beta_\alpha)$ for all $\alpha \leq \mathrm{cf}(\beta)$.

The successor stage simply utilizes the induction hypothesis, so let us show how to handle the limit stage of the recursion. Suppose $\alpha \leq \mathrm{cf}(\beta)$ is a limit ordinal, and $\langle\langle s_\eta, k_\eta\rangle \mid \eta < \alpha\rangle$ is defined.

Fix $\gamma < \alpha$. Since $\langle\langle s_\eta, k_\eta\rangle \restriction \mathbb{Q}_{\beta_\gamma} \mid \gamma \leq \eta < \alpha\rangle$ is an increasing sequence of conditions in $\mathbb{Q}'_{\beta_\gamma}$ of length $< \lambda^+$, we may define $q_\gamma = \langle s^\gamma, k^\gamma\rangle$ as its least upper bound. Note that $\langle q_\gamma \mid \gamma < \alpha\rangle$ is in $\prod_{\gamma<\alpha} \mathbb{Q}'_{\beta_\gamma}$, and that $q_{\gamma_1} \restriction \mathbb{Q}_{\beta_\gamma} = q_{\gamma_2} \restriction \mathbb{Q}_{\beta_\gamma}$ whenever $\gamma_1 \leq \gamma_2 \leq \gamma < \alpha$. We now define $\langle s_\alpha, k_\alpha\rangle$, and then argue that it is indeed a legitimate condition. Thus, let $\langle s_\alpha, k_\alpha\rangle$ be the condition satisfying:

- $s_\alpha := s^0$;
- s_α decides the support of k_α to be $\bigcup_{\gamma<\alpha} \mathrm{supp}(k^\gamma) \cup \mathrm{supp}(k)$;
- for all $i \in \mathrm{supp}(k_\alpha) \cap \beta_\alpha$:
$$k_\alpha(i) = k^\gamma(i), \text{ where } \gamma := \min\{\gamma' \mid i \in \mathrm{supp}(k^{\gamma'})\};$$
- for all $i \in \mathrm{supp}(k_\alpha) \setminus \beta_\alpha$:
$$k_\alpha(i) = k(i).$$

Evidently, for all $\gamma < \alpha$, we have $\langle s_\alpha, k_\alpha\rangle \restriction \mathbb{Q}_{\beta_\gamma} = q_\gamma \restriction \mathbb{Q}_{\beta_\gamma}$. Since $\mathrm{cf}(\beta_\alpha) < \lambda$, the forcing $\mathbb{P}_{\beta_\alpha}$ is the inverse limit of $\langle \mathbb{P}_\tau \mid \tau < \beta_\alpha\rangle$, and hence $\langle s_\alpha, k_\alpha\rangle \restriction \mathbb{Q}_{\beta_\alpha} \in \mathbb{Q}_{\beta_\alpha}$. By arguments which, by now, are standard, we moreover have $\langle s_\alpha, k_\alpha\rangle \restriction \mathbb{Q}_{\beta_\alpha} \in \mathbb{Q}'_{\beta_\alpha}$.

Finally, as $\langle s_\alpha, k_\alpha\rangle \restriction \mathbb{Q}_{\beta_\alpha} \geq \langle s, k\rangle \restriction \mathbb{Q}_{\beta_\alpha}$, $k_\alpha \restriction (\lambda^{++} \setminus \beta_\alpha) = k \restriction (\lambda^{++} \setminus \beta_\alpha)$ and:
$$\langle s, k\rangle \restriction \mathbb{Q}_{\beta_\alpha} \Vdash k \restriction (\lambda^{++} \setminus \beta_\alpha) \in \mathbb{KAD}(\dot{S}(\lambda^+)) \restriction (\lambda^{++} \setminus \beta_\alpha),$$
we get that
$$\langle s_\alpha, k_\alpha\rangle \restriction \mathbb{Q}_{\beta_\alpha} \Vdash k_\alpha \restriction (\lambda^{++} \setminus \beta_\alpha) \in \mathbb{KAD}(\dot{S}(\lambda^+)) \restriction (\lambda^{++} \setminus \beta_\alpha),$$
and hence $\langle s_\alpha, k_\alpha\rangle$ is a legitimate condition of $\mathbb{Q}(\lambda^+)$. Thus, the recursion may indeed be carried out, and we end up with a condition $\langle s_{\mathrm{cf}(\beta)}, k_{\mathrm{cf}(\beta)}\rangle \in \mathbb{Q}'_\beta$ which is stronger than $\langle s, k\rangle$, as requested. ☐

The next theorem is the core of our proof. We encourage the reader to notice the role of the fact that S concentrates on the critical cofinality, i.e., that $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$.[3]

Theorem 1.8. *Suppose $\lambda > \mathrm{cf}(\lambda) = \omega$ is a strong limit, and $2^\lambda = \lambda^+$.*
If $G * H$ is $\mathbb{Q}(\lambda^+)$-generic, then letting $S := \bigcup G$, we have:
$$V[G][H] \models S \textit{ is stationary.}$$

Proof. Fix a name $\dot{E}$ and condition $\langle s^*, k^*\rangle \in \mathbb{Q}'(\lambda^+)$ forcing that $\dot{E}$ is a club subset of λ^+. Clearly, we may assume that $0 \in \mathrm{supp}(k^*)$ and that $|\mathrm{supp}(k^*)| = \lambda$. Fix a large enough regular cardinal χ and an elementary submodel $M \prec \langle H(\chi), <_\chi\rangle$ satisfying:

- $|M| = \lambda$;

[3]Recall that by [19], $2^\lambda = \lambda^+$ entails $\diamondsuit_S$ for every S which does not concentrate on the critical cofinality.

- $M \cap \lambda^+ \in E^{\lambda^+}_\omega$;
- $\langle \mathbb{Q}(\lambda^+), \mathbb{Q}'(\lambda^+), \langle s^*, k^*\rangle, \dot{E}\rangle \in M$.

Notice that since λ is a strong limit, we have ${}^Z2 \subseteq M$ for all $Z \in M$ with $|Z| < \lambda$. For a set $Z \in [\lambda^{++}]^\lambda$, let $\psi_Z : \lambda \to Z$ be the $<_\chi$-least surjection. Fix $\{\lambda_n \mid n < \omega\} \in M$ which is a cofinal subset of λ. Evidently, for every $Z \in [\lambda^{++}]^\lambda \cap M$ and $n < \omega$, we have $\psi_Z``\lambda_n \in M$.

Definition 1.8.1 (Deciding the club)**.** For $\alpha < \lambda^+$ and a condition $\langle s, k\rangle \geq \langle s^*, k^*\rangle$, let $\langle s, k\rangle^\alpha$ denote the $<_\chi$-least extension of $\langle s, k\rangle$ such that:

- $\langle s, k\rangle^\alpha \in \mathbb{Q}'(\lambda^+)$;
- $\langle s, k\rangle^\alpha$ decides a value for $\min(E \setminus \alpha)$;
- if $\langle s, k\rangle^\alpha = \langle s', k'\rangle$, then $\sup(s') \geq \alpha$.

Notice that if $\langle s, k\rangle \in M$ and $\alpha \in M \cap \lambda^+$, then $\langle s, k\rangle^\alpha \in M$, as well.

To slightly simplify the next definition, for $\alpha \in \mathrm{Ord}$ and $m < \omega$, we designate the open interval $\mathrm{Int}(\alpha, m) = \{\beta \mid \alpha + m < \beta < \alpha + \omega_1\}$.

Definition 1.8.2 (Branching extensions)**.** For a condition $\langle s, k\rangle \in \mathbb{Q}'(\lambda^+)$, a set $Z \in [\lambda^{++}]^{\leq\lambda}$, a function $g : Z \to 2$, and an ordinal $\gamma < \lambda^+$, we shall define $\langle s, k\rangle^g_\gamma$.

If $\gamma < \sup\{\max(c^k_i) \mid i \in \mathrm{supp}(k) \cap Z\}$, we just let $\langle s, k\rangle^g_\gamma := \emptyset$. Otherwise, $\langle s, k\rangle^g_\gamma = \langle s', k'\rangle$ is the $\leq_\chi$-least extension of $\langle s, k\rangle$ such that:

- $\langle s', k'\rangle \in \mathbb{Q}'(\lambda^+)$;
- $s' = s$;
- $\mathrm{supp}(k') = \mathrm{supp}(k) \cup Z$;
- for all $i \in \mathrm{supp}(k')$:

$$(x^{k'}_i, c^{k'}_i) = \begin{cases} (x^k_i, c^k_i), & i \notin Z \\ (\mathrm{Int}(\gamma, g(i)), \{\gamma + \omega_1\}), & i \in Z \setminus \mathrm{supp}(k) \\ \left(x^k_i \cup \mathrm{Int}(\gamma, g(i)), c^k_i \cup \{\gamma + \omega_1\}\right), & i \in Z \cap \mathrm{supp}(k). \end{cases}$$

To see that the definition is good, just notice that for all $i \in \mathrm{supp}(k')$, if $s' \cap c^{k'}_i \neq \emptyset$, then $i \in \mathrm{supp}(k)$, and $x^{k'}_i \cap \delta = x^k_i \cap \delta$ for all $\delta \in s' \cap c^{k'}_i = s \cap c^k_i$.

Evidently, if $\{g, \gamma, \langle s, k\rangle\} \subseteq M$, then $\langle s, k\rangle^g_\gamma \in M$.

Definition 1.8.3 (Mixing two conditions)**.** Given $q_0 = \langle s_0, k_0\rangle, q_1 = \langle s_1, k_1\rangle$ in $\mathbb{Q}'(\lambda^+)$, and $\beta < \lambda^{++}$ such that $\langle s_0, k_0\rangle \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta \leq \langle s_1, k_1\rangle \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta$, let $\mathrm{mix}(q_1, \beta, q_0)$ be the $<_\chi$-least condition $\langle s', k'\rangle$ such that:

- $\langle s', k'\rangle \in \mathbb{Q}'(\lambda^+)$;
- $s' = s_1$;
- $\mathrm{supp}(k') = \mathrm{supp}(k_0) \cup (\mathrm{supp}(k_1) \cap \beta)$;

- for all $i \in \operatorname{supp}(k')$:

$$(x_i^{k'}, c_i^{k'}) = \begin{cases} (x_i^{k_1}, c_i^{k_1}), & i < \beta \\ (x_i^{k_0}, c_i^{k_0}), & i \geq \beta \wedge \max(c_i^{k_0}) \geq \sup(s') \\ \left(x_i^{k_0} \cup \operatorname{Int}(\sup(s'), 0), c_i^{k_0} \cup \{\sup(s') + \omega_1\}\right), & \text{otherwise} \end{cases}$$

It is not hard to see that the definition is good, and that $\operatorname{mix}(q_1, \beta, q_0) \in M$ whenever $\{q_1, \beta, q_0\} \subseteq M$. Notice that $\operatorname{mix}(q_1, \beta, q_0)$ makes sense, also in the case $\beta = 0$.

Claim 1.8.4. *Suppose $g_1, g_0 \in {}^Z 2$ for a given set $Z \in [\lambda^{++}]^{\leq\lambda}$, and that $q \in \mathbb{Q}'(\lambda^+)$.*

If $\beta \leq \Delta(g_1, g_1)$ and $\alpha, \gamma < \lambda^+$, then $\operatorname{mix}(q_\gamma^{g_1}, \beta, q_\gamma^{g_0}) = q_\gamma^{g_1}$, and $\operatorname{mix}((q_\gamma^{g_1})^\alpha, \beta, q_\gamma^{g_0})$ is a well-defined extension of $q_\gamma^{g_0}$.

Proof. This follows immediately from the above definitions, and we encourage the reader to digest these definitions by verifying this claim. $\square$

Put $\tau := M \cap \lambda^+$ and pick $\{\tau_n \mid n < \omega\} \subseteq M \cap \lambda^+$ with $\sup_{n<\omega} \tau_n = \tau$. Denote $Z_{-1} := \emptyset$. We now recursively define a sequence $\langle (\gamma_n, Y_n, Z_n, \mathcal{F}_n) \mid n < \omega\rangle$ in such a way that for all $n < \omega$:

(1) $\gamma_n < \lambda^+$;
(2) $\{Y_n \mid n < \omega\} \subseteq [\lambda^{++}]^\lambda$ and $\{Z_n \mid n < \omega\} \subseteq [\lambda^{++}]^{<\lambda}$ are increasing chains that converges to the same set;
(3) for every $g \in {}^{Z_n}2$, we define a condition $q_g \in \mathbb{Q}'(\lambda^+)$ in such a way that $m < n$ implies $q_{g \restriction Z_m} \leq q_g$, and we let $\mathcal{F}_n := \{q_g \mid g \in {}^{Z_n}2\}$;
(4) for every $g \in {}^{Z_n}2$, there exists some $r \in \mathbb{Q}'(\lambda^+)$ such that
$$q_g \geq r_{\gamma_n}^{g \restriction (Z_n \setminus Z_{n-1})} \geq r \geq \langle s^*, k^*\rangle;$$
(5) for every $g_0, g_1 \in {}^{Z_n}2$ and $\beta < \lambda^{++}$, if $g_0 \restriction \beta = g_1 \restriction \beta$, then
$$q_{g_0} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta = q_{g_1} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta;$$
(6) if $\langle s, k\rangle \in \mathcal{F}_n$, then $\langle s, k\rangle$ decides a value for $\min(E \setminus \tau_n)$, and $\sup(s) \geq \tau_n$.

We commence with letting $\gamma_0 := \sup\{\max(c_i^{k^*}) \mid i \in \operatorname{supp}(k^*)\}$, $Y_0 := \operatorname{supp}(k^*)$, and $Z_0 := \psi_{Y_0}``\lambda_0$. Next, we would like to define q_g for all $g \in {}^{Z_0}2$. Let $\{g_j \mid j < 2^{\lambda_0}\}$ be the $<_\chi$-least injective enumeration of ${}^{Z_0}2$. We shall define an upper triangular matrix of conditions, $\{q_{jl} \mid j \leq l < 2^{\lambda_0}\}$, in such a way that for all $j \leq j' \leq l \leq l' < 2^{\lambda_0}$, we would have:

(a) $r_{\gamma_0}^{g_j} \leq q_{jj} \leq q_{jl} \leq q_{jl'}$, where $r := \langle s^*, k^*\rangle$;
(b) $p^{\tau_0} \leq q_{jj}$ for some condition $p \in \mathbb{Q}'(\lambda^+)$;
(c) $q_{jl'} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta = q_{j'l'} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta$ whenever $g_j \restriction \beta = g_{j'} \restriction \beta$.

Once we have that, for each $g \in {}^{Z_0}2$, we pick the unique j such that $g_j = g$, and let q_g be the least upper bound of the increasing sequence $\langle q_{jl} \mid j \leq$

$l < 2^{\lambda_0}\rangle$. Then item (a) takes care of requirement (4), item (b) establishes requirement (6), and item (c) yields requirement (5).

Thus, the j_{th} row of the matrix is responsible for the condition q_{g_j}. The actual definition of the matrix, however, is obtained along the columns. Namely, we define $\{q_{jl} \mid j \leq l < 2^{\lambda_0}\}$ by induction on $l < 2^{\lambda_0}$.

Induction base: Let $q_{00} := (r^{g_0}_{\gamma_0})^{\tau_0}$.

Successor step: Suppose $l < 2^{\lambda_0}$ and $\{q_{jl} \mid j \leq l\}$ has already been defined. We would like to define $\{q_{jl'} \mid j \leq l'\}$, for $l' := l+1$.

Put $\alpha := \sup\{\Delta(g_j, g_{l'}) + 1 \mid j < l'\}$. For all $\beta < \alpha$, let j_β be the least such that $j_\beta < l'$ and $g_{j_\beta} \restriction \beta = g_{l'} \restriction \beta$.

Fix $\beta < \alpha$. By Definition 1.8.2, we have $r^{g_{j_\beta}}_{\gamma_0} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta = r^{g_{l'}}_{\gamma_0} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta$, and by property (a), we have $r^{g_{j_\beta}}_{\gamma_0} \leq q_{j_\beta l}$. It follows that $r^{g_{l'}}_{\gamma_0} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta \leq q_{j_\beta l} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta$, and hence $\mathrm{mix}(q_{j_\beta l}, \beta, r^{g_{l'}}_{\gamma_0})$ is a well-defined extension of $r^{g_{l'}}_{\gamma_0}$.

By property (c), we get that $\beta < \gamma < \alpha$ implies $q_{j_\beta l} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta = q_{j_\gamma l} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta$ and $\mathrm{mix}(q_{j_\beta l}, \beta, r^{g_{l'}}_{\gamma_0}) = \mathrm{mix}(q_{j_\gamma l}, \beta, r^{g_{l'}}_{\gamma_0}) \leq \mathrm{mix}(q_{j_\gamma l}, \gamma, r^{g_{l'}}_{\gamma_0})$.

For all $\beta < \alpha$, let $p_\beta := \mathrm{mix}(q_{j_\beta l}, \beta, r^{g_{l'}}_{\gamma_0})$. Then, we have just established that $\langle p_\beta \mid \beta < \alpha\rangle$ is an increasing sequence of conditions, with $p_0 \geq r^{g_{l'}}_{\gamma_0}$, and $p_\beta \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta = p_\gamma \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta$ for all $\beta < \gamma < \alpha$.

By $\mathrm{cf}(\alpha) \leq l' < \lambda$, let p_α be the least upper bound of the sequence, $\langle p_\beta \mid \beta < \alpha\rangle$. Then, for all $j < l'$:

$$(\star) \qquad \begin{array}{c} p_\alpha \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_{\Delta(g_j,g_{l'})} = p_{\Delta(g_j,g_{l'})} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_{\Delta(g_j,g_{l'})} = \\ q_{j_{\Delta(g_j,g_{l'})} l} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_{\Delta(g_j,g_{l'})} = q_{jl} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_{\Delta(g_j,g_{l'})} \end{array}.$$

Put $q_{l'l'} := (p_\alpha)^{\tau_0}$. Then (b) is clearly satisfied. By $q_{l'l'} \geq p_\alpha \geq p_0 \geq r^{g_{l'}}_{\gamma_0}$, we also have (a). By $(\star)$, we now get that for all $j < l'$:

$$q_{l'l'} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_{\Delta(g_j,g_{l'})} \geq p_\alpha \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_{\Delta(g_j,g_{l'})} = q_{jl} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_{\Delta(g_j,g_{l'})},$$

thus, to establish property (c), we just assign $q_{jl'} := \mathrm{mix}(q_{l'l'}, \Delta(g_j, g_{l'}), q_{jl})$ for all $j < l'$.

Limit step: Suppose $l' < 2^{\lambda_0}$ is some ordinal and $\{q_{jl} \mid j \leq l < l'\}$ has already been defined. For all $j < l'$, let q_j be the least upper bound of the increasing sequence, $\langle q_{jl'} \mid j \leq l' < l\rangle$. To compare with the successor step, we now work against $\{q_j \mid j < l'\}$, instead of $\{q_{jl} \mid j < l'\}$, where l was the immediate predecessor of l'.

Put $\alpha := \sup\{\Delta(g_j, g_{l'}) + 1 \mid j < l'\}$, and for all $\beta < \alpha$, let j_β be the least such that $j_\beta < l'$ and $g_{j_\beta} \restriction \beta = g_{l'} \restriction \beta$. Then (a) and (c) implies:

(a') $r^{g_{j_\beta}}_{\gamma_0} \leq q_{j_\beta j_\beta} \leq q_{j_\beta}$ for all $\beta < \alpha$;

(c') $q_{j_\beta} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta = q_{j_\gamma} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta$ for all $\beta < \gamma < \alpha$.

For all $\beta < \alpha$, let $p_\beta := \mathrm{mix}(q_{g_{j_\beta}}, \beta, r^{g_{l'}}_{\gamma_0})$, and let p_α be the least upper bound of the increasing sequence $\langle p_\beta \mid \beta < \alpha\rangle$. Then for all $j < l'$:

$$(\star') \qquad \begin{array}{c} p_\alpha \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_{\Delta(g_j,g_{l'})} = p_{\Delta(g_j,g_{l'})} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_{\Delta(g_j,g_{l'})} = \\ q_{j_{\Delta(g_j,g_{l'})}} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_{\Delta(g_j,g_{l'})} = q_j \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_{\Delta(g_j,g_{l'})} \end{array}.$$

Thus, put $q_{l'l'} := (p_\alpha)^{\tau_0}$, and for all $j < l'$, let $q_{jl'} := \mathrm{mix}(q_{l'l'}, \Delta(g_j, g_{l'}), q_j)$.

This completes the construction of $\mathcal{F}_0 = \{q_g \mid g \in {}^{Z_0}2\}$.

Next, suppose $\langle(\gamma_m, Y_m, Z_m, \mathcal{F}_m) \mid m \leq n\rangle$ has already been defined for some $n < \omega$. Let:

- $\gamma_{n+1} := \sup\{\max(c^k_i) \mid \langle s, k\rangle \in \mathcal{F}_n \text{ for some s}, i \in \mathrm{supp}(k)\}$;
- $Y_{n+1} := \bigcup\{\mathrm{supp}(k) \mid \langle s, k\rangle \in \mathcal{F}_n \text{ for some } s\}$;
- $Z_{n+1} := \bigcup\{\psi_{Y_k}``\lambda_{n+1} \mid k \leq n+1\}$;
- $\mathcal{F}_{n+1} := \{q_g \mid g \in {}^{Z_{n+1}}2\}$.

Clearly, our main task is defining q_g for all $g \in {}^{Z_{n+1}}2$. Let $\{g_j \mid j < 2^{\lambda_{n+1}}\}$ be the $<_\chi$-least bijective enumeration of ${}^{Z_{n+1}}2$. We shall now define an upper triangular matrix $\{q_{jl} \mid j \leq l < 2^{\lambda_{n+1}}\}$ in such a way that for all $j \leq j' \leq l \leq l' < 2^{\lambda_{n+1}}$, we would have:

(a) $(q_{g_j\restriction Z_n})^{g_j\restriction(Z_{n+1}\setminus Z_n)}_{\gamma_{n+1}} \leq q_{jj} \leq q_{jl} \leq q_{jl'}$;
(b) $p^{\tau_{n+1}} \leq q_{jj}$ for some condition $p \in \mathbb{Q}'(\lambda^+)$;
(c) $q_{jl'} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta = q_{j'l'} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta$ whenever $g_j \restriction \beta = g_{j'} \restriction \beta$.

As in the base case, once the matrix is defined, for each $g \in {}^{Z_{n+1}}2$, we pick the unique j such that $g_j = g$, and let q_g be the least upper bound of the increasing sequence $\langle q_{jl} \mid j \leq l < 2^{\lambda_{n+1}}\rangle$.

<u>Induction base:</u> Let $q_{00} := ((q_{g_0\restriction Z_n})^{g_0\restriction(Z_{n+1}\setminus Z_n)}_{\gamma_{n+1}})^{\tau_{n+1}}$.

<u>Successor step:</u> Suppose $l < 2^{\lambda_{n+1}}$ and $\{q_{jl} \mid j \leq l\}$ has already been defined. We would like to define $\{q_{jl'} \mid j \leq l'\}$, for $l' := l + 1$.

Put $\alpha := \sup\{\Delta(g_j, g_{l'}) + 1 \mid j < l'\}$, and for all $\beta < \alpha$, let j_β be the least such that $j_\beta < l'$ and $g_{j_\beta} \restriction \beta = g_{l'} \restriction \beta$. Fix $\beta < \alpha$. By (5), we have $q_{g_{l'}\restriction Z_n} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta = q_{g_{j_\beta}\restriction Z_n} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta$, and hence:

$$(q_{g_{l'}\restriction Z_n})^{g_{l'}\restriction(Z_{n+1}\setminus Z_n)}_{\gamma_{n+1}} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta = (q_{g_{j_\beta}\restriction Z_n})^{g_{j_\beta}\restriction(Z_{n+1}\setminus Z_n)}_{\gamma_{n+1}} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta.$$

In addition, by property (a), we have $(q_{g_{j_\beta}\restriction Z_n})^{g_{j_\beta}\restriction(Z_{n+1}\setminus Z_n)}_{\gamma_{n+1}} \leq q_{j_\beta l}$. Denote $r_{l'} := (q_{g_{l'}\restriction Z_n})^{g_{l'}\restriction(Z_{n+1}\setminus Z_n)}_{\gamma_{n+1}}$. It follows that $r_{l'} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta \leq q_{j_\beta l} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta$, and hence $\mathrm{mix}(q_{j_\beta l}, \beta, r_{l'})$ is a well-defined extension of $r_{l'}$.

By property (c), we get that $\beta < \gamma < \alpha$ implies $q_{j_\beta l} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta = q_{j_\gamma l} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta$ and $\mathrm{mix}(q_{j_\beta l}, \beta, r_{l'}) = \mathrm{mix}(q_{j_\gamma l}, \beta, r_{l'}) \leq \mathrm{mix}(q_{j_\gamma l}, \gamma, r_{l'})$.

For all $\beta < \alpha$, let $p_\beta := \mathrm{mix}(q_{j_\beta l}, \beta, r_{l'})$. Then, we have just established that $\langle p_\beta \mid \beta < \alpha\rangle$ is an increasing sequence of conditions, with $p_0 \geq (q_{g_{l'}\restriction Z_n})^{g_{l'}\restriction(Z_{n+1}\setminus Z_n)}_{\gamma_{n+1}}$, and $p_\beta \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta = p_\gamma \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta$ for all

$\beta < \gamma < \alpha$. By $\mathrm{cf}(\alpha) < \lambda$, let p_α be the least upper bound of the sequence, $\langle p_\beta \mid \beta < \alpha\rangle$. Then, for all $j < l'$, we have $(\star)$ as above, so let $q_{l'l'} := (p_\alpha)^{\tau_{n+1}}$, and assign $q_{jl'} := \mathrm{mix}(q_{l'l'}, \Delta(g_j, g_{l'}), q_{jl})$ for all $j < l'$.

Limit step: Suppose $l' < 2^{\lambda_{n+1}}$ is some ordinal and $\{q_{jl} \mid j \leq l < l'\}$ has already been defined. For all $j < l'$, let q_j be the least upper bound of the increasing sequence, $\langle q_{jl'} \mid j \leq l' < l\rangle$. Put $\alpha := \sup\{\Delta(g_j, g_{l'}) + 1 \mid j < l'\}$, and for all $\beta < \alpha$, let j_β be the least such that $j_\beta < l'$ and $g_{j_\beta} \restriction \beta = g_{l'} \restriction \beta$. Then, the following holds:

(a') $(q_{g_{j_\beta}\restriction Z_n})^{g_{j_\beta}\restriction(Z_{n+1}\setminus Z_n)}_{\gamma_{n+1}} \leq q_{j_\beta j_\beta} \leq q_{j_\beta}$ for all $\beta < \alpha$;

(c') $q_{j_\beta} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta = q_{j_\gamma} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta$ for all $\beta < \gamma < \alpha$.

Fix $\beta < \alpha$. By (5), we have $q_{g_{l'}\restriction Z_n} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta = q_{g_{j_\beta}\restriction Z_n} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta$, and hence:

$$(q_{g_{l'}\restriction Z_n})^{g_{l'}\restriction(Z_{n+1}\setminus Z_n)}_{\gamma_{n+1}} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta = (q_{g_{j_\beta}\restriction Z_n})^{g_{j_\beta}\restriction(Z_{n+1}\setminus Z_n)}_{\gamma_{n+1}} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta.$$

Denote $r_{l'} := (q_{g_{l'}\restriction Z_n})^{g_{l'}\restriction(Z_{n+1}\setminus Z_n)}_{\gamma_{n+1}}$. By item (a') and the above equation, we get that $r_{l'} \restriction \mathbb{S}(\lambda^+)*\mathbb{P}_\beta \leq q_{j_\beta} \restriction \mathbb{S}(\lambda^+)*\mathbb{P}_\beta$, and hence $\mathrm{mix}(q_{j_\beta}, \beta, r_{l'})$ is a well-defined extension of $r_{l'}$. By (c'), if we write $p_\beta := \mathrm{mix}(q_{j_\beta}, \beta, r_{l'})$, then $\langle p_\beta \mid \beta < \alpha\rangle$ is an increasing sequence of conditions, with $p_0 \geq (q_{g_{l'}\restriction Z_n})^{g_{l'}\restriction(Z_{n+1}\setminus Z_n)}_{\gamma_{n+1}}$, and $p_\beta \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta = p_\gamma \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta$ for all $\beta < \gamma < \alpha$.

Let p_α be the least upper bound of the sequence, $\langle p_\beta \mid \beta < \alpha\rangle$. Then, for all $j < l'$, we have $(\star')$ as above, so let $q_{l'l'} := (p_\alpha)^{\tau_{n+1}}$ and $q_{jl'} := \mathrm{mix}(q_{l'l'}, \Delta(g_j, g_{l'}), q_j)$ for all $j < l'$.

This completes the construction of $\mathcal{F}_{n+1} = \{q_g \mid g \in {}^{Z_{n+1}}2\}$.

Claim 1.8.5. *$\{\gamma_n, Z_n, Y_n, \mathcal{F}_n\} \subseteq M$ for all $n < \omega$.*

Proof. Easy. □

Let $Z := \bigcup_{n<\omega} Z_n$, and for every function $g : Z \to 2$, let q_g denote the least upper bound of the increasing sequence $\langle q_{g\restriction Z_n} \mid n < \omega\rangle$. We now state and prove several claims that should gradually clarify the role of the above construction.

Claim 1.8.6. *For every $g \in {}^Z2$, if $q_g = \langle s, k\rangle$, then* $\mathrm{supp}(k) = Z$.

Proof. Fix $n < \omega$ and $\langle s_n, k_n\rangle \in \mathcal{F}_n$. By property (4) and Definition 1.8.2, we have $Z_n \setminus Z_{n-1} \subseteq \mathrm{supp}(k_n)$. By definition of Y_{n+1}, we also have $\mathrm{supp}(k_n) \subseteq Y_{n+1}$. It follows that if $g \in {}^Z2$ and $q_g = \langle s, k\rangle$, then

$$Z = \bigcup_{n<\omega}(Z_n \setminus Z_{n-1}) \subseteq \mathrm{supp}(k) \subseteq \bigcup_{n<\omega} Y_{n+1} = Z. \quad \square$$

Claim 1.8.7. *For every $g_0, g_1 \in {}^Z2$ and $\beta < \lambda^{++}$, if $g_0 \restriction \beta = g_1 \restriction \beta$, then $q_{g_0} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta = q_{g_1} \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\beta$.*

Proof. For each $i < 2$, q_{g_i} is the least upper bound of $\langle q_{g_i \restriction Z_n} \mid n < \omega\rangle$. Now appeal to property (5) of the construction. $\square$

By the previous claim, the next definition is good.

Definition 1.8.8. Given $\beta < \lambda^{++}$, $h : Z \cap \beta \to 2$, and $i \in Z \cap \beta$, pick a function $g : Z \to 2$ that extends h, and define $(x_i^h, c_i^h) := (x_i^k, c_i^k)$, where k is such that $q_g = \langle s, k\rangle$.

Claim 1.8.9. *For all $\beta < \lambda^{++}$, $h : Z \cap \beta \to 2$, and $i \in Z \cap \beta$, we have:*

$$\sup(x_i^h) = \max(c_i^h) = \tau.$$

Proof. Pick $g : Z \to 2$ extending h. Fix $n < \omega$, and denote $\langle s_n, k_n\rangle = q_{g\restriction Z_n}$. By property (6), we have $\sup(s) \geq \tau_n$. By $\langle s_n, k_n\rangle \in \mathbb{Q}'(\lambda^+) \cap M$, we have

$$\tau = \lambda^+ \cap M > \sup(x_i^{k_n}) = \sup(c_i^{k_n}) \geq \sup(s) \geq \tau_n.$$

The conclusion now follows. $\square$

Claim 1.8.10. *There exists $\bar{s} \subseteq \tau$ such that for all $g \in {}^Z2$, if $\langle s, k\rangle = q_g$, then $s = \bar{s}$.*

Proof. The existence of $\bar{s}$ follows from Claim 1.8.7, but let us argue that $\bar{s} \subseteq \tau$. Pick an arbitrary function $g \in {}^Z2$. Fix $n < \omega$. Denote $\langle s_n, k_n\rangle = q_{g\restriction Z_n}$. Then $s_n \in M$ and hence $s_n \subseteq \tau$. As $\bar{s}$ is equal to $\bigcup_{n<\omega} s_n$, we conclude that $\bar{s} \subseteq \tau$. $\square$

Claim 1.8.11. *For every $g \in {}^Z2$, $q_g \Vdash \check{\tau} \in \dot{E}$.*

Proof. For all $n < \omega$, by property (6) and $\mathcal{F}_n \subseteq M$, there exists α_n with $\tau_n \leq \alpha_n < \lambda^+ \cap M = \tau$ such that $q_{g\restriction Z_n} \Vdash \check{\alpha}_n \in \dot{E}$. Since $\sup\{\tau_n \mid n < \omega\} = \tau$, we get that $\sup\{\alpha_n \mid n < \omega\} = \tau$, and hence q_g forces that τ is an accumulation point of the club E. $\square$

Thus, our main task is to argue the existence of some $g \in {}^Z2$ such that q_g is compatible with $\langle \bar{s} \cup \{\tau\}, \emptyset\rangle$. Evidently, such a condition will force that S meets E. A key fact for insuring the existence of such function, is the next claim.

Claim 1.8.12. *Suppose $A \subseteq \tau$.*

For every $i \in Z$, there exists some $m < 2$ such that $x_i^k \neq A$ whenever $\langle s, k\rangle = q_g$ and $\{(i, m)\} \subseteq g \in {}^Z2$.

Proof. Suppose $i \in Z$. Let $n < \omega$ be the unique ordinal such that $i \in Z_n \setminus Z_{n-1}$. Since $\mathrm{Int}(\gamma_n, 0) \neq \mathrm{Int}(\gamma_n, 1)$ and $\gamma_n + \omega_1 < \tau = M \cap \lambda^+$, let us pick some $m < 2$ such that $A \cap \mathrm{Int}(\gamma_n, 0) \neq \mathrm{Int}(\gamma_n, m)$. Now, if $\langle s, k\rangle = q_g$ for some $g \in {}^Z2$ with $g(i) = m$, then by property (4), $q_g \geq r_{\gamma_n}^{g\restriction(Z_n \setminus Z_{n-1})} \geq r$ for some condition r, and by Definition 1.8.2, this means that $i \in \mathrm{supp}(k)$, and:

$$x_i^k \cap \mathrm{Int}(\gamma_n, 0) = \mathrm{Int}(\gamma_n, g(i)) = \mathrm{Int}(\gamma_n, m) \neq A \cap \mathrm{Int}(\gamma_n, 0).$$

In particular $x_i^k \neq A$. $\square$

Let D denote the set of all $\langle s,k\rangle$ such that all items of Definition 1.4, except to (5), are satisfied, where instead, we require that for all $\alpha \in \mathrm{supp}(k)$:

(5a) either $\sup(x^k_\alpha) = \max(c^k_\alpha) \geq \sup(s)$, or
(5b) $\alpha \in \mathrm{supp}(k^*)$ and $(x^k_\alpha, c^k_\alpha) = (x^{k^*}_\alpha, c^{k^*}_\alpha)$.

Then $D \supseteq \mathbb{Q}'(\lambda^+)$ is a dense set, and by a proof similar to the one of Lemma 1.5, every increasing sequence of length $< \lambda^+$ of elements of D has a least upper bound (within D).

Definition 1.8.13. Given $\alpha \in Z$ and a function $h : Z \cap (\alpha+1) \to 2$, we define the condition $q_h = \langle s', k'\rangle$ as follows. Fix $g : Z \to 2$ extending h.

If $q_g = \langle s,k\rangle$, then:

- $s' = s$;
- $\langle s', k'\rangle \in D$;
- $\mathrm{supp}(k') = (\mathrm{supp}(k) \cap (\alpha+1)) \cup \mathrm{supp}(k^*)$;
- for all $i \in \mathrm{supp}(k')$:

$$(x^{k'}_i, c^{k'}_i) = \begin{cases} (x^k_i, c^k_i), & i \leq \alpha \\ (x^{k^*}_i, c^{k^*}_i), & \text{otherwise} \end{cases}.$$

Then q_h is well-defined, by Claim 1.8.7.

For each $\alpha < \lambda^{++}$, let D_α denote the dense-open subset of $\mathbb{S}(\lambda^+) * \mathbb{P}_\alpha$ that decides $\overrightarrow{A_\alpha}(\tau)$, and let $D^{\rightarrow}_\alpha := \{p \in D \mid p \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_\alpha \in D_\alpha\}$ denote its cylindric extension.

Put $\theta := \mathrm{otp}(Z)$ and let $\{\varepsilon_\alpha \mid \alpha < \theta\}$ be the increasing enumeration of Z. We now define by induction an increasing sequence of conditions, $\langle p_\alpha = \langle s_\alpha, k_\alpha\rangle \mid \alpha < \theta\rangle$ and a chain of functions $\{h_\alpha : Z \cap (\varepsilon_\alpha+1) \to 2 \mid \alpha < \theta\}$ in such a way that for all $\alpha < \theta$:

(a) $q_{h_\alpha} \leq p_\alpha \in D^{\rightarrow}_{\varepsilon_\alpha}$;
(b) $\bar{s} \cup \{\tau\} \sqsubseteq s_\alpha$;
(c) $\mathrm{supp}(k_\alpha) \subseteq \mathrm{supp}(k^*) \cup (\varepsilon_\alpha + 1)$;
(d) $(x^{k_\alpha}_i, c^{k_\alpha}_i) = (x^{k^*}_i, c^{k^*}_i)$ for all $i \in \mathrm{supp}(k_\alpha) \setminus (\varepsilon_\alpha+1)$.

<u>Induction base:</u> Since $0 \in \mathrm{supp}(k^*) \subseteq Z$, we have $\varepsilon_0 = 0$. Pick $\langle s_0, \emptyset\rangle \in D_0$ with $\bar{s} \cup \{\tau\} \sqsubseteq s_0$.[4] In particular, $\langle s_0, \emptyset\rangle$ decides $\overrightarrow{A_0}(\tau)$ to be, say, A^τ_0. By Claim 1.8.12, pick a function $h_0 : \{\varepsilon_0\} \to 2$ such that $A^\tau_0 \neq x^{h_0}_{\varepsilon_0}$, and let $p_0 := \langle s_0, k_0\rangle$ be the condition in D such that $\mathrm{supp}(k_0) = \mathrm{supp}(k^*)$ and for all $i \in \mathrm{supp}(k_0)$:

$$(x^{k_0}_i, c^{k_0}_i) := \begin{cases} (x^{h_0}_i, c^{h_0}_i), & i = \varepsilon_0 \\ (x^{k^*}_i, c^{k^*}_i), & \text{else} \end{cases}.$$

[4]Recall that $D_0 \subseteq \mathbb{S}(\lambda^+) * \mathbb{P}_0$, and that $\mathbb{P}_0$ is the trivial forcing $\langle\{\emptyset\}, \{(\emptyset,\emptyset)\}\rangle$.

Let us show that p_0 is well-defined. Suppose not. Fix $i \in \mathrm{supp}(k_0)$ and $\delta \in s_0 \cap c_i^{k_0}$ such that $\overrightarrow{A_i}(\delta) = x_i^{k_0} \cap \delta$. Clearly, if $i > \varepsilon_0$, then $\delta \in s_0 \cap c_i^{k^*}$ implies that $\delta \in s^*$, contradicting the fact that $\langle s^*, k^* \rangle$ is a legitimate condition. So $i = \varepsilon_0 = 0$. Since $\delta \in c_0^{h_0}$, we get from Claim 1.8.9, that either $\delta = \tau$ or $\delta \in \bar{s}$. It is impossible that $\delta = \tau$, because the choice of h_0 insured that $x_0^{k_0} \cap \delta \neq \overrightarrow{A_0}(\delta)$. So, $\delta \in \bar{s}$.

Pick a function $g : Z \to 2$ extending h_0, and let $\langle s^n, k^n \rangle := q_{g \restriction Z_n}$ for all $n < \omega$. Then $\bar{s} = \bigcup_{n<\omega} s^n$, $x_0^{h_0} = \bigcup_{n<\omega} x_0^{k^n}$ and $c_0^{h_0} = \bigcup_{n<\omega} c_0^{k^n} \cup \{\tau\}$. Pick $n < \omega$ such that $\delta \in s_n$. By $\langle s^n, k^n \rangle \in \mathbb{Q}'(\lambda^+)$, we have $\sup(x_0^{k^n}) = \max(x_0^{k^n}) \geq \sup(s^n) \geq \delta$, and then $\overrightarrow{A_0}(\delta) = x_0^{h_0} \cap \delta = x_0^{k^n} \cap \delta$ while $\delta \in s^n \cap c_0^{k^n}$, contradicting the basic fact that $\langle s^n, k^n \rangle$ is a legitimate condition.

Thus, p_0 is well-defined and satisfies requirements (a)–(d).

Successor step: Suppose that for some $\alpha < \theta$, p_α has already been defined. Pick $p' = \langle s', k' \rangle \geq p_\alpha$ in $D_{\overrightarrow{\varepsilon_{\alpha+1}}}$. Then $p' \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_{\varepsilon_{\alpha+1}}$ decides $\overrightarrow{A_{\varepsilon_{\alpha+1}}}(\tau)$ to be, say, $A^\tau_{\varepsilon_{\alpha+1}}$. By Claim 1.8.12, pick a function $h_{\alpha+1} : Z \cap (\varepsilon_{\alpha+1} + 1) \to 2$ extending h_α such that $A^\tau_{\varepsilon_{\alpha+1}} \neq x^{h_{\alpha+1}}_{\varepsilon_{\alpha+1}}$. Finally, let $p_{\alpha+1} = \langle s_{\alpha+1}, k_{\alpha+1} \rangle$ be the condition in D such that:

- $s_{\alpha+1} = s'$;
- $\mathrm{supp}(k_{\alpha+1}) = (\mathrm{supp}(k') \cap \varepsilon_{\alpha+1}) \cup \{\varepsilon_{\alpha+1}\} \cup \mathrm{supp}(k^*)$;
- for all $i \in \mathrm{supp}(k_\alpha)$, we have:

$$(x_i^{k_{\alpha+1}}, c_i^{k_{\alpha+1}}) := \begin{cases} (x_i^{k'}, c_i^{k'}), & i < \varepsilon_{\alpha+1} \\ (x_i^{h_{\alpha+1}}, c_i^{h_{\alpha+1}}), & i = \varepsilon_{\alpha+1} \\ (x_i^{k^*}, c_i^{k^*}), & \text{else} \end{cases}.$$

Assume indirectly that $p_{\alpha+1}$ is not well-defined. Fix $i \in \mathrm{supp}(k_{\alpha+1})$ and $\delta \in s_{\alpha+1} \cap c_i^{k_{\alpha+1}}$ such that $\overrightarrow{A_i}(\delta) = x_i^{k_{\alpha+1}} \cap \delta$. If $i \neq \varepsilon_{\alpha+1}$, then we get a contradiction to the fact that $\langle s^*, k^* \rangle$ and $\langle s', k' \rangle$ are legitimate conditions. If $i = \varepsilon_{\alpha+1}$, then Claim 1.8.9 and the choice of the function $h_{\alpha+1}$ insures that $\delta \in s_{\alpha+1} \cap \tau = \bar{s}$. But then, by $\delta \in \bar{s}$ and the exact same argument of the successor step, we have $\overrightarrow{A_{\varepsilon_{\alpha+1}}}(\delta) \neq x^{k_{\alpha+1}}_{\varepsilon_{\alpha+1}} \cap \delta$.

Clearly, $p_{\alpha+1}$ satisfies requirements (a)–(d).

Limit step: Suppose $\langle p_\beta \mid \beta < \alpha \rangle$ has already been defined for some limit ordinal $\alpha < \theta$, and let $\{h_\beta : Z \cap (\varepsilon_\beta + 1) \to 2 \mid \beta < \alpha\}$ be the witnessing functions to property (a). Since $\alpha < \theta < \lambda^+$, let p be the least upper bound of $\langle p_\beta \mid \beta < \alpha \rangle$, and let $h := \bigcup_{\beta<\alpha} h_\beta$. Clearly, $p \geq q_h$. Fix $p' = \langle s', k' \rangle \geq p$ in $D_{\overrightarrow{\varepsilon_\alpha}}$. Then $p' \restriction \mathbb{S}(\lambda^+) * \mathbb{P}_{\varepsilon_\alpha}$ decides $\overrightarrow{A_{\varepsilon_\alpha}}(\tau)$ to be, say, $A^\tau_{\varepsilon_\alpha}$. Pick $h_\alpha : Z \cap (\varepsilon_\alpha + 1) \to 2$ extending h such that $A^\tau_{\varepsilon_\alpha} \neq x^{h_\alpha}_{\varepsilon_\alpha}$, and let $p_\alpha = \langle s_\alpha, k_\alpha \rangle$ be the condition in D such that:

- $s_\alpha = s'$;
- $\mathrm{supp}(k_\alpha) = (\mathrm{supp}(k') \cap \varepsilon_\alpha) \cup \{\varepsilon_\alpha\} \cup \mathrm{supp}(k^*)$;

- for all $i \in \operatorname{supp}(k_\alpha)$, we have:

$$(x_i^{k_\alpha}, c_i^{k_\alpha}) := \begin{cases} (x_i^{k'}, c_i^{k'}), & i < \varepsilon_\alpha \\ (x_i^{h_\alpha}, c_i^{h_\alpha}), & i = \varepsilon_\alpha \\ (x_i^{k^*}, c_i^{k^*}), & \text{else} \end{cases}.$$

Then p_α is well-defined and satisfies all requirements.

This completes the construction. Put $g := \bigcup_{\alpha<\theta} h_\alpha$, and let p be an upper bound for the increasing sequence, $\langle p_\alpha \mid \alpha < \theta\rangle$. Then $p \geq q_g$ and $p \geq \langle \bar{s} \cup \{\tau\}, \emptyset\rangle$, and hence $p \Vdash \check{\tau} \in \dot{E} \cap \dot{S}$. $\square$

1.2. **Applications.** Utilizing the poset from the previous subsection, and the existence of a supercompact cardinal, we now consider three models in which diamond fails on a set that reflects stationarily often.

Theorem 1.9. *It is relatively consistent with the existence of a supercompact cardinal that all of the following holds simultaneously:*

(1) GCH;
(2) $\mathrm{AP}_{\aleph_\omega}$;
(3) $\mathrm{Refl}(E^{\aleph_{\omega+1}}_\omega)$;
(4) $\diamondsuit_S$ *fails for some stationary* $S \subseteq E^{\aleph_{\omega+1}}_\omega$.

Proof. We take as our ground model, the model from [7, §5]. That is, GCH holds, κ is a supercompact cardinal, there exists $\langle C_\alpha \mid \alpha < \kappa^{+\omega+1}\rangle$ which is a very weak square sequence, and there exists $\langle D_\alpha \mid \alpha \in E^{\kappa^{+\omega+1}}_{\geq\kappa}\rangle$ which is a partial square sequence. We shall not define these concepts here, instead, we just mention two important facts. The first is that the properties of these sequences are indestructible under cofinality-preserving forcing; the second is that in the generic extension by the Lévy collapse, $\mathrm{Col}(\omega_1, <\kappa)$, these two sequences are combined to witness $\mathrm{AP}_{\aleph_\omega}$.

Let $\mathbb{P}$ denote the iteration of length $\kappa+1$ with backward Easton support, where for every inaccessible $\alpha \leq \kappa$, we force with $\mathbb{Q}(\alpha^{+\omega+1})$ from Definition 1.4, and for accessible $\alpha < \kappa$, we use trivial forcing.

Let G be $\mathbb{P}$-generic over V. Then by Lemmas 1.6, 1.7, and a well-known argument of Silver (see [1, §11]), κ remains supercompact in $V[G]$. Also, the very weak square sequence and the partial square sequence remains as such.

By Theorem 1.8, there exists in $V[G]$, a stationary subset $S \subseteq E^{\kappa^{+\omega+1}}_\omega$ such that $\diamondsuit_S$ fails. Finally, let H be $\mathrm{Col}(\omega_1, <\kappa)$-generic over $V[G]$. Work in $V[G][H]$. Then $\aleph_2 = \kappa$, $\aleph_\omega = \kappa^{+\omega}$, $\aleph_{\omega+1} = \kappa^{+\omega+1}$, and $\mathsf{GCH} + \mathrm{AP}_{\aleph_\omega}$ holds. Since $\mathrm{Col}(\omega_1, <\kappa)$ satisfies the κ-c.c., S remains stationary, and $\diamondsuit_S$ still fails (for if $\{\dot{A}_\delta \mid \delta \in S\}$ is a name for a $\diamondsuit_S$-sequence in $V[G][H]$, then $\{\mathcal{A}_\delta := \{A \subseteq \delta \mid \exists p \in \mathrm{Col}(\omega_1, <\kappa)(p \Vdash \check{A} = \dot{A}_\delta)\} \mid \delta \in S\}$ would be a $\diamondsuit^-_S$-sequence in $V[G]$. See [12].)

Finally, since κ is $\kappa^{+\omega+1}$-supercompact in $V[G]$, an argument of Shelah yields that $\mathrm{Refl}(E^{\aleph_{\omega+1}}_{\omega})$ holds (see [1, §10]). $\square$

The next theorem shows that it possible to have the failure of $\diamondsuit_S$ for a set S which reflects in an even stronger sense.

Theorem 1.10. *It is relatively consistent with the existence of a supercompact cardinal that there exists a singular cardinal λ for which all of following holds simultaneously:*

(1) *λ is a strong limit of countable cofinality and $2^\lambda = \lambda^+$;*
(2) *there exists a stationary $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ such that:*
   (a) *$\{\alpha \in E^{\lambda^+}_{>\omega} \mid S \cap \alpha$ contains a club in $\alpha\}$ is stationary;*
   (b) *$\diamondsuit_S$ fails.*

Proof. Start with a model of MM. Put $\lambda := \beth_\omega$. Since λ is a singular strong limit, we get from [5] that $2^\lambda = \lambda^+$, and hence $\mathbb{Q}(\lambda^+)$ is well-defined, so let us work in the generic extension, $V^{\mathbb{Q}(\lambda^+)}$.

Since $\mathbb{Q}(\lambda^+)$ is $\aleph_2$-directed closed, we get from Larson's theorem [13] that MM is preserved, and by the additional "good properties" of $\mathbb{Q}(\lambda^+)$, the cardinals structure is preserved, as well. Then $\lambda = \beth_\omega$, $2^\lambda = \lambda^+$, and there exists a stationary $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$, such that $\diamondsuit_S$ fails. Finally, clause (a) is an immediate consequence of the fact that MM implies *Friedman's problem* (see [5]). $\square$

Analysis of the models from Theorems 1.9 and 1.10 yields that these models satisfies a certain strong form of reflection, namely, $\mathrm{Refl}^*([\lambda^+]^\omega)$, and hence there exists no *very good scale* (or even a *better scale*) for λ in these models. We now consider a third model, establishing that a very good scale has no effect on the validity of diamond for reflecting stationary sets.

Theorem 1.11. *It is relatively consistent with the existence of a supercompact cardinal that there exists a singular cardinal κ for which all of following holds simultaneously:*

(1) *κ is a strong limit of countable cofinality and $2^\kappa = \kappa^+$;*
(2) *there exists a very good scale for κ;*
(3) *$\diamondsuit_S$ fails for some $S \subseteq E^{\kappa^+}_{\mathrm{cf}(\kappa)}$ that reflects stationarily often.*

Proof. Start with a model of GCH, in which there exists a supercompact cardinal, κ. Let $\lambda := \kappa^{+\omega}$.

Step 1. Let $\mathbb{P}_1$ denote the iteration of length $\kappa + 1$ with backward Easton support, where for every inaccessible $\alpha \leq \kappa$, we force with $\mathbb{Q}(\alpha^{+\omega+1})$, and for accessible $\alpha < \kappa$, we use trivial forcing. Let V_1 denote the generic extension by $\mathbb{P}_1$. Then, in V_1, GCH holds, κ is supercompact, and there exists a stationary $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ on which $\diamondsuit_S$ fails.

Step 2. Work in V_1. Fix a normal ultrafilter $\mathcal{U}$ over $\mathcal{P}_\kappa(\lambda^+)$, and the corresponding embedding $j : V_1 \to M \simeq \mathrm{Ult}(V_1, \mathcal{U})$. Let $\mathbb{P}_2$ denote the iteration of length $\kappa + 1$ with backward Easton support, where for every inaccessible $\alpha \leq \kappa$, we force with $\mathrm{Add}(\alpha, \alpha^{+\omega+1})$, adding $\alpha^{+\omega+1}$-Cohen functions from α to α, and for accessible $\alpha < \kappa$, we use trivial forcing.

Let G be $\mathbb{P}_2$-generic over V_1, and work in $V_2 := V_1[G]$. Then, by standard arguments (see [1, §11]), we may find a set G^* such that the embedding j lifts to an embedding $j^* : V_1[G] \to M[G^*]$. Let $\langle F_\alpha : \kappa \to \kappa \mid \alpha < \lambda^+\rangle$ denote the generic functions introduced by the κ_{th}-stage of $\mathbb{P}_2$. Clearly, we may choose G^* in such a way that $F_{j^*(\alpha)}(\kappa) = \alpha$ for all $\alpha < \lambda^+$. Thus, $2^\kappa = 2^\lambda = \lambda^+$. As $\mathbb{P}_2$ has the κ^+-c.c., S remains stationary, and $\diamondsuit_S$ still fails (by the chain condition of $\mathbb{P}_2$, if $\diamondsuit_S$ holds in V_2, then $\diamondsuit^-_S$ holds in V_1. However, $\diamondsuit^-_S$ entails $\diamondsuit_S$, and the latter fails in V_1.) Let $T := \{\alpha \in E^{\lambda^+}_{<\kappa} \mid \mathrm{cf}(\alpha) > \omega, S \cap \alpha \text{ is stationary}\}$. Since S is a stationary subset of $E^{\lambda^+}_{<\kappa}$, and κ is λ^+-supercompact (as witnessed by j^*), the set T is stationary.

Note that the cardinals structure in V_2 is the same as in V.

Step 3. Work in V_2. Let $\mathcal{U}^* := \{X \subseteq \mathcal{P}_\kappa(\kappa^{+\omega+1}) \mid j^*``\lambda^+ \in j^*(X)\}$. Then $\mathcal{U}^*$ is a normal ultrafilter extending $\mathcal{U}$. For every $n < \omega$, let $\mathcal{U}_n$ denote the projection of $\mathcal{U}^*$ to $\mathcal{P}_\kappa(\kappa^{+n})$. Next, let $\langle \mathcal{Q}, \leq, \leq^*\rangle$ denote the variation of supercompact Prikry forcing from [8, Definition 2.9]. That is, instead of working with a single measure, we work with the sequence of measures $\langle \mathcal{U}_n \mid n < \omega\rangle$. By [8, §2], we have:

(a) $\langle \mathcal{Q}, \leq\rangle$ satisfies the λ^+-c.c.;
(b) $\langle \mathcal{Q}, \leq\rangle$ does not add new bounded subsets to κ;
(c) in the generic extension by $\langle \mathcal{Q}, \leq\rangle$, $(\kappa^{+n})^V$ changes its cofinality to ω for every $n < \omega$.

Let V_3 denote the generic extension by $\langle \mathcal{Q}, \leq\rangle$. Work in V_3. Then κ is a strong limit cardinal of countable cofinality, and $2^\kappa = (\lambda^+)^{V_2} = \kappa^+$. Since $\langle \mathcal{Q}, \leq\rangle$ has the λ^+-c.c., the sets S and T remains stationary subset of κ^+, and $\diamondsuit_S$ still fails.

Claim 1.11.1. *S reflects stationarily often.*

Proof. Recall that we work in V_3. Put $T^* := \{\alpha < \kappa^+ \mid \mathrm{cf}(\alpha) > \omega, S \cap \alpha \text{ is stationary}\}$. Since T is stationary, to show that S reflects stationarily often, it suffices to establish that $T \subseteq T^*$. For this, it suffices to prove that if $\alpha \in E^{\kappa^+}_{>\omega} \cap E^{\kappa^+}_{<\kappa}$ and C is a club subset of α, then there exists a club $C' \subseteq C$ lying in V_2. But this is obvious: fix, in V_2, a continuous function $\pi : \mathrm{cf}^{V_2}(\alpha) \to \alpha$ whose image is cofinal in α. Put $C' := C \cap \mathrm{Im}(\pi)$. Then C' is a club subset of C, and $\pi^{-1}[C']$ is a club subset of $\mathrm{cf}^{V_2}(\alpha)$. By property (c) and $\alpha \in E^{\kappa^+}_{>\omega} \cap E^{\kappa^+}_{<\kappa}$, we have $\mathrm{cf}^{V_2}(\alpha) = \mathrm{cf}^{V_3}(\alpha)$, so property (b) entails that $\pi^{-1}[C']$ is in V_2, and hence C' is in V_2, as requested. $\square$

Claim 1.11.2. *There exists a very good scale for κ.*

Proof. Let $\langle P_n \mid n < \omega\rangle$ denote the supercompact Prikry sequence introduced by $\langle \mathcal{Q}, \leq\rangle$ over V_2. For each $n < \omega$, consider the inaccessible cardinal $\kappa_n = P_n \cap \kappa$. For each $\alpha < \kappa^+ = (\lambda^+)^{V_2}$, define a function $t_\alpha \in \prod_{n<\omega} \kappa_n^{+\omega+1}$ as follows:

$$t_\alpha(n) := \begin{cases} F_\alpha(\kappa_n), & F_\alpha(\kappa_n) < \kappa_n^{+\omega+1} \\ 0, & \text{otherwise} \end{cases}, \quad (n < \omega).$$

Then, by Proposition 2.21 of [8], $\langle t_\alpha \mid \alpha < \kappa^+\rangle$ is a very good scale. □

□

1.3. **Uncountable cofinality.** It is worth mentioning that via straightforward modifications to the proofs of subsection 1.1, it is possible to handle singular cardinals of uncountable cofinality, as well. More specifically, we have:

Theorem 1.12. *Suppose λ is a strong limit singular cardinal, and $2^\lambda = \lambda^+$ Then there exists a notion of forcing $\mathbb{Q}'(\lambda^+)$, satisfying:*

1. *$\mathbb{Q}'(\lambda^+)$ is λ^+-directed closed;*
2. *$\mathbb{Q}'(\lambda^+)$ has the λ^{++}-c.c.;*
3. *$|\mathbb{Q}'(\lambda^+)| = \lambda^{++}$;*
4. *in $V^{\mathbb{Q}'(\lambda^+)}$, $\diamondsuit_S$ fails for some stationary $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$.*

In particular, it is possible to obtain the failure of diamond on a stationary subset of λ^+, in the presence of a supercompact cardinal in the interval $(\mathrm{cf}(\lambda), \lambda)$.

Remark. It follows from the proof in [15, §4], that the above $\mathbb{Q}'(\lambda^+)$ is not isomorphic to Cohen's notion of forcing, $\mathrm{Add}(\lambda^+, \lambda^{++})$.

2. Negation of guessing

In [4], Džamonja and Shelah considered a particular consequence of diamond and established the consistency of its negation. To state their result, we need the following two definitions.

Definition 2.1. For a function $f : \lambda^+ \to \mathrm{cf}(\lambda)$, let κ_f denote the minimal cardinality of a family $\mathcal{P} \subseteq [\lambda^+]^{\mathrm{cf}(\lambda)}$ with the property that whenever $Z \subseteq \lambda^+$ is such that $\bigwedge_{\beta<\mathrm{cf}(\lambda)} |Z \cap f^{-1}\{\beta\}| = \lambda^+$, then there exist some $a \in \mathcal{P}$ with $\sup(f[a \cap Z]) = \mathrm{cf}(\lambda)$.

It is obvious that the function $f : \lambda^+ \to \{0\}$ satisfies $\kappa_f = 0$. Also notice that any partition of λ^+ into $\mathrm{cf}(\lambda)$ many sets of cardinality λ^+ induces *a non-trivial function*, that is, a function $f \in {}^{\lambda^+}\mathrm{cf}(\lambda)$ with $\kappa_f > 0$.

Definition 2.2. For a singular cardinal λ, we say that λ^+-*guessing* holds iff $\kappa_f \leq \lambda^+$ for all $f \in {}^{\lambda^+}\mathrm{cf}(\lambda)$.

We refer the reader to [4] for background and motivation for this definition, but let us just mention the obvious fact that $\diamondsuit_{\lambda^+}$ implies λ^+-guessing. In [4, §2], the consistency of the negation of λ^+-guessing has been established:

Theorem 2.3 (Džamonja-Shelah). *It is relatively consistent with the existence of a supercompact cardinal that there exist a strong limit singular cardinal λ and a function $f : \lambda^+ \to \mathrm{cf}(\lambda)$ such that $\kappa_f = 2^\lambda > \lambda^+$.*

Here, we reduce the large cardinal hypothesis significantly by establishing that any model with a strong limit singular cardinal λ with $2^\lambda > \lambda^+$ will do. Moreover, in such a model, any non-trivial function is a counterexample.

Theorem 2.4. *If λ is a strong limit singular cardinal, then*

$$\{\kappa_f \mid f \in {}^{\lambda^+}\mathrm{cf}(\lambda)\} = \{0, 2^\lambda\}.$$

Proof. Since λ is a strong limit, the next lemma tells us that any non-trivial function, f, satisfies $\kappa_f = \lambda^{\mathrm{cf}(\lambda)} = 2^\lambda$. □

Lemma 2.5. *Suppose λ is an infinite cardinal, and $2^{\mathrm{cf}(\lambda)} + \lambda^{<\mathrm{cf}(\lambda)} \leq \lambda^+$.*
Then $\kappa_f = \lambda^{\mathrm{cf}(\lambda)}$ for every non-trivial function $f \in {}^{\lambda^+}\mathrm{cf}(\lambda)$.

Proof. Suppose $f : \lambda^+ \to \mathrm{cf}(\lambda)$ is a function with $\kappa_f > 0$. Fix $\beta < \mathrm{cf}(\lambda)$, and let $A_\beta := \{\delta < \lambda^+ \mid f(\delta) = \beta\}$. Since $\kappa_f > 0$, we have $|A_\beta| = \lambda^+ \geq \lambda^{<\mathrm{cf}(\lambda)}$, so let us fix a surjection $\psi_\beta : A_\beta \to {}^{<\mathrm{cf}(\lambda)}\lambda$ such that $(\psi_\beta)^{-1}\{\eta\}$ has cardinality λ^+ for all $\eta \in {}^{<\mathrm{cf}(\lambda)}\lambda$. Clearly, if $\mathcal{P} \subseteq [\lambda^+]^{\mathrm{cf}(\lambda)}$ is a family of size $< \lambda^+$, then there exists a set $Z \subseteq \lambda^+$ such that $|Z \cap A_\beta| = \lambda^+$ for all $\beta < \lambda^+$, while $Z \cap a = \emptyset$ for all $a \in \mathcal{P}$. This shows that $\kappa_f \geq \lambda^+$.

For all $\delta < \lambda^+$, put $b_\delta := (f(\delta), \psi_{f(\delta)}(\delta))$. For every function $g \in {}^{\mathrm{cf}(\lambda)}\lambda$, denote by $g^* : \mathrm{cf}(\lambda) \to {}^{<\mathrm{cf}(\lambda)}\lambda$ the function satisfying $g^*(\beta) := g \restriction \beta$ for all $\beta < \mathrm{cf}(\lambda)$. Also denote $Z_g := \{\delta < \lambda^+ \mid b_\delta \in g^*\}$.

Claim 2.5.1. $\bigwedge_{\beta<\mathrm{cf}(\lambda)} |Z_g \cap f^{-1}\{\beta\}| = \lambda^+$ *for every $g \in {}^{\mathrm{cf}(\lambda)}\lambda$.*

Proof. For every $\beta < \mathrm{cf}(\lambda)$, letting $\eta := g^*(\beta)$, we get that $Z_g \cap f^{-1}\{\beta\} = \{\delta \in A_\beta \mid b_\delta \in g^*\} = (\psi_\beta)^{-1}\{\eta\}$. □

Let $\{g_i \mid i < \lambda^{\mathrm{cf}(\lambda)}\}$ be an injective enumeration of ${}^{\mathrm{cf}(\lambda)}\lambda$. By $\lambda^+ \leq \kappa_f \leq \lambda^{\mathrm{cf}(\lambda)}$, we avoid trivialities and assume that $\lambda^{\mathrm{cf}(\lambda)} > \lambda^+$. Thus, it suffices to establish the following.

Claim 2.5.2. *For all $a \in [\lambda^+]^{\mathrm{cf}(\lambda)}$, $I_a := \{i < \lambda^{\mathrm{cf}(\lambda)} \mid \sup(f``a \cap Z_{g_i}) = \mathrm{cf}(\lambda)\}$ has cardinality $\leq \lambda^+$.*

Proof. Assume indirectly that $a \in [\lambda]^{\mathrm{cf}(\lambda)}$ is such that $|I_a| > \lambda^+$. By $|I_a| \geq (2^{\mathrm{cf}(\lambda)})^+$ and the Erdös-Rado theorem, let us pick a set $I' \subseteq I_a$ with $|I'| > \mathrm{cf}(\lambda)$ and an ordinal $\gamma < \mathrm{cf}(\lambda)$ such that $g_{i_0}(\gamma) \neq g_{i_1}(\gamma)$ for all distinct $i_0, i_1 \in I'$. Shrinking further, pick $I'' \subseteq I'$ with $|I''| > \mathrm{cf}(\lambda)$ and an ordinal

$\beta > \gamma$ such that $\beta \in (f[a \cap Z_{g_i}])$ for all $i \in I''$. Finally, for all $i \in I''$, pick $\delta_i \in a \cap Z_{g_i}$ such that $f(\delta_i) = \beta$. Since $|a| = \mathrm{cf}(\lambda) < |I''|$, there exist $i_0, i_1 \in I''$ with $i_0 \neq i_1$ and $\delta_{i_0} = \delta_{i_1}$. For $n < 2$, by $\delta_{i_n} \in Z_{g_{i_n}}$, we have $(\beta, \psi_\beta(\delta_{i_n})) \in g^*_{i_n}$, and hence:

$$g_{i_0} \restriction \beta = g^*_{i_0}(\beta) = \psi_\beta(\delta_{i_0}) = \psi_\beta(\delta_{i_1}) = g^*_{i_1}(\beta) = g_{i_1} \restriction \beta,$$

which contradicts the existence of $\gamma < \beta$ with $g_{i_0}(\gamma) \neq g_{i_1}(\gamma)$. □

□

By combining the arguments of the above proof with the ones from [14], it is possible to obtain a lower bound on κ_f even without assuming $2^{\mathrm{cf}(\lambda)} \leq \lambda^+$. Namely, if λ is a singular cardinal and $\lambda^{<\mathrm{cf}(\lambda)} \leq \lambda^+$, then $\kappa_f \geq \mathrm{pp}(\lambda)$ for every non-trivial function $f \in {}^{\lambda^+}\mathrm{cf}(\lambda)$. In particular, if $\lambda > \mathrm{cf}(\lambda) = \omega$, then $\kappa_f \geq \mathrm{pp}(\lambda)$ for every non-trivial f. It follows:

Corollary 2.6. *The following are equivalent:*

1. *Shelah's strong hypothesis;*
2. *λ^+-guessing holds for all singular cardinal λ.*

3. The sup-function on stationary subsets of $[\lambda^+]^\omega$

In this section, we shall supply a negative answer to a following question.

Question (König-Larson-Yoshinobu, [11])**.** Let $\lambda > \omega_1$ be a successor cardinal. Is it possible to prove in ZFC that every stationary $\mathcal{B} \subseteq [\lambda]^{\aleph_0}$ can be thinned out to a stationary $\mathcal{A} \subseteq \mathcal{B}$ on which the sup-function is 1-1?

We refer the reader to [11] for motivation and background concerning this question. Recall that a set $\mathcal{A} \subseteq \mathcal{P}(\lambda)$ is said to be *stationary* (in the generalized sense) iff for every function $f : [\lambda]^{<\omega} \to \lambda$, there exists some $A \in \mathcal{A}$ such that $f``[A]^{<\omega} \subseteq A$. Now, it is obvious that if $\mathrm{cf}([\lambda]^\omega, \subseteq) > \lambda$, then $\mathcal{B} := [\lambda]^\omega$ is a counterexample to the above question. In particular, any model on which the singular cardinals hypothesis fails, gives a negative answer. Thus, in this section, we shall focus on answering the above question in the context of GCH.

Definition 3.1. Given a set $\mathcal{X} \subseteq \mathcal{P}(\lambda)$, denote $S(\mathcal{X}) := \{\sup(X) \mid X \in \mathcal{X}\}$.

Definition 3.2. For an infinite cardinal λ and a stationary set $S \subseteq E^{\lambda^+}_{<\lambda}$, consider the following three principles.

(a) $(1)_S$ asserts that there exists a stationary $\mathcal{X} \subseteq [\lambda^+]^{<\lambda}$ such that:
  - the sup-function on $\mathcal{X}$ is 1-to-1;
  - $S(\mathcal{X}) \subseteq S$.

(b) $(\lambda)_S$ asserts that there exists a stationary $\mathcal{X} \subseteq [\lambda^+]^{<\lambda}$ such that:
  - the sup-function on $\mathcal{X}$ is $(\leq \lambda)$-to-1;
  - $S(\mathcal{X}) \subseteq S$.

(c) $\clubsuit^-_S$ asserts that there exists a sequence $\langle \mathcal{A}_\delta \mid \delta \in S\rangle$ such that:
- for all $\delta \in S$, $\mathcal{A}_\delta \subseteq [\delta]^{<\lambda}$ and $|\mathcal{A}_\delta| \leq \lambda$;
- if Z is a cofinal subset of λ^+, then the following set is stationary:

$$\{\delta \in S \mid \exists A \in \mathcal{A}_\delta(\sup(A \cap Z) = \delta)\}\,.$$

The principle $\clubsuit^-_S$ has been considered in [15], and was found to be the GCH-free version of $\diamondsuit_S$. From this, we easily get the following.

Lemma 3.3. *Suppose λ is an infinite cardinal, and $S \subseteq E^{\lambda^+}_{<\lambda}$ is stationary. Then* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4)*, and if* $2^\lambda = \lambda^+$*, then moreover,* (4) $\Rightarrow$ (1)*, where:*

(1) $\diamondsuit_S$;
(2) $(1)_S$;
(3) $(\lambda)_S$;
(4) $\clubsuit^-_S$.

Proof. (1) $\Rightarrow$ (2) By $\diamondsuit_S$, pick a collection $\{f_\delta : [\delta]^{<\omega} \to \delta \mid \delta \in S\}$ such that for every $f : [\lambda^+]^{<\omega} \to \lambda^+$ there exists some $\delta \in S$ with $f \restriction [\delta]^{<\omega} = f_\delta$. For each $\delta \in S$, pick a cofinal $Y_\delta \subseteq \delta$ of minimal order-type, and find $X_\delta \supseteq Y_\delta$ with $|X_\delta| = |Y_\delta|$ such that $f_\delta``[X_\delta]^{<\omega} \subseteq X_\delta$. It is now easy to see that $\mathcal{X} := \{X_\delta \mid \delta \in S\}$ is as requested.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (4) Let $\mathcal{X}$ exemplify $(\lambda)_S$. For each $\delta \in S$, let $\mathcal{A}_\delta := \{X \in \mathcal{X} \mid \sup(X) = \delta\}$. To see that $\langle \mathcal{A}_\delta \mid \delta \in S\rangle$ witness $\clubsuit^-_S$, we fix a cofinal subset $Z \subseteq \lambda^+$ and a club $C \subseteq \lambda^+$, and argue that there exists $\delta \in C \cap S$ and $A \in \mathcal{A}_\delta$ such that $\sup(Z \cap A) = \delta$. Define a function $f : [\lambda^+]^{<\omega} \to \lambda^+$ as follows:

$$f(\sigma) := \begin{cases} \min(Z \setminus \sup(\sigma)), & |\sigma| \text{ is odd} \\ \min(C \setminus \sup(\sigma)), & |\sigma| \text{ is even} \end{cases}.$$

Since $\mathcal{X}$ is stationary, we may pick some $X \in \mathcal{X}$ such that $f``[X]^{<\omega} \subseteq X$. Put $\delta := \sup(X)$. Then $X \in \mathcal{A}_\delta$, $f``[X]^7$ is a cofinal subset of $Z \cap \delta$, and $f``[X]^4$ is a cofinal subset of $C \cap \delta$. In particular, $\sup(X \cap Z) = \sup(X \cap C) = \delta$, so $\delta \in C$ and we are done.

Finally, if $2^\lambda = \lambda^+$, then a theorem from [15] stating that $\diamondsuit_S \Leftrightarrow 2^\lambda = \lambda^+ + \clubsuit^-_S$, yields (4) $\Rightarrow$ (1). $\square$

Remark. The preceding lemma improves an unpublished result by Matsubara and Sakai, who established the implication (3) $\Rightarrow$ (1) under stronger cardinal arithmetic assumptions.

Since, under mild cardinal arithmetic hypothesis, the above principles coincide, it is interesting to study whether these principles can be separated. The reader who is only interested in the promised solution to the abovementioned question, may now skip to Theorem 3.6 below.

Proposition 3.4. *If the singular cardinals hypothesis fails, then for some singular cardinal λ, $\clubsuit^-_{E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}}$ holds, while $(\lambda)_{E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}}$ fails.*

Proof. By [15], $\clubsuit^-_{E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}}$ holds for every infinite cardinal λ, so let us focus on the second component. Suppose that the singular cardinal hypothesis fails. By a theorem of Shelah [18], in this case, there exists a singular cardinal λ such that $\mathrm{cf}(\lambda) = \omega$, and $\mathrm{cov}(\lambda, \lambda, \mathrm{cf}(\lambda)^+, 2) > \lambda^+$. This means, that for every $\mathcal{X} \subseteq [\lambda]^{<\lambda}$ of size λ^+, there exists some $x \in [\lambda]^\omega$, such that $x \not\subseteq X$ for all $X \in \mathcal{X}$. Now assume indirectly that $\mathcal{X}$ witnesses $(\lambda)_{E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}}$. In particular, $\mathcal{X} \subseteq [\lambda^+]^{<\lambda}$, and $|\mathcal{X}| = \lambda^+$. Pick a function $x : \omega \to \lambda$ with $\mathrm{Im}(x) \not\subseteq X$ for all $X \in \mathcal{X}$. Define $f : [\lambda^+]^{<\omega} \to \lambda^+$ by $f(\sigma) := x(|\sigma|)$ for all $\sigma \in [\lambda^+]^{<\omega}$. Since $\mathcal{X}$ is stationary, there exists some $X \in \mathcal{X}$ such that $f``[X]^{<\omega} \subseteq X$. In particular, $\mathrm{Im}(x) \subseteq X$, contradicting the choice of x. □

Proposition 3.5. $(1)_S \not\Rightarrow \diamondsuit_S$ *for any uncountable cardinal λ and any stationary $S \subseteq E^{\lambda^+}_\omega$.*

Proof. Suppose λ is an uncountable cardinal, $S \subseteq E^{\lambda^+}_\omega$ and $(1)_S$ holds, as witnessed by a stationary set $\mathcal{X}$. Use Cohen forcing to blow up the continuum above λ^+, then $\diamondsuit_S$ fails. Finally, since Cohen forcing is proper, $\mathcal{X}$ remains stationary, so it still witnesses $(1)_S$. □

Answering the above-mentioned question in the negative, we now prove:

Theorem 3.6. *It is relatively consistent with* ZFC *that the* GCH *holds and there exists a stationary subset $\mathcal{B} \subseteq [\aleph_{\omega+1}]^\omega$ that cannot be thinned out to a stationary $\mathcal{A} \subseteq \mathcal{B}$ on which the* sup*-function is injective.*

Proof. Start with a model of GCH in which $\diamondsuit_S$ fails for some stationary $S \subseteq E^{\aleph_{\omega+1}}_\omega$ (by appealing to the forcing from [17, §2], or by forcing with $\mathbb{Q}(\aleph_{\omega+1})$ from section 1.)

Claim 3.6.1. $\mathcal{B} := \{X \in [\aleph_{\omega+1}]^\omega \mid \sup(X) \in S\}$ *is stationary.*

Proof. Suppose $f : [\aleph_{\omega+1}]^{<\omega} \to \aleph_{\omega+1}$ is a given function. Since $\{\delta \in \aleph_{\omega+1} \mid f``[\delta]^{<\omega} \subseteq \delta\}$ is a club, and $S$ is stationary, we may fix some $\delta \in S$ such that $f``[\delta]^{<\omega} \subseteq \delta$. Pick $Y \in [\delta]^\omega$ with $\sup(Y) = \delta$. Then there exists $X \supseteq Y$ such that $|X| = |Y|$ and $f``[X]^{<\omega} \subseteq X$. Such an X is in $\mathcal{B}$, so we are done. □

Now, suppose $\mathcal{A} \subseteq \mathcal{B}$ is stationary on which the sup-function is injective. Then $(1)_{S(\mathcal{A})}$ holds, and as $S(\mathcal{A}) \subseteq S$, also $(1)_S$ holds, contradicting Lemma 3.3 and the fact that $\diamondsuit_S$ fails. □

Let us emphasize that the above theorem does not require large cardinals. Assuming large cardinals, one can obtain a stronger counterexample:

Theorem 3.7. *It is relatively consistent with the existence of a supercompact cardinal that the* GCH *holds and there exists a stationary subset* $\mathcal{B} \subseteq [\aleph_{\omega+1}]^\omega$ *which is large in the following two senses:*

1. *if* $X \in [\aleph_{\omega+1}]^\omega$ *and* $\sup(X) \in S(\mathcal{B})$*, then* $X \in \mathcal{B}$*;*
2. *for every stationary* $\mathcal{A} \subseteq \mathcal{B}$*, there exists some* $X \in [\aleph_{\omega+1}]^{\omega_1}$ *with* $\omega_1 \subseteq X$ *and* $\mathrm{cf}(\mathrm{otp}(X)) = \omega_1$ *such that* $\mathcal{A} \cap [X]^\omega$ *is stationary.*

still, the sup*-function is not injective on any stationary subset of* $\mathcal{B}$*.*

Proof. Consider the model from Theorem 1.9; GCH holds, and there exists a stationary subset $S \subseteq E^{\aleph_{\omega+1}}_\omega$ such that $\diamondsuit_S$ fails. Put $\mathcal{B} := \{X \in [\aleph_{\omega+1}]^\omega \mid \sup(X) \in S\}$. Then (1) is obvious, and by Lemma 3.3, the sup-function is not injective on any stationary subset of $\mathcal{B}$.

Finally, clause (2) follows from the general fact from [5] that if κ is a supercompact cardinal, then $V^{\mathrm{Col}(\omega_1,<\kappa)} \models \mathrm{Refl}^*([\kappa^{+\omega+1}]^\omega)$. $\square$

Note that $\mathrm{SAP}_{\aleph_\omega}$ (and hence $\square^*_{\aleph_\omega}$) necessarily fails in a model consisting of such $\mathcal{B}$, hence, the high consistency strength.

Discussion. To get a finer understanding of Theorem 3.6, we now discuss a more direct argument which allows to point our finger at the role of the injectivity of the sup-function.

We start with a model of GCH, and let $\lambda := \aleph_\omega$. We consider the forcing notion $\mathbb{X}(\lambda^+)$, as an alternative to $\mathbb{S}(\lambda^+)$ from section 1. A condition $\mathcal{X}$ is in $\mathbb{X}(\lambda^+)$ iff $\mathcal{X} \subseteq [\lambda^+]^\omega$ and $S(\mathcal{X})$ is a bounded subset of λ^+. A condition $\mathcal{X}'$ is stronger than $\mathcal{X}$ iff $\mathcal{X}' \supseteq \mathcal{X}$ and $S(\mathcal{X}') \sqsupseteq S(\mathcal{X})$. To study the injectivity of the sup-function, we also consider $\mathbb{X}^1(\lambda^+)$, where $\mathcal{X} \in \mathbb{X}^1(\lambda^+)$ iff $\mathcal{X} \in \mathbb{X}(\lambda^+)$ and for all $\tau \in S(\mathcal{X})$, there exists a *unique* $X \in \mathcal{X}$ with $\sup(X) = \tau$.

Let $\dot{S}(\lambda^+) := \{\langle \check{\sup(X)}, \mathcal{X}\rangle \mid X \in \mathcal{X} \in \mathbb{X}(\lambda^+)\}$ be the canonical name for the generic subset of $E^{\lambda^+}_\omega$ introduced by $\mathbb{X}(\lambda^+)$ and by $\mathbb{X}^1(\lambda^+)$. Now, instead of forcing with $\mathbb{Q}(\lambda^+) = \mathbb{S}(\lambda^+) * \mathbb{KAD}(\dot{S}(\lambda^+))$, we shall force with $\mathbb{P}(\lambda^+) := \mathbb{X}(\lambda^+) * \mathbb{KAD}(\dot{S}(\lambda^+))$. To compare, we also define $\mathbb{P}^1(\lambda^+) := \mathbb{X}^1(\lambda^+) * \mathbb{KAD}(\dot{S}(\lambda^+))$.

The same arguments as in section 1 shows that $\mathbb{P}(\lambda^+)$ and $\mathbb{P}^1(\lambda^+)$ satisfies the λ^{++}-c.c., and that it has a dense subset in which every increasing sequence of conditions of length $< \lambda^+$ has a least upper bound.

We now sketch the changes to be made to the proof of Theorem 1.8, to show that if $G * H$ if $\mathbb{P}(\lambda^+)$-generic, letting $\mathcal{X} := \bigcup G$, then $V[G][H] \models \mathcal{X}$ is stationary. Instead of fixing a name for a club $E \subseteq \lambda^+$, we fix a name for a function $e : [\lambda^+]^{<\omega} \to \lambda^+$. Instead of deciding the value for $\min(E \setminus \alpha)$, we decide $e \restriction [\alpha]^{<\omega}$, utilizing the fact that $\mathbb{P}(\lambda^+)$ does not add bounded subsets of λ^+. Then the analogue of Claim 1.8.11 is that for all $g \in {}^Z 2$, q_g forces that τ is a closure point of e, and moreover, q_g decides $e \restriction [\tau]^{<\omega}$ to be, say, $e^g : [\tau]^{<\omega} \to \tau$.

While in the previous section, we didn't care about $E \cap \tau$, and only focused on the fact that τ is a (closure) point of E, here we really need to know $e \restriction [\tau]^{<\omega}$. Notice, however, that if $g_0 \neq g_1$, then it is possible that $e^{g_0} \neq e^{g_1}$. This is a subtle point, and we shall get back to it at the end of our discussion.

From here, we continue smoothly until we reach to the construction of the sequence of conditions $\langle p_\alpha = \langle \mathcal{X}_\alpha, k_\alpha\rangle \mid \alpha < \theta\rangle$ and the chain of functions $\{h_\alpha : Z \cap (\varepsilon_\alpha + 1) \to 2 \mid \alpha < \theta\}$. At the induction base, instead of choosing $\langle s_0, \emptyset\rangle \in D_0$ with $\bar{s} \cup \{\tau\} \sqsubseteq s_0$, we first pick an arbitrary cofinal subset $X \in [\tau]^\omega$ and then choose $\langle \mathcal{X}_0, \emptyset\rangle \in D_0$ which is stronger than $\langle \bar{\mathcal{X}} \cup \{X\}, \emptyset\rangle$. Once the construction is completed, we let $g := \bigcup_{\alpha<\theta} h_\alpha$, and let $\langle \mathcal{X}', k'\rangle$ be an upper bound for the increasing sequence, $\langle p_\alpha \mid \alpha < \theta\rangle$.

Then $\langle \mathcal{X}', k'\rangle \geq q_g$, and q_g decides $e \restriction [\tau]^{<\omega}$ to be $e^g : [\tau]^{<\omega} \to \tau$. Pick a cofinal $X' \subseteq [\tau]^\omega$ which is closed under e^g. Then $\langle \mathcal{X}' \cup \{X'\}, k'\rangle$ is a legitimate condition (because $\sup(X') = \tau \in S(\mathcal{X}_0) \sqsubseteq S(\mathcal{X}'))$, and it forces that there exists some $X \in \mathcal{X}$ with e^g "$[X]^{<\omega} \subseteq X$, as requested.

So, in $V[G][H]$, $\mathcal{X}$ is stationary subset of $[\aleph_{\omega+1}]^\omega$, GCH holds, and $\diamondsuit_{S(\mathcal{X})}$ fails.

Now, what would have gone wrong had we force with $\mathbb{P}^1(\lambda^+)$, instead of $\mathbb{P}(\lambda^+)$? We know that for all $g \in {}^Z2$, there exists some k_g such that $q_g = \langle \bar{\mathcal{X}}, k_g\rangle$, and that $S(\bar{\mathcal{X}}) \subseteq \tau$. Clearly, there is no way of insuring that for some g, there already exists an $X \in \bar{\mathcal{X}}$ which is closed under e^g, but this is a density argument, so we may consider extensions of q_g.

Now, for all g, since τ is a closure point of e^g, there indeed exists a cofinal subset $X_g \in [\tau]^\omega$, which is closed under e^g, and it is tempting to just take $\langle \bar{\mathcal{X}} \cup \{X_g\}, k_g\rangle$. So, here is the problem — how do we know that the latter is a legitimate condition? As $\tau \in c_i^{k_g}$ for all $i \in Z$, we need, in particular, to establish that $x_i^{k_g} \cap \tau \neq \overrightarrow{A}_i(\tau)$ for all $i \in Z$. In the above construction, we done so by throwing a countable cofinal subset of τ to $\mathcal{X}_0$, thus, insuring that $\tau \in S(\mathcal{X}_0) \sqsubseteq S(\mathcal{X}')$. This time, we are allowed to throw only a single cofinal subset of τ to $\mathcal{X}_0$, so we need to throw cofinal subset of τ which is closed under e^g for all $g \in {}^Z2$, at once. But, this turns out to be impossible.

4. Open problems

Let λ denote a singular cardinal. Probably the most interesting open question in this area is the following question of Shelah:

Question 1. Does $2^\lambda = \lambda^+$ imply $\diamondsuit_{E^{\lambda^+}_{\mathrm{cf}(\lambda)}}$? Does GCH imply $\diamondsuit_{E^{\lambda^+}_{\mathrm{cf}(\lambda)}}$?

By [21] and the fact that $E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ reflects stationarily often, a negative answer to the above question witnesses the failure of $\square^*_\lambda$, so large cardinals are necessary.

Question 2. Suppose $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ reflects stationarily often, must $\mathrm{NS}_{\lambda^+} \restriction S$ be non-saturated?

By [15], a negative answer to the above question witnesses the failure of $\square^*_\lambda$ (actually, of SAP_λ). Note that by [9], $\mathrm{NS}_{\lambda^+} \restriction E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ is indeed non-saturated. Also note that if one does not require reflection, then by results of Woodin and Foreman (see [6, §8]), $\mathrm{NS}_{\aleph_{\omega+1}} \restriction S$ can consistently be saturated for some stationary (non-reflecting) $S \subseteq E^{\aleph_{\omega+1}}_{\omega}$.

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